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Reflecting on fraction addition: Using the mirror formula to visualise unit fraction addition

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This article proposes a practical method of teaching the addition of unit fractions using a series of mirror equation experiments.

Introduction

Mathematical knowledge and skills are crucial for success in many of today's endeavours. Mathematical modelling prepares students to develop a working knowledge of dynamic information systems (Shahbari and Peled, 2017). Addition of unit fractions is one of the important, yet more difficult, concepts for students to master initially. Our experience is that these concepts are often taught through repetitious textbook examples, often with very little or no real-life application, making them abstract in nature. Visual and manipulative models of the relationships between fractions provide a crucial tool for developing student understanding (Scaptura et al. 2007; Suh et al. 2005) and obtaining proficiency with fractions (Tsai and Li, 2017).

Fractions can be challenging to teach and learn (Shahbari and Peled, 2017) and their real-life relevancy is often difficult to articulate (Suh et al. 2005). Research suggests that fractions, particularly unit fractions, are more effectively understood by students when taught in an immediate context (Ganor-Stern et al. 2011).

The combination of the importance of, and difficulty with, manipulating fractions as students begin to learn about adding fractions, highlights the value of research into developing activities that enable students to visualise and manipulate unit fraction addition (Kërënxhi and Gjoci, 2017; Tsai and Li, 2017). Research also demonstrated that mathematical and scientific skills development can be optimised by activities taking a 'middle ground' between student-led inquiry and teacher-led instruction (Godino et al. 2016). Practical teaching methods provide opportunities for students to evaluate data and become more capable of modifying their conclusions based on new evidence (Holmes et al. 2015).

This article demonstrates an example of a practical method for teaching unit fraction addition, utilising student-led inquiry and teacher-led instruction. Students performed and evaluated the results of an optical mirror equation experiment.

The abstract problem

A potentially major difficulty when learning fractions involves the basic operations with fractions, specifically addition and subtraction of fractions with different denominators (Shahbari and Peled, 2017). When this concept is taught using repetitious textbook exercises the student potentially develops considerable procedural knowledge (Kërënxhi and Gjoci, 2017; Suh et al. 2005). However, students can be provided with the skills to perform unit fraction addition in real-life examples such as those used in optics and electronic circuitry. These are not always obvious or present in textbooks (Kërënxhi and Gjoci, 2017; Holmes et al. 2015).

A visible solution

Students in a multi-age group of 13 to 17 year olds from the STEM club (Da Vinci club) at Esperance Anglican Community School (Esperance, Western Australia), participated in a series of 3 × 30 minute experimental lessons. These lessons aimed to develop an understanding of the addition of unit fractions by analysing the results of a mirror equation experiment. The lessons employed a mix of teacher-led instruction and student-led inquiry (Godino et al. 2016). The students were shown how to setup the apparatus, shown the mirror equation and the meaning of each variable was explained.

Lesson 1

The mirror equation (Serway, 1992) (Equation 1) applies to both concave and convex mirrors and describes the relationship between the object and image distances from the mirror and the mirror's focal length. The equation is:

$$\frac{1}{\text{(object distance from mirror)}} + \frac{1}{\text{(image distance from mirror)}} = \frac{1}{\text{(focal length of mirror)}} \qquad [1]$$

For ease of use, the students were taught to write the equation algebraically (equation 2) and it is shown diagrammatically in Figure 1. The object X is at a distance x from the concave mirror M. The mirror has a focal length f. The image Y is formed at a distance y from the mirror.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{f} \qquad [2]$$

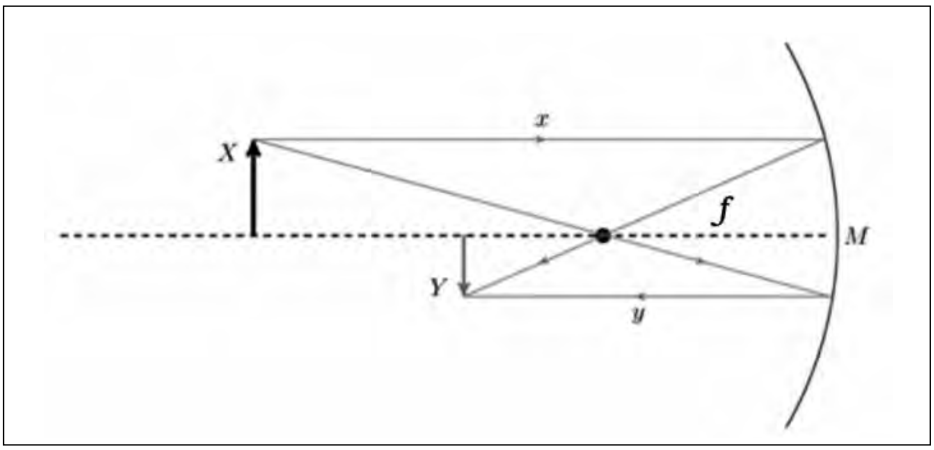

Figure 1. Diagram showing the variables of the mirror equation. The object *X* is at a distance x from the concave mirror *M* of focal length *f*. The image *Y* is formed at a distance *y* from the mirror.

