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Children's mathematical reasoning:
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Reasoning underpins students' mathematical understanding and promotes creative thinking. It is regarded as a key mathematical proficiency. This article discusses the reasoning actions that primary children employed and teachers noticed for "What else belongs?" a task focused on forming and testing conjectures.

Reasoning involves more than explaining

Reasoning is one of the proficiencies in the *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority [ACARA], 2016) and it plays a critical role in developing students' understanding and promoting creative thinking in mathematics (Carpenter, Franke & Levi, 2003). It has been included in previous state-based mathematics curriculums (for example, Victorian CSFII) where it was a component of 'working mathematically' (Victorian Curriculum and Assessment Authority, 2002). Reasoning as defined in the *Australian Curriculum: Mathematics* requires students to develop:

> ... an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising… (ACARA, 2016).

This definition lists the variety of reasoning actions needed to develop and communicate a convincing argument which Pedemonte (2007) claimed to be at the heart of developing mathematics knowledge. Various researchers and writers of mathematics curriculum and learning materials have focussed on two main elements of reasoning: generalising and justifying (proving). Stylianides (2010) argued that generating and validating new knowledge through inquiry-based activity involves these two elements of reasoning. Much of the reasoning research involving primary children has concerned generalising growing patterns for early algebra learning (for example, Cooper & Warren, 2008; Hourigan & Leavy, 2015; Mulligan & Mitchell, 2009), whilst Carpenter and colleagues (2003) explored ways to support primary students to justify. Frameworks have been developed to either identify steps in the process of generalising, levels of generalisation statements, or levels of justification or proof (Carpenter, et al. 2003; Ellis, 2007; Lannin, Ellis & Elliot, 2011; Stylianides, 2010).

From our work with teachers (Loong, Vale, Bragg & Herbert, 2013) and other studies (Clarke, Clarke & Sullivan, 2012) we know that teachers often associate reasoning with explaining and many do not make reference to any other reasoning actions listed in the *Australian Curriculum: Mathematics.* This is perhaps understandable as many of the examples included in the curriculum concern explaining, but the year level statements in the *Australian Curriculum: Mathematics* also include other reasoning actions, for example "justifying representations" (Year 1), "using known facts to derive strategies" (Year 2), "comparing and contrasting" (Year 2), "generalising from number properties" (Years 3 and 4), "deriving… communicating…evaluating" (Year 5) and "explaining why" (Year 6).

We have been working with teachers in a number of schools through the Mathematical Reasoning Professional Learning Research Program to develop tasks and lessons to elicit and challenge children's mathematical reasoning, and to support teachers to include opportunities for reasoning in their mathematics lessons. We focussed especially on tasks that provide opportunities for comparing and contrasting, forming conjectures (that is, generalising), and testing, validating and justifying these conjectures. In this article we discuss the reasoning actions demonstrated by children in Years 3 and 4 from four different schools who worked on the "What else belongs?" task (Small, 2011) during demonstration lessons.

Reasoning task: {30, 12, 18} What else belongs?

The learning goals for the lessons using this task included “generalising from number properties” (Years 3 and 4). The lesson had six parts: launching the problem to the whole class; children working in pairs on the problem with the teacher roving the class probing, supporting and challenging children’s reasoning; whole-class orchestrated discussion where the children’s conjectures, explanations and justifications were discussed and challenged; children working in pairs to create their own set of numbers; whole class discussion; and finally, the children completing a self-assessment.

At the beginning of the lesson the set of numbers {30, 12, 18} was displayed to the whole class and the problem posed, ‘What is the same about these numbers?’. When launching the problem the teacher did not take verbal responses but rather used stimulating questioning to encourage children to compare and contrast what they knew about these numbers:

> “Think about what you know about each of these numbers. [Pause] I wonder, could these numbers belong together? [Pause] I’m wondering what reasons there might be for these numbers belonging together in this group. [Pause] Why do you think these numbers belong together? [Pause] What is your reason? [Pause] I wonder if there is more than one reason...?”

These questions were intended to encourage children to form conjectures about possible common properties of these numbers. The children then worked with a partner to solve the problem and respond to three questions:

1. These numbers belong together because …”
2. Other numbers that belong with this group are...
3. How do you know that all these numbers fit with your reason? Use words, numbers or drawings to explain.

The first of these questions called on children to generalise, that is, to compare and contrast the properties of the numbers in the set in order to form a conjecture about a common property for the numbers; the second question invited them to extend the generality by identifying other numbers that could join the set. We chose this set of numbers because there are a number of possible common properties and the children might identify more than one property. The third question required the children to test their conjecture and justify their responses to the previous two questions, that is, to defend and justify their generalisation.

