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The multiplicative situation

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The relationships between three critical elements, and the associated mathematical language, to assist students to make the critical transition from additive to multiplicative thinking are examined in this article by Chris Hurst.

Multiplicative thinking — a very 'big idea' of number

Multiplicative thinking is a critical stage in children's mathematical understanding and is indeed a 'big idea' of number (Siemon et al., 2011). Wright (2011) suggests that children often need to reconceptualise their thinking about what is involved in multiplication and division in order to understand the 'multiplicative situation'. This article suggests that the level of reconceptualisation required could depend on how multiplicative relationships are taught. Multiplication and division are often taught as separate entities whereas in reality, the 'multiplicative situation' is rich in connections and links. If teachers know this and teach to it, they are more likely to help children understand this important idea. This article considers that there are three critical elements to be considered — an understanding of the 'multiplicative situation', a deep understanding of multiplicative arrays, and the notion of factors and multiples and associated language — and the myriad connections that exist between those three elements. However, it is important to briefly consider some rationale for moving from additive to multiplicative thinking.

Additive to multiplicative

The three elements mentioned above are critical to the development of multiplicative thinking. One of the elements is the multiplicative array. Downton (2008) used the term 'composites' to refer to an equal grouping structure that greatly assists children to think multiplicatively. It is suggested here that this would be further enhanced by the use of grid arrays to promote the idea of a composite or 'entity'. Figure 1 shows the progression from additive to multiplicative thinking through the use of the array. The final right hand grid is considered as a more powerful representation that enables the link with area to be made.

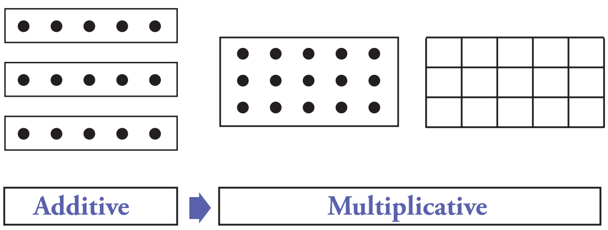


Figure 1. Additive (separate groups) to multiplicative (array).

It is essential that children move from additive to multiplicative thinking. Devlin (2008) described these two stages of development as preceding the third stage, or exponential thinking, and suggests that if children are explicitly taught in the first instance that multiplication is 'repeated addition', they are likely to remember that to their detriment when they need to be thinking multiplicatively. He notes an essential difference between the two ideas—"Adding numbers tells you how many things (or parts of things) you have when you combine collections. Multiplication is useful if you want to know the result of scaling some quantity" (Devlin, 2008, p. 1). The idea of multiplication as scaling rather than adding is a key aspect of thinking that needs to be developed. 'Repeated addition' may seem innocuous enough but it is often harder to 'unlearn' a concept, or 're-conceptualise' as Wright (2011) suggested, than learn it appropriately in the first place. Nunes and Bryant (1996, p. 153) note that multiplication and division can certainly be done through repeated addition and subtraction but they hasten to add that "several new concepts emerge in multiplicative reasoning, which are not needed in the understanding of additive situations". Devlin comments that this is often what happens with beginning mathematics instruction and working with small, positive whole numbers but it is important to move beyond the notion of 'repeated addition' and to see multiplication as a 'scaling' concept.

This is reinforced by Watson who noted the following:

> A shift from seeing additively to seeing multiplicatively is expected to take place during late primary or early secondary school. Not everyone makes this shift successfully, and multiplication seen as 'repeated addition' lingers as a dominant image for many students. This is unhelpful for learners who need to work with ratio, to express algebraic relationships, to understand polynomials, to recognise and use transformations and similarity, and in many other mathematical and other contexts. (Watson, n.d.)

Given the importance of moving beyond additive to multiplicative thinking, let us now consider the three elements mentioned in the introduction.