The equation can be solved both experimentally and theoretically. The real-life application of the use of this equation is in industry (optics), for example telescopes and microscopes.

Students set up the experiment, using a torch as the light source—a light emitting diode (LED) torch is very effective as it has a visible pattern of individual LEDs. A concave mirror was secured in a fixed position using a retort stand and clamp, and a rigid clean grey surface was used for the reflected image. Working as a group, and moving the surface and torch towards and away from the mirror, the students quickly found a position where the object and image were the same distance from the mirror and the image was sharp or in focus (Figure 2).

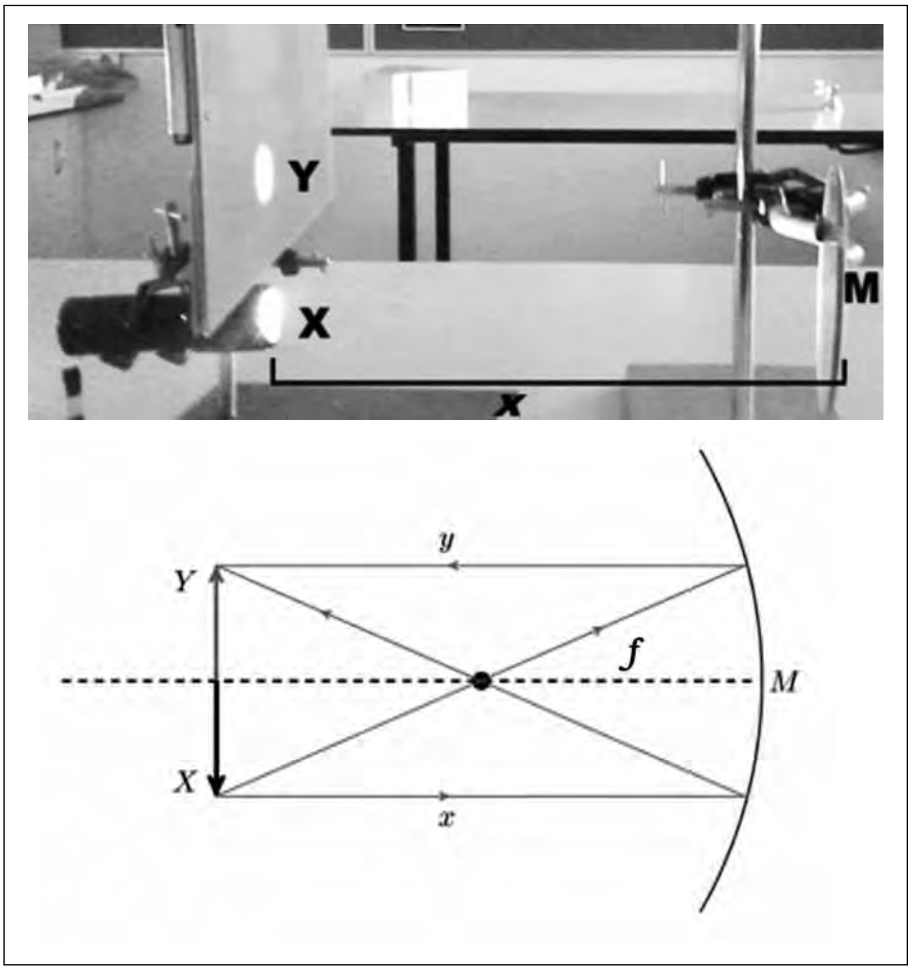

Figure 2. Experimental setup for the light source or object (*X*) and the image (*Y*) at the same distance from the concave mirror *M* of focal length *f* (top). The setup is shown diagrammatically underneath.