Comparing and contrasting

During the paired working time, the children noticed similarities in the numbers, such as their closeness or the size of the numbers. Many children searched and noticed common properties, such as: the numbers were two digit numbers, even, or belonged to a counting pattern (for example, Figure 1). Comparing and contrasting actions were elicited when the teacher inquired into the children’s work on the task. For example:

- “We’ve been looking at the numbers and working out what’s different and we’ve so far worked out that they’re all over 10, under 40 and they each have 2 digits.” (Ryan , School D).
- “Well we’re trying to figure out well like something about the numbers like see if we can like split them to see if there is something to do with them.” (Shanti, School D)

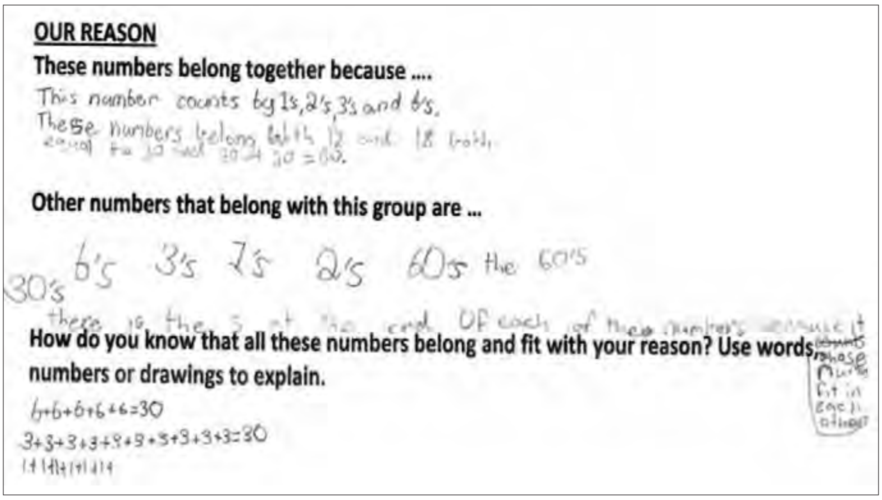


Figure 1. Noticing common properties and relationships (Dane & Lloyd).

Madison and Ron randomly recorded the following number facts on the worksheet: “12 × 1 = 12, 4 × 3 = 12, 12 × 5 = 60, 2 × 14 = 60 (sic), 3 × 6 = 18; 5 × 6 = 30, 3 × 10 = 30” but did not notice that 3 was a factor. An enabling prompt from their partner or the teacher about “what is the same?” and “what is different?” in their list of number facts may have helped them to notice the common factor.

Some children noticed the additive relationship between the numbers, for example, 12 + 18 = 30 (see Figure 1). The teachers observing these lessons commented that often it was the higher achieving children who noticed the additive relationship and sometimes they did not continue to search for common properties. In subsequent demonstration lessons we changed the numbers in the set to {36, 12, 18} to avoid this problem.

Forming conjectures and generalisations

As shown in Figures 1 to 6 the children formed conjectures about the similarity or commonality using phrases such as: “all two-digit”; “each number is [created] by 2s they are evan (sic) numbers”; “all in the threes [twos, sixes] counting pattern”; “all in the count by 2s pattern”; “count by 6s”; “in the 6 column 6 times tables”; “all in the groups of 6”; “6 fits into each number”; or “in the times patterns”.

The teachers observing the lessons commented on the difficulty that the children had in using formal mathematical terms such as even, digit, factor or multiple when discussing and recording their conjectures (see Bragg, Herbert, Loong, Vale & Widjaja, 2016). Nevertheless, the children did demonstrate understanding of these properties. This can be seen by the relationship between the terms that they used to describe them and the way in which they identified other numbers that could belong to the group. For example, Figures 2 and 3 show students using counting patterns, that is repeated addition, to find other numbers that could belong to the group.

Explaining, validating and justifying conjectures

Whilst roving the room the teacher reminded the children that they need to be able to explain their conjecture and convince others that their reason works for all numbers. The teacher also used this time to identify different conjectures for presentation during the whole class discussion.