One situation . . . three quantities

First, Jacob & Mulligan (2014) specifically use the term 'multiplicative situation' to describe the relationship that exists between multiplication and division. Perhaps they have chosen to use that term in order to emphasise the link, rather than consider the two terms individually. This is a key to how teachers should think about and teach children about multiplicative thinking. It might seem an insignificant point to make but it is better to teach children about ONE situation—the multiplicative situation—not about multiplication and its inverse, division. The multiplicative situation is about the three key quantities—the number of equal groups, the number in each group, and the total amount. If we know the group size and the number of groups, we multiply. If we know the total amount and one of the other quantities, we divide to find the one we don't know. The idea of the multiplicative situation as one situation, is a more powerful way of thinking than simply considering multiplication and division as the inverse of one another.

The multiplicative array

Second, the case for the use of the multiplicative array has been well made for some time (Jacob & Mulligan, 2014; Jacob & Willis, 2003; Kinzer & Stanford, 2014; Siemon et al., 2011; Young-Loveridge, 2005). The power of the multiplicative array lies in its ability to focus children's attention on the three quantities involved in the multiplicative situation at the same time (Siemon et al., 2011) and therefore the array is critically important in developing multiplicative thinking. Young-Loveridge (2005) specifically described the power of arrays to develop flexible partitioning of numbers, and Jacob & Mulligan (2014) showed the strength of the array in helping children understand a range of multiplicative story problems. As well, Kinzer & Stanford discussed how arrays can help children develop an understanding of the distributive property noting that "Students who make friends with the distributive property early on will find that is a friend for life."(2014, p. 304). They base this claim on the distributive property being a key element in developing an algorithm for multi-digit multiplication, amongst other things.

Factors, multiples and other language

Third, the language associated with multiplicative thinking is important. Specifically, the terms 'factor' and 'multiple' are vital in helping children to understand the situation and to be able to explain why the properties of multiplication work and why the inverse relationship with division works. Children often learn these ideas in a procedural way and explain them in terms of "You're just switching the numbers around" (commutative property) or "You're just splitting up the number" (distributive property). The group size and number of groups need to be considered as factors and the total amount as the multiple. If both of the factors are known, we multiply to find the total, or multiple. If one factor and the multiple (or total) are known, we divide to find the other factor.

Downton noted the significance of language saying that "placing emphasis on the relationship between multiplication and division and then language associated with both operations, before any use of symbols and formal recording, needs to be a priority" (2008, p. 177). This article highlights the many links and connections within multiplicative thinking, and for these links to be understood, discussion, reasoning, inferring, and justifying need to be at the forefront of classroom activity. Hence, language assumes a vital role and it needs to be supported with the extensive use of materials to assist children to learn from a conceptual standpoint rather than learn procedures.

Specific connections within the multiplicative situation

So, just what connections and links can be found within the multiplicative situation? How can the explicit teaching and highlighting of these connections enrich children's understanding? How can these connections and ideas be shown to children?

Factor-multiple relationship

To begin with, the factor-multiple relationship needs to be shown in two ways. (See Figure 2.)

Note the use of the terms 'scaled up or down by a factor of 3 or 5'. This is important as it highlights the essential difference between multiplicative and additive thinking described by Devlin (2008) ates thinking about ratio and proportion. As noted earlier, the development of understanding of the concept of the multiplicative situation is a complex process that takes time. The specific use of the term 'scaling' is included here to show the connection that exists, but its introduction in the teaching process requires caution. In keeping with the idea of 'big idea thinking' and the connections that are inherent, there is an opportunity here for teachers to make such a connection explicit for their students. That is, the idea of 'scaling' is multiplicative in nature and essentially the same as using scales on maps. For example, on a map with a scale of 1:100, real measurements and distances are 'scaled down' by 'a factor of 100'.

