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Gardner's two children problems and variations:

Puzzles with conditional probability and sample spaces

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Introduction

For 26 years Martin Gardner wrote a famous and very widely read mathematical puzzle column in the magazine *Scientific American*. In 1959, he published what he called *The Two Children Problem*, which presented two questions like this:

> Problem 1. Mr. Jones has two children. The older child is a boy. What is the probability that both children are boys?[1]

> Problem 2. Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?

Martin Gardner gave the answer $\frac{1}{2}$ for Mr Jones' question and the answer $\frac{1}{3}$ for Mr Smith's question, and in so doing caused a commotion. How could the two probabilities be different?

Over fifty years later, we are familiar with this difference in probabilities. However, a second commotion began at the ninth "Gathering 4 Gardner" (G4G9) in 2010, a biennial convention of mathematicians, magicians and puzzle enthusiasts inspired by the work of Martin Gardner (www.gathering-4gardner.org). Puzzler Gary Foshee presented the following problem

> Problem 3. Mr Ng has two children. One is a boy born on a Tuesday. What is the probability that both children are boys?

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1   The original was about girls, though we have changed to boys to allow for consistency within the presentation of the various problems discussed in the paper.

At first glance, Problem 3 seems the same as Problem 2, because it seems ridiculous that the day of the week on which one of the boys is born has anything to do with the sex of his sibling, and so the answer was expected to be $\frac{1}{3}$ and the question was expected to be uninteresting. However, Foshee commented: "The first thing you think is 'What has Tuesday got to do with it?' Well, it has everything to do with it." Foshee's answer was $\frac{13}{27}$. An internet search on "Tuesday Birthday Problem" finds many hits testifying to the ongoing interest in this paradox (see, for example, Juul, n.d.).

The surprising result is that adding an extra condition such as "born on a Tuesday" increases the probability that the other sibling is a boy from the base of 33% for Problem 2, through Foshee's example of 48% ($\frac{13}{27}$) for Problem 3 and up to the limit of 50% for Problem 1. The condition "born on a Tuesday" (48%) could have been replaced with "not born on Tuesday" (36%), "born after midday" (43%), "born in autumn" (47%) or "born on the 2nd of January" (49.97%), or "has red hair" or "loves liquorice". The probability will still be more than the expected $\frac{1}{3}$ (33%), which in turn is more than the basic probability of 25% of having two boys in a two child family. This paper will explain the paradox of Problems 2 and 3 and many other variations of the theme.

In the next section, we look at the solutions to problems 2 and 3 and their generalisations and compare the results. Finally, we use these problems to draw attention to the importance of both conditional probability and sample spaces for teaching of probability at all levels.

Assumptions about birthdays

Before considering the problems more closely, we note that various assumptions will be made. Teachers may find it useful to have these facts at hand when they are teaching probability using sex of children. First, we assume boys and girls are equally likely to be born, so the probability that a baby is a boy is one half (also for a girl). In fact, the Australian Bureau of Statistics (2011) gives the actual probability that a baby born in Australia in 2011 was a boy as 51.4% and a girl 48.6%. Teachers can redo the analysis presented here to show that this does not affect the phenomenon under discussion: the numbers are only slightly different. Second, we assume that the sex of one child in a family does not affect the sex of later children; in other words having boys or girls does not "run in families". Empirical data from the USA supports this assumption (Rodgers & Doughty, 2001). Third, when considering the time of birth of a child we assume each time of the day, day of the week, month of the year is equally probable. This assumption is not strictly true. For example, data from 2011 (Division of Vital Statistics, 2013) in the USA shows that only about two thirds as many children are born on a Saturday or Sunday as on a weekday. This is due to the high incidence of births by planned caesarean section. Australian data is likely to reveal the same effect. Fourth, we assume that the time of the birth (e.g., on Tuesday) has no effect on the sex of the child. Fifth, we assume that there are no twins. All of these assumptions affect the precise calculations, but none of them affect the conundrum about the probabilities.

Most importantly, however, we are assuming that each of the problems is fundamentally asking for the probability of having two sons given that you know there are two children *and* at least one is a boy *and* (except in Problem 1) there is an additional condition.

Probability analysis of Problem 3 and variations

For the probability analysis we have chosen to represent the problem situation with two-way tables (see Figure 1) although tree diagrams would have been equally effective. The probabilities are readily computed by considering the number of equally likely outcomes in total and that are favourable, or alternatively by utilising the conditional probability formula

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Table 1 shows the probability analyses for Problem 2 and Problem 3 and three variations (Problems 4, 5 and 6). The problems are arranged in order of decreasing likelihood of the additional condition, showing that the conditional probability increases. The first table gives the solution for Problem 2, which has no extra condition. All four cells are equally likely, under our assumptions. The 3 shaded cells are the reduced sample space, where there is at least one boy. The one lighter-shaded cell is favourable (two boys). The probability is therefore $\frac{1}{3}$. The third column also shows the calculation using the conditional probability formula. In Problem 4, 16 equally likely outcomes are created for the two child family, with seven (dark) shaded cells fitting the "at least one boy" condition and the three lighter cells also fitting the "born after midday" condition. The probability is therefore $\frac{3}{7}$. The other problems are solved in the same manner.