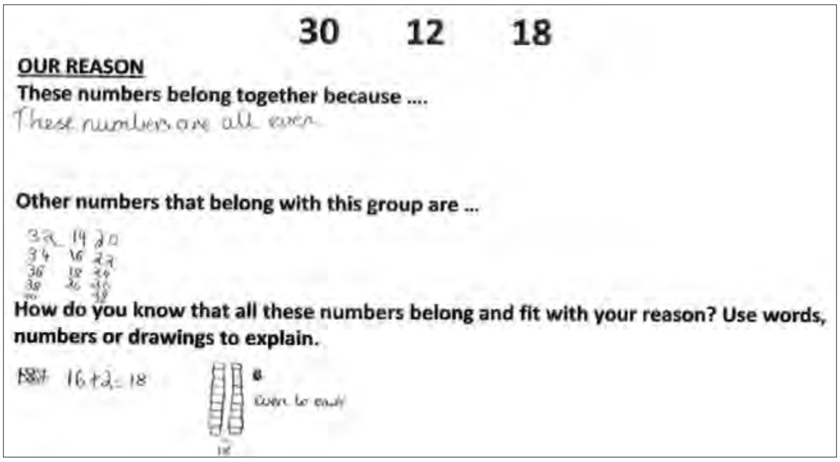


Figure 2. Recording a conjecture and explaining ‘even’ using an example (Tanika & Trish).

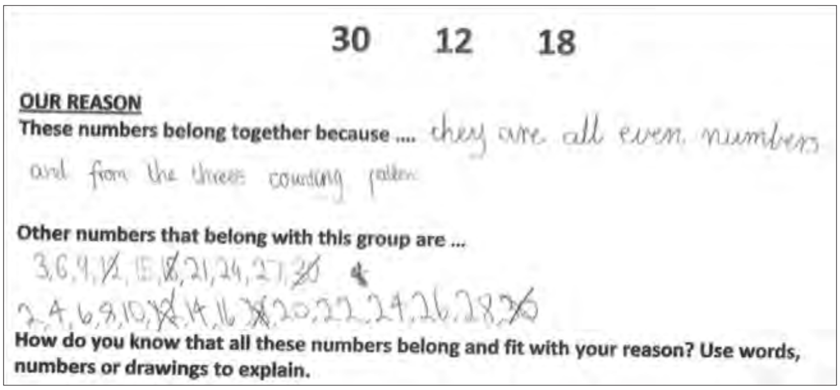


Figure 3. Forming and testing two conjectures about common properties (Jackson & Milly).

When explaining their conjecture to each other, the teacher or the class, some children used analogy. In the following case the children were searching for other numbers that could belong:

Teacher: OK. How do you know these are odd numbers?

Naomi: Because two of them have a partner and there is one left out.

Teacher: Oh, so then how do you know this is an even number? … OK so can you just show me again how you are doing that?

Dennis: [pairing his fingers on each hand] Two partners, two partners, groups of two …

Others used diagrams and number facts. Tanika and Trish (Figure 2) gave two explanations of 18 as an even number: by adding 2 to 16 and a diagram to show that even numbers can be split equally. "Even to each" is interpreted as meaning the two groups are the same. Other children also used 'even' and 'equal' to mean 'the same'. However, whilst Tanika and Trish, and others explained the meaning of even, they did not verify that each number in the original set was even.

Many children were able to validate their conjecture by verifying that their rule worked for all numbers in their written response. Jackson and Milly (Figure 3) did this when they included a number count for the second question and crossed out the numbers in the set to show that they belonged. Bonnie and Eve (Figure 4) used diagrams to show that each number in the set is even. Like Tanika and Trish, they argued that even numbers can be split equally in two, and drew 30 as (5 × 2) + (5 × 2) + (5 × 2). Jessie and Elisabeth (Figure 5) used multiplication facts to verify that "they are all in the 6 groups of".

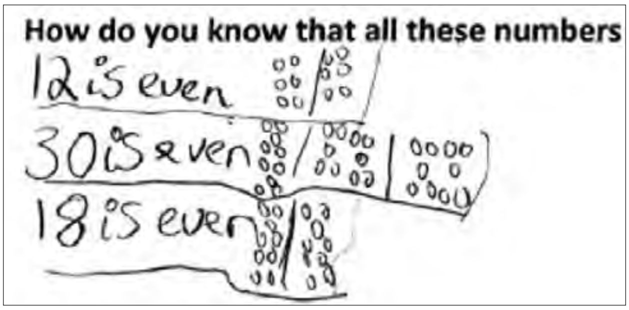


Figure 4. Justifying conjecture using diagrams (Bonnie & Eve).