Commutative property

In terms of the properties of multiplication, the array is a powerful representation as has already been indicated. It is widely noted that the array can simply be rotated by ninety degrees to represent the commutative property (e.g., Siemon, Breed & Virgona, 2006) but an even more powerful way of viewing this is to maintain the same orientation for the array and consider the different numbers of equal groups as shown in the second part of Figure 3.

Factor of 15	Factor of 15	Multiple of 3 & 5	
Number of groups	Number in each group	Total of all groups	
5	× 3	=	15
3 is scaled up by a factor of 5			

Multiple of 3 & 5	Factor of 15	Factor of 15	
Total of all groups	Number of groups	Number in each group	
15	÷ 5	=	3
15 is scaled down by a factor of 5			

Figure 2. Factor and multiple relationships.

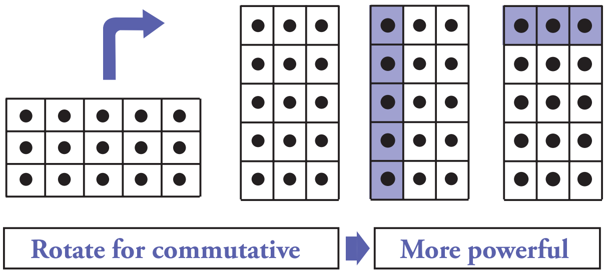


Figure 3. Two ways of illustrating the commutative property using arrays.

To support children to understand the commutative property, teachers could use a simple task like 'Building Arrays'. In this, two 5 × 3 arrays could be cut into pieces — one grid is cut into three groups of five and the other into five groups of three. The first grid is then rebuilt as a vertical array as shown in Figure 3 (with the groups of five squares being in columns) and the second grid can be overlaid on the first grid (with the groups of three squares being in rows). This is a very powerful representation of the commutative property without even having to rotate the array! It should be noted that the arrays in Figure 3 are shown as dot/grid arrays — it is suggested that an even stronger representation would have the dots removed.

The *Australian Curriculum: Mathematics* describes the development of the 'Understanding' proficiency in terms of students connecting ideas that are related, when students represent concepts in various ways, and when they identify similarities between different aspects of a concept. The statement about the proficiency of Fluency discusses flexible use of procedures and recognizing robust ways of answering questions (ACARA, 2013). The Building Arrays task described above helps develop those aspects of the Understanding and Fluency proficiencies.

Distributive property

The distributive property can be equally well shown with an array (see Figure 4). That is 13 × 6 can be considered as (10 × 6) + (3 × 6) or in combination with the commutative property, 6 × 13 can be considered as (6 × 10) + 3 × 10). As discussed by Kinzer & Stanford (2014), this use of the array can also be beneficial for helping children learn the 'harder number facts' by splitting them, as well as developing the algorithm for multiplying larger numbers. Also, this use of the array is important for developing a standard algorithm for double digit multiplication for an example like 36 × 28. The second part of Figure 4 demonstrates this by splitting 36 into 30 and 6, and 28 into 20 and 8, to derive the four parts of the algorithm as shown by the shaded parts of the array.