In Figure 2, the distance of the object and image form the mirror (measured in millimetres and rounded to centimetres) were both approximately 36 cm (Equation 3), providing:

$$x \approx y \approx 36 \qquad [3]$$

Substituting in for the original equation (Equation 2), the students were asked to predict the focal length of the mirror (Equation 4).

$$\frac{1}{36} + \frac{1}{36} = \frac{1}{f} \qquad [4]$$

Students were not to use calculators at this point to calculate the answer as the aim was to provide an understanding of how to handle fractions. At this stage, depending on the amount of prior knowledge and mathematical ability, some students suggested that two possible solutions were likely to occur. Generally, the older students initiated the discussion and the younger students started to gain an understanding and joined in the discussion to develop the solution. A common mistake that can occur is where the student adds both the numerator and denominator (Equation 5).

$$\frac{1}{36} + \frac{1}{36} = \frac{2}{72} = \frac{1}{36} \qquad [5]$$

The final step of using equivalent fractions to simplify the fraction may already be known by the student, or can be guided by the teacher. The incorrect process in Equation 5 would determine the focal length of the mirror to be 36 cm; however, at this stage, the students were alerted to an impossibility where the answer equals either part of the sum (Equation 6).

$$\frac{1}{36} + \frac{1}{36} = \frac{1}{36} \qquad [6]$$

To determine the focal length experimentally, a second experiment was now performed. A light box with multiple slits was employed to produce parallel light beams that were focussed to a point by the mirror. (Figure 3). This light source was employed as it produces parallel light beams, enabling ready determination of the focal length.

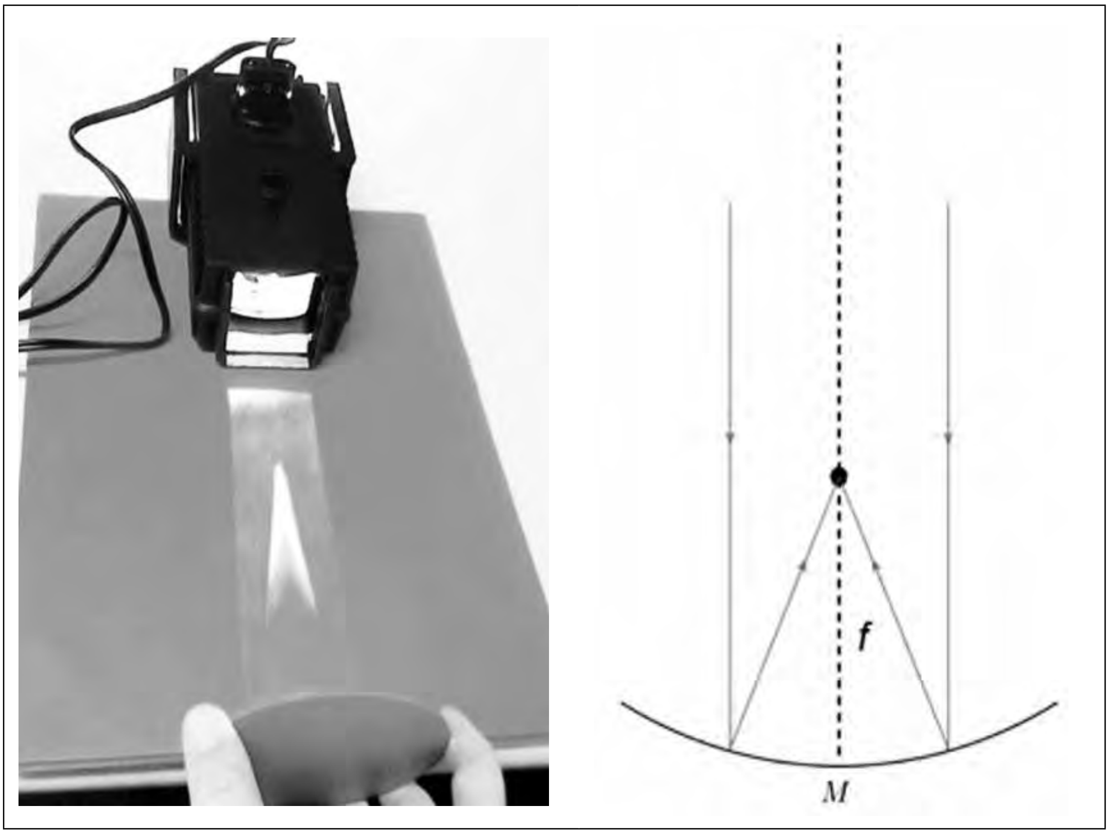

Figure 3. Experiment to determine the focal length of the mirror used in the investigation where a parallel beam of light converges at the focal point of the concave mirror (left) and shown diagrammatically (right).

In this experiment, the focus was found to be approximately 18 cm from the mirror. The students were then shown the following revised equation (Equation 7), now complete with measured values.

$$\frac{1}{36} + \frac{1}{36} = \frac{1}{18} \qquad [7]$$

Most students quickly identified that the relationship in Equation 8 representing the results from the experiment was correct. This step provided an avenue for students to reflect on the correct method for adding unit fractions (Equation 8).

$$\frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18} \qquad [8]$$

Students recognised that the simplification in Equation 8 was a similar process as the simplification in Equation 5. This practical experiment reinforced the approach below necessary to add fractions:

- First represent the fractions as equivalent fractions with the same denominator
- Fractions with the same denominator can be added by adding their numerators
- Equivalent fractions in lowest terms are often preferred.