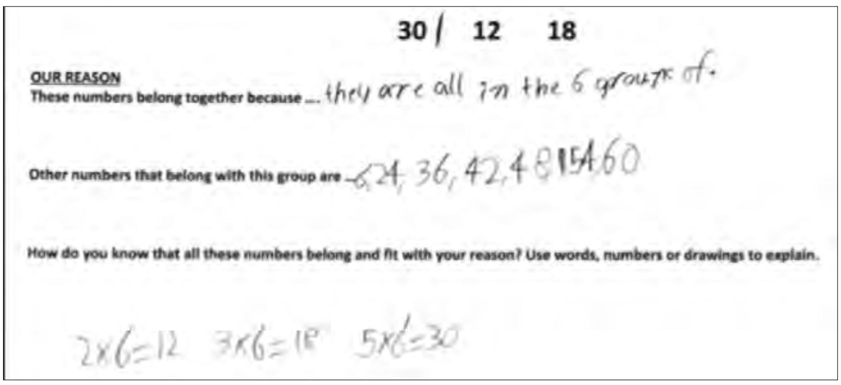


Figure 5. Justifying conjecture using known facts (Jessie & Elisabeth).

During the whole class discussion the teacher challenged the selected pairs of students to justify their conjecture: "Convince us that this reason works for all the numbers [30, 12, 18, 6]." So in addition to explaining the meaning of the common property, the teacher also discussed Question 3, which called on them to verify that their conjecture worked for all the numbers and others that they included in the set, for example, 6. For instance, during whole class discussion Dane and Lloyd (Figure 1) were challenged to justify their conjectures. Another child suggested that they count by sixes to verify that each number was in the sixes' counting pattern and the teacher then conducted a choral count. Following the choral count Dane then explained the meaning of "fits in". He said, "6 goes into that [pointing at 30] 5 times, 6 goes into that [pointing at 12] 2 times, 6 goes into that [pointing at 18] 3 times." By providing these facts for each number, Dane verified the conjecture for each number in the group. Even though these two students did not verify their conjecture for each number in their written response, they did so when asked to convince others.

The whole class discussion enabled children to hear different explanations and justifications for the same conjecture. The responses to Questions 2 and 3 discussed here illustrate the difference between explaining and justifying; the key point being that explaining a property using an example is not sufficient to validate a conjecture, in other words, to justify a generalisation.

Challenging children's reasoning for creative thinking

A number of the children's written responses and justifications provided further opportunity for creative thinking and mathematical reasoning. Brett and Jarrod (Figure 6) used a more general property to find other numbers to join the group. They identified "all in the 3s counting pattern" as the common property and listed "3, 6, 9, 15, 21, 24, 27, 60, 72, 84, 96, 108, 120" as other numbers that could join the group. These two boys counted by 3, leaving out the numbers already in the group, doubled 30 to list 60 as another number that belonged, and then counted on by 12 to list more numbers that belonged.

When this pair of students were invited to report and justify their reason, Brett offered 216 as another number that was in the counting pattern and explained: "Yeah, and I just got to 108 and doubled it." His response revealed that rather than using the 3s counting pattern to find other numbers that belonged to the group, Brett used a conjecture, that is, the doubling of any number in the counting pattern to find another number that belonged. They had also used the conjecture that multiples of 12 are also multiples of 3 when they counted on by 12s from 60 to find other numbers. Their responses on the worksheet and during whole class discussion show that these children demonstrated knowledge and use of relations between properties and a more general structure of these properties. During this lesson the teacher did not take up the opportunity to challenge these children to justify their implied general conjecture. However it is suggested here that the creative thinking of these children provides a great opportunity to build on their reasoning by posing another problem for this pair of students, or for the whole class:

> "If you double a number in a counting pattern then that number will also be in the counting pattern. True or false. Why?"

A teacher might support some learners in the class by suggesting that they explore particular counting patterns such as threes, and sixes or other counting patterns. Alternately the challenge might relate to the second conjecture implied in the students' response:

> "Numbers in the 12s count (or numbers that are multiples of 12) are also in the 3s count (multiples of 3). True or false. Justify."

In responding to these problems, children could use examples to test this conjecture but they would need to identify relationships to explain why and verify the conjecture. They might do this using drawings (such as Figure 4), or structured number counts (see Figure 7), or logical argument using symbolic statements, such as: $24 = 12 \times 2$. Since $12 = 3 \times 4$, then $24 = 3 \times 4 \times 2 = 3 \times (4 \times 2)$.