The table demonstrates how the probability varies depending on the likelihood of the final condition. The more likely the condition, the closer the probability is to $\frac{1}{3}$, the less likely the condition the closer the probability is to $\frac{1}{2}$.

It is in fact possible to obtain all values between $\frac{1}{2}$ and $\frac{1}{3}$. If the probability of the final condition is p then the probability that there are two sons is

$$\frac{2-p}{4-p} \tag{1}$$

A derivation of this formula is presented in Appendix A, though we encourage readers to have a go before looking at our solution.

Uses in teaching

These problems and their accompanying graphical representations provide a wonderful source of motivation or extension for Year 11 or 12 students during the topic of probability, and possibly in earlier years. The two-way tables shown in Table 1 are particularly useful for illustrating the reduced sample spaces that are the key to conditional probability.

Alternatively students could be presented with Problems 2 and 3 and asked to explore the variations presented in the paper and then find situations which produce specific probabilities. More advanced students may be able to deduce the general formula (1). There are other directions for variations, such as the probability of two boys given at least one child (of either sex) is born on a Tuesday (back to 25%).

The general formula and the graphical representations also provide an alternate view point of limits and their existence, as students can see that although

$$\lim_{p \to 0} \frac{2-p}{4-p} = \frac{1}{2}$$

Table 1. Two way table solutions of Problems 2, 3, 4, 5 and 6.

Condition	Tabular representation	Probability calculation
Problem 2. Two boys given at least one is a boy.		Number of favourable outcomes: 1 Total number of possible outcomes: 3 Probability: $\frac{1}{3}$ (33%) $P\left(\text{both boys} \mid \text{at least one is a boy}\right) = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3}$
Problem 4. Two boys given at least one is a boy born after midday. (BAm indicates boy born in the morning, BPm indicates boy born in afternoon, etc.)		Number of favourable outcomes: 3 Total number of possible outcomes: 7 Probability: $\frac{3}{7}$ (42%) $P\left(\begin{array}{l}\text{both boys} \mid \\ \text{at least one is a boy born after midday}\end{array}\right) = \frac{\left(\frac{3}{16}\right)}{\left(\frac{7}{16}\right)} = \frac{3}{7}$
Problem 5. Two boys given at least one is a boy born in autumn. (BSu indicates boy born in summer, BA indicates boy born in autumn, etc.)		Number of favourable outcomes: 7 Total number of possible outcomes: 15 Probability: $\frac{7}{15}$ (46%) $P\left(\begin{array}{l}\text{both boys} \mid \\ \text{at least one is a boy born in autumn}\end{array}\right) = \frac{\left(\frac{7}{64}\right)}{\left(\frac{15}{64}\right)} = \frac{7}{15}$
Problem 3. Two boys given at least one is a boy born on a Tuesday. (BM indicates boy born on Monday, BTu indicates boy born on Tuesday etc.)		Number of favourable outcomes: 13 Total number of possible outcomes: 27 Probability: $\frac{13}{27}$ (48%) $P\left(\begin{array}{l}\text{both boys} \mid \\ \text{one is a boy born on a Tuesday}\end{array}\right) = \frac{\left(\frac{13}{196}\right)}{\left(\frac{27}{196}\right)} = \frac{13}{27}$
Problem 6. Two boys given at least one is a boy born in February.		Number of favourable outcomes: 23 Total number of possible outcomes: 47 Probability: $\frac{23}{47}$ (49%) $P\left(\begin{array}{l}\text{both boys} \mid \\ \text{one is a boy born in February}\end{array}\right) = \frac{\left(\frac{23}{576}\right)}{\left(\frac{47}{576}\right)} = \frac{23}{47}$

and indeed
$$\frac{2-0}{4-0} = \frac{1}{2}$$

the real situation with $p = 0$ does not give a probability of $\frac{1}{2}$, the probability is actually undefined. For example, it is meaningless to determine the probability of having two boys given at least one is a boy born with one hundred heads, as the condition would never be met. This is different from a situation that gives a probability of zero. For example: If today is Monday, what is the probability that tomorrow will be Wednesday? Here the condition "today is Monday" is possible, allowing the probability of tomorrow being Wednesday to be well defined, albeit impossible!

Resolving the paradox

The calculations above show why Foshee's answer of $\frac{13}{27}$ is correct, but do they explain the paradox? Does knowing the day a child is born really affect the sex of the sibling? The two-way tables show that if there was no intersection of the horizontal and vertical shading, the probability would always be a half, because the shaded horizontal row is half light and half dark, and so is the shaded column. As the intersection becomes relatively smaller, the probability moves up towards a half. So the key to understanding the paradox is understanding the intersection.

The intersection only arises because the condition does not specify which child is involved. As soon as the child who is born on Tuesday is specified (e.g., as the older child), the conditional probability is a half—it is just a more complicated version of Problem 1. In the two way tables, there will only be one shaded row or column. Note that specifying the child as the older is only one possibility: the mathematics will be the same whether the child is specified as being the older, or the one first in alphabetical order or the one who loves liquorice the most. The two-way tables in Figure 1 use first and second born, but any distinguishing feature could be used.