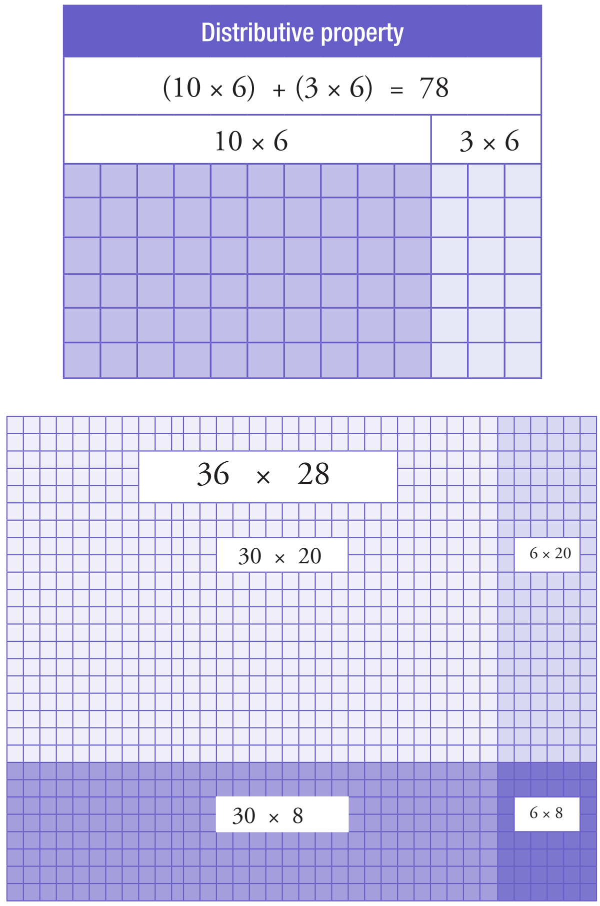


Figure 4. Distributive property shown by an array with development of standard algorithm.

Arrays, fraction concepts and division

Fractions are notorious for the perception that they are difficult to teach and understand. However, it is suggested that much of this perception arises from procedural teaching and a lack of linking and connecting of fractions to other ideas. One such idea is the division construct for fractions which is often ignored. Again, there is some interesting language such as the notions of 'thirding' and 'fifthing', as described by Siemon et al. (2011), and which is a part of developing the understanding of this idea, as shown by Figures 5 and 6. The understanding that these connections exist within the multiplication situation, and the making of those connections explicit to children, are characteristics of strong teaching.

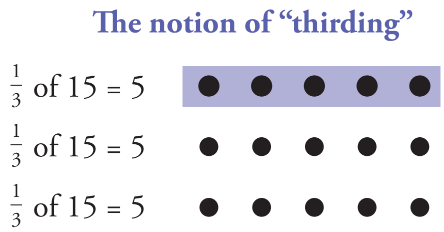


Figure 5. Using an array to demonstrate division by 3 or 'thirding'.

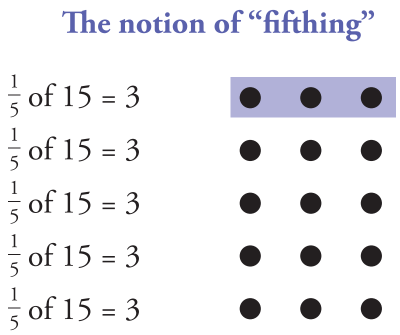


Figure 6. Using an array to demonstrate division by 5, or 'fifthing'.

This representation can be taken a step further to help develop the idea of fraction equivalence, as shown in Figure 7.

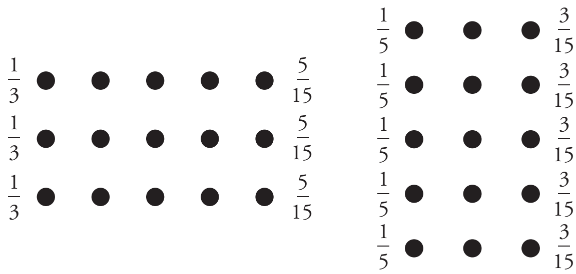


Figure 7. Using an array to demonstrate equivalent fractions.

Arrays, fraction language, and 'times as many'

The importance of language has already been discussed, and children need to be exposed to a range of ways of considering the multiplicative situation and discussing it. Too often, multiplication facts are learned in isolation and not linked to other representations. Askew (1999) used the notion of 'free gifts' as a way of considering 'free' or additional facts that come from knowing one fact. This notion can be applied here, as shown in Figure 8, to learn about fraction concepts based on known facts.