Lesson 2

This lesson works best when the students have had some prior practice in common denominators and simplifying fractions; the experiment can be used to reinforce the concepts in a practical manner. The setup is almost identical to Lesson 1 except that the distance from the mirror to the object and from the mirror to the image are different (Figure 4). The mirror is maintained at a fixed location and the torch is moved until an image resembling the LED pattern in the torch is reproduced on the more distant wall.

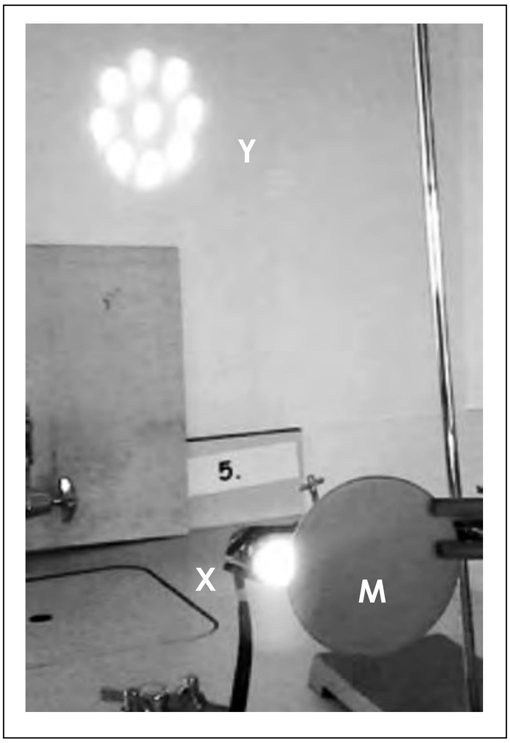

Figure 4. Oblique view of the experiment where the object (*X*) and the image (*Y*) are at different positions from the mirror (*M*).

In Figure 4, the object and the image were approximately 20cm and 150cm respectively from the mirror. Knowing that the focal length of the mirror was 18cm, the original mirror equation becomes:

$$\frac{1}{20} + \frac{1}{150} = \frac{1}{18} \qquad [9]$$

The lowest common multiple (LCM) for the denominators is 900. In order to determine this, the students were told to list each of the equation's denominators and multiply each by consecutive integers until there was a multiple that was the same for each of the denominators. Although calculating this could be challenging, it is achievable for students studying this topic. Converting the fractions on the left hand side of the equation to have a common denominator of 900 gives (Equation 10):

$$\frac{45}{900} + \frac{6}{900} = \frac{51}{900} \qquad [10]$$

The students surmised that the slight difference between the values, as observed in equation 11, is due to measurement error, where the measurements taken using standard rulers may not have been as precise as possible. In recognition of experimental error, the right hand side of the equation can be simplified to:

$$\frac{51}{900} \approx \frac{50}{900} = \frac{1}{18} \qquad [11]$$

Evidence of the development of the student's learning of fraction addition is that they were able to tackle with little guidance this second scenario where x and y were different.

Extension lessons

The addition of fractions using the mirror equation can be extended by investigating different object and image distance combinations, as well as using mirrors with different focal lengths. Importantly, the object (light source) distances need to be greater than the focal length of the selected mirror in order to obtain real images that can be projected on to a screen or flat surface.

These experiments could be performed with lenses on a light bench, using the lens maker's equation, which has the same form as the mirror equation. There are considerable avenues for related extension work, depending on the lesson time and the skill level of the students. These include:

- Fraction subtraction, by rearranging of the mirror formula.
- Ratio (Shahbari and Peled, 2017), with the magnification factor of the image.
- Reciprocals, including 'lens power', measured in Dioptres.
- Online simulations, where students also gain conceptual understanding (Suh et al. 2005).
- Investigations of the use of technology, such as smartphones, to improve the accuracy and precision of the measurements (Igoe et al. 2017).

Further extension lessons are also available with a similar type of equation used in electronics. For example, in the study of total resistance in parallel circuits, where the number of terms in the unit fraction equation is limited only by the number of individual resistors (Khan, 2016).

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_{Total}} \qquad [12]$$

Conclusion

Students were given the opportunity to learn and practise unit fraction addition in the context of the mirror equation. Based on teacher-led instruction, the students learned how to add fractions using experimental data on the mirror equation. This was studied through visual and physical manipulation of experiments, the development of conceptual knowledge on the addition of unit fractions, and then applying the method to more complex experimental setups. This article is a practical method for teaching a fundamental mathematical topic that is often a source of difficulty for students and teachers.

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