Imagine there are 19 600 fathers of two children standing in a stadium. The number 19 600 has been chosen for numerical convenience. We expect 100 fathers for each of the equally probable gender-birth day two-child combination cells displayed in the tabular representation of Problem 3 in Figure 1. We expect 4900 fathers of two girls, 9800 fathers of a boy and a girl and 4900 fathers of two boys. We ask all those with a least one son to remain in the stadium, and the others (those with two daughters) to leave. We expect 14 700 fathers ($\frac{3}{4}$ of the group) to remain. At this stage, if we select a father at random, he has a probability of
$$\frac{4900}{4900+9800} = \frac{1}{3}$$

of having two sons (Problem 2). The probability is greater than the initial $\frac{1}{4}$ (before some fathers left) because this sample of fathers is 'boy-enriched'. Now ask the fathers to remain in the stadium if they have a son with a birthday on Tuesday. Fathers with two sons have a greater chance (about twice) of staying than fathers with only one son, and so the group of fathers that remains has relatively more fathers of two sons than before. If we now select a father at random, the probability of him having two sons will be even more than $\frac{1}{3}$ (Problem 3). This analysis shows that the paradox occurs because it is easy to think that children are being sampled, whereas in fact the probabilities are about families (fathers).

The paradox in this problem also arises because the problem statement does not fully make clear what is sampled and the sampling procedure.

A comprehensive discussion of this is given by Khovanova (2012). If we change the sampling procedure in quite reasonable ways, other answers are possible. To show an example, return to our hall, once again containing the original 19 600 fathers of two children. For this second sampling procedure, start by asking each father to flip a coin to select one of his children and provide information about the gender and birthday of the chosen child. One fourteenth of the fathers (in this case 1400) will make the statement that: "The selected child is a son born on a Tuesday." This statement will come from none of the 4900 fathers of two girls, 1 out of 7 of the 4900 fathers of two boys, 1 out of 7 of the 4900 fathers with a boy and a girl who selected the boy and none of the 4900 fathers of a boy and a girl who selected the girl. In total, there are 1400 fathers who make the statement and of these, 700 have two boys. If this is the correct sampling procedure, the solution to Problem 3 is $\frac{1}{2}$.

Now consider a third sampling procedure. Return to the hall with the original 19 600 fathers. Ask the 4900 fathers with two daughters to leave. The remaining fathers are asked to provide information about their son's day of birth, but fathers with two sons must randomly select one of their boys before making the statement. There will be 2100 fathers who make the statement, "I have a son born on a Tuesday," made up of one seventh of the 9800 fathers with a son and a daughter and one seventh of the 4900 fathers with two sons. Consequently, this sampling procedure will result in obtaining a solution of $\frac{1}{3}$ to Problem 3.

A fourth sampling procedure is effectively the same as the first. The fathers who do not have a son born on a Tuesday are asked to leave. Looking at Figure 1, we see that 16 900 fathers leave (100 fathers per unshaded cell) and 2700 fathers remain. Again looking at the two-way table, we see that 1400 of the remaining fathers have a son and a daughter and 1300 have two sons. In this case the solution of Problem 3 is again $\frac{13}{27}$.

These different sampling procedures result in different correct answers to Problem 3. That is, the Problem 3 paradox arises in part because the actual sampling procedure is not fully specified and the problem solver needs to interpret it, and this was what Khovanova (2012) considered as "Martin Gardner's Mistake". This also happens in some school mathematics probability questions. It is clearly essential that all probability calculations and conclusions from statistics need to consider carefully both what is being sampled and how it is being sampled.

Paradoxes are a wonderful way to get teachers, students and their parents thinking hard about mathematics and solving problems and variations. Foshee's puzzle is a new contribution which can be used in schools. Watch for new puzzles that emerge to celebrate the centenary of Martin Gardner's birth on October 21st, 2014.

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Appendix A: Derivation of the general formula

Let $p = \frac{m}{n}$ be the probability of the final condition. The two-way table in Table 2 can represent the situation.

Table 2. Two-way table for the probability that a family has two boys, given that there is at least one son and he has a characteristic with probability p.

The lighter shaded region contains $2mn - m^2$ equally likely outcomes and the total shaded section contains $(2^2 mn) - m^2$ equally likely outcomes and so the probability is

$$\frac{2mn - m^2}{4mn - m^2} = \frac{2n - m}{4n - m} = \frac{\left(\dfrac{2n - m}{n}\right)}{\left(\dfrac{4n - m}{n}\right)} = \frac{\left(2 - \dfrac{m}{n}\right)}{\left(4 - \dfrac{m}{n}\right)} = \frac{2 - p}{4 = p}$$

This proof is for rational values of p, but it can be generalised to all values.

From Helen Prochazka's

ScrApbook

"Statistics is hard. But that's not just an issue of individual understanding; it's also becoming one of the nation's biggest political problems. We live in a world where the thorniest policy issues increasingly boil down to arguments over what the data means. If you don't understand statistics, you don't know what's going on — and you can't tell when you're being lied to."

Clive Thompson, commentator