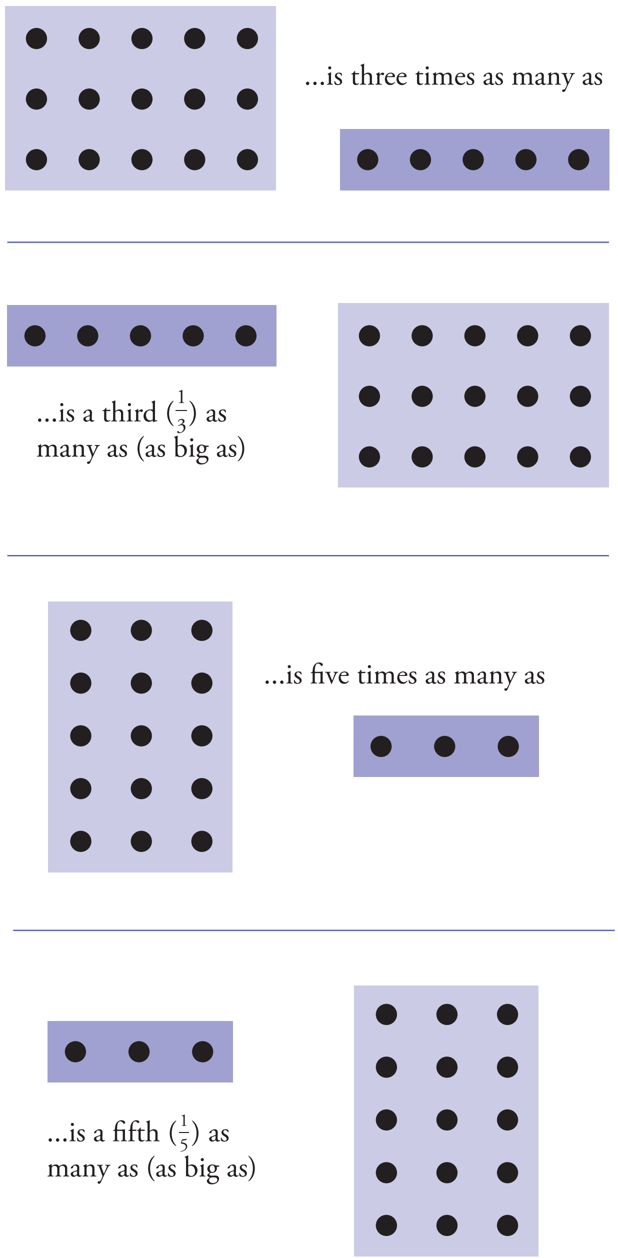


Figure 8. Using arrays to develop 'times as many' and fraction language.

Arrays and area

The concept of area is easily shown by the grid array and can be explicitly developed as multiplication facts are learned in a conceptual way through the use of arrays that incorporate the commutative property, inverse relationship and associated language. Again, this is another example of Askew's notion of 'free gifts'.

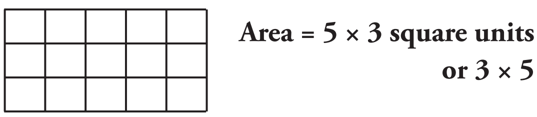


Figure 9. Using an array to demonstrate area alongside commutativity.

The language of 'times as many' and 'a third as big as' is clearly important and forms an important bridge to the understanding of ratio. It has already been pointed out that young children are capable of understanding multiplicative situations with sharing or splitting as the basis.

Arrays and combination problems

Further to this, Jacob & Mulligan say that "Even young children need to be exposed to a range of multiplication and division problems" (2014, p. 36). Such a range of problem types that characterise the multiplicative situation are described in various sources (Department of Education, Western Australia, 2013; Van de Walle, Karp & Bay-Williams, 2013). One example of problem types that can be represented well by an array is the combination problem based on a familiar canteen menu scenario as shown in Figure 10. The array can be a powerful tool for developing a multiplicative view of this type of problem rather than an additive one.