+3

+12

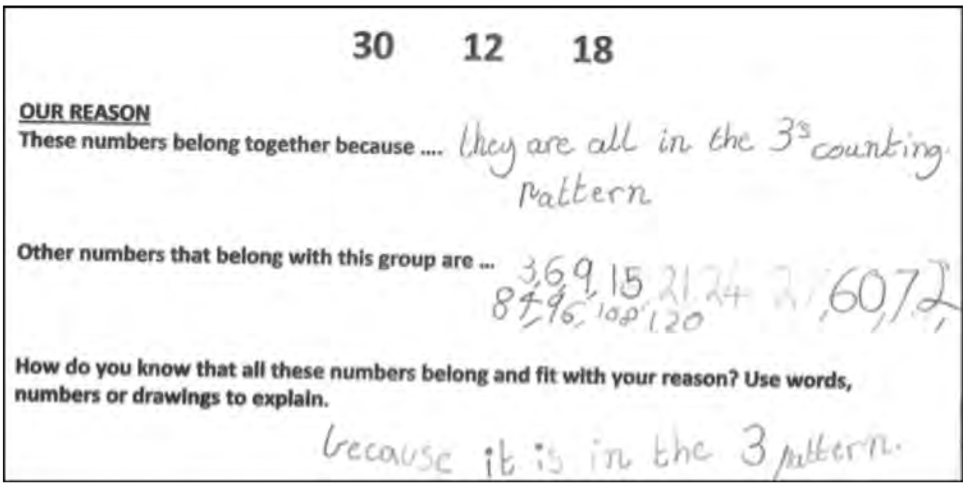
3	6	9	12
15	18	21	24
27	30	33	36
39	42	45	48
51	54		

Figure 7. A structured count by 3s to show the relationship to count by 12s.

Conclusion

In the lessons that used the "What else belongs?" task, the children provided a range of responses to the task, displayed diverse understandings of number properties and also demonstrated various reasoning actions. The task therefore was open-ended, and this made it accessible for a range of students in the classroom, as well as providing opportunities for creative thinking.

The teachers observing the lessons noticed the importance of providing opportunity for the children to work in pairs to explore and discuss possible conjectures to develop their mathematical reasoning.


30 12 18

OUR REASON

These numbers belong together because .... they are all in the 3s counting pattern

Other numbers that belong with this group are ... 3,6,9,15,21,24, ,60,72, 84,96,108,120

How do you know that all these numbers belong and fit with your reason? Use words, numbers or drawings to explain.

because it is in the 3 pattern.

Figure 6. Extending generalisation of number properties (Brett & Jarrod).

They also noted that in their self-assessment responses the children wrote that they valued the opportunity of talking and sharing with their partners. They also commented on the questioning techniques used by the teacher to promote and elicit children's reasoning when interacting with pairs of students or during whole class discussion. Observing the difficulty that the children had in using formal mathematical terms alerted the teachers to the need to develop children's mathematical language in their teaching (Bragg et al., 2016). They believed that an increased mathematical vocabulary would assist the children to communicate with each other and to communicate their reasoning. Some teachers thought they would need to revise particular mathematical terms before using tasks such as "What else belongs?"

The illustrations of children's reasoning presented in this article show that children communicate their reasoning using a variety of representations including analogy, diagrams, verbal and written statements, and number facts recorded symbolically. The task and this lesson also illustrated the range of reasoning actions which can be elicited and developed in a primary mathematics lesson, including comparing and contrasting, forming conjectures, generalising, explaining, validating and justifying. We have used findings from this study to develop a framework of these different reasoning actions (see Vale, Widjaja, Herbert, Loong & Bragg, 2016). Comparing and contrasting involves analysing and searching for what is the same and what is different, to notice similarities, commonalities or relationships. Generalising involves forming conjectures, that is, recording a statement about the common property using words, symbols, number sentences or rules and identifying further examples that fit with the property or rule. Justifying occurs at different levels: explaining involves using a single example to provide meaning or definition of the property or relationship; verifying involves showing that a property or relationship holds for each member of the original group; and using logical argument.

"What else belongs?" (Small, 2011) and "Which one doesn't belong?" (http://wodb.ca) can be used to provide opportunities for reasoning for a variety of mathematics concepts at different primary year levels. Many other tasks for forming and testing conjectures, and proving or disproving statements are available, for example *Top Drawer Teachers* (Australian Association of Mathematics Teachers, 2014) and the *Magic V Task* (Bragg, Loong, Widjaja, Vale, & Herbert, 2015; Widjaja, 2014).

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