Arrays and ratios

Multiplicative thinking underpins the development of proportional reasoning and the concept of ratio and this is associated with Devlin's (2008) view of multiplication as a 'scaling concept'. It is inherently linked to the notion of 'times as many' and as such, informs the development of understanding of percentage. Figure 11 shows how an array can be used to highlight yet another of the rich connections within the concept of multiplicative thinking and specifically the multiplicative situation. The concept of ratio can be quite a difficult one to master, probably because of the two types of ratio to be considered. Rathouz, Cengiz, Krebs & Rubenstein (2014) make the distinction in terms of an additive comparison of the two parts (part-to-part ratio) and a multiplicative comparison of the one part to the total (part-to-whole ratio). The distinction can be seen quite simply with the aid of an array, as shown in Figure 11.

Conclusion

It is well established that multiplicative thinking is a critical stage in the development of children's mathematical understanding as it underpins much of the mathematics that follows.

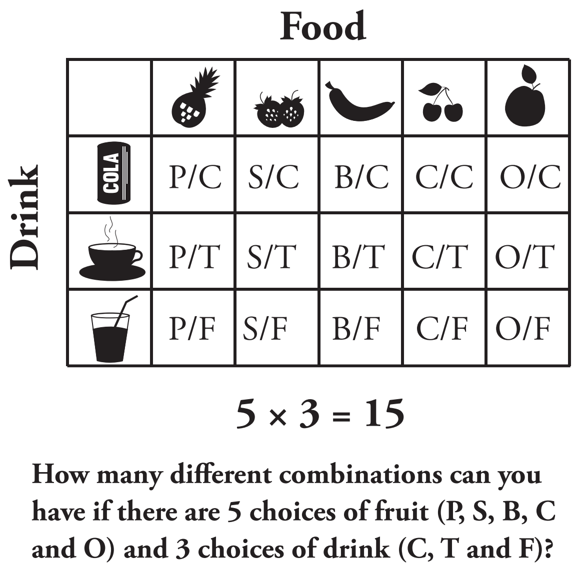
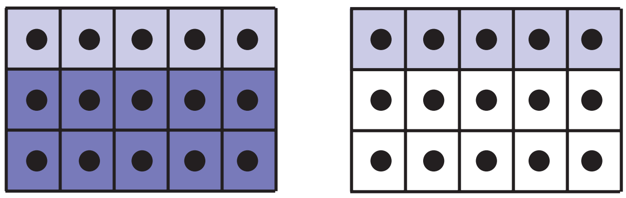


Figure 10. Using an array to demonstrate a combination problem.



Ratio of part:part is 1:2

That is $\frac{1}{3}$ and $\frac{2}{3}$

or $\frac{10}{15}$ and $\frac{5}{15}$

Ratio of part:whole is 1:3 or 5:15

Figure 11. Using an array to demonstrate two types of ratio.

Understanding the multiplicative situation and the three quantities involved, having a flexible knowledge of the terms factor and multiple, and knowing about the extended use of the multiplicative array, are three key elements of multiplicative thinking. It is important to realise that such conceptual development takes place over time and requires a whole-school approach beginning in the early years and extending throughout primary school. This article has attempted to show the myriad connections and links within the idea of multiplicative thinking as well as how aspects of it link to and inform the development of other ideas. The power of the multiplicative array as an enabling tool is central to children making sense of these connections and links.

Initially, teachers may not necessarily see the connections between ideas presented in this article and the elements of the multiplicative situation, but a major purpose of the article is to point out that these connections do exist. The multiplicative array can indeed be a powerful way of representing the distributive property, combination problems, and even equivalent fractions, as has been shown here. The construct of fraction as division is very much a part of the multiplicative situation—when fifteen is divided by 5, it is the same as 'fifthing' the original quantity. Similarly, the concept of division as sharing into equal groups is inextricably linked to equivalent fractions. The connections based on the multiplicative situation are there. If teachers understand that and make the connections explicit to children, they are enabling them to understand mathematics conceptually and in a truly connected way.

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