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CHAPTER 5.

MEASUREMENT MATTERS: FRACTION NUMBER LINES AND LENGTH CONCEPTS ARE RELATED*

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As busy teachers, sometimes we ask ourselves, is spending time on number lines important? Am I teaching this for its own sake or can I link it to other mathematical knowledge to make for more numerate students? What do my students need to know to use a ruler correctly? Does length measurement understanding link to fraction understanding?

In our work with assessment tasks for fractions and length measurement, we found that measurement matters. Success on fraction number line tasks was related to success on broken ruler tasks, but not with whether a student could use an ordinary ruler: conceptual understanding of length measurement was more important than a tools and procedures knowledge of length measurement.

Number lines

When we use number lines as fraction models, there are conventions about how to read them and how to draw them with which we are familiar, but this is not always straight forward to our students. Kieren (1993) has described different mathematical contexts for fractions, but the one that concerns us in this chapter is the *measure* sub-construct. The length examples of the measure sub-construct encompass three main representations:

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- a part-whole segment of a line. When we illustrate fractional parts by folding paper strips, a quarter refers to a segment of the paper that could be iterated four times to fill the whole strip. This single quarter segment might be the first, second, third or fourth one of these segments. Often we call this aspect of the measure sub-construct part-whole.
- a point on a line in which the line is assumed to be the whole. A quarter is a quarter of the way along a line, and the left hand edge of the line is the assumed zero point. Again, we often call this interpretation part-whole.
- a point on a number line. Fraction number lines are another more formal length measure representation; they have a labelled zero point and can have improper fractions on them. On a number line, $\frac{1}{4}$ is one quarter of the distance between 0 and 1.

Given that using number lines is complicated, one approach is to help children use related knowledge from different domains. Both number lines and rulers are examples of scales and as such they both have zero-points. Are children making connections between related concepts in the different domains of measurement and fractions?

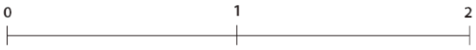
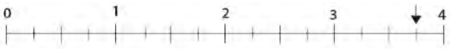
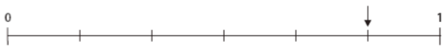
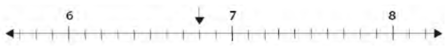
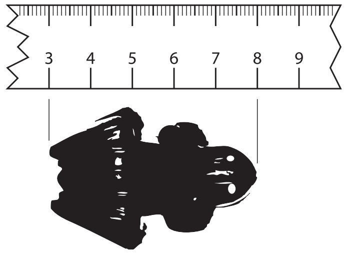
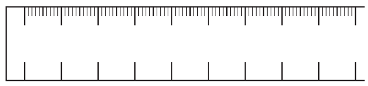
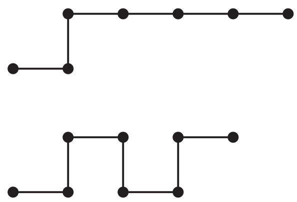
Broken rulers

The model we use for length measurement is based on Lehrer's (2003) eight key concepts for *spatial* measures (length, area, volume, angle). These key concepts are:

- unit-attribute relations
- iteration
- tiling
- identical units
- standardisation
- proportionality
- additivity
- origin (zero point).

The concepts that concern us here are *additivity* and *origin*. Additivity describes the concept that "the total distance between two points is equivalent to the sum of any arbitrary set of segments that subdivide the line segment" (Lehrer, 2003, p.181). With the conservation of length, the whole is equal to the sum of the parts even when the parts are rearranged. Length measurement can include straight paths, bent paths (including curves) and perimeters, as well as scales in which a zero point is important. Broken ruler tasks—where a ruler is "broken" and the child cannot measure

Table 5.1. Fraction and length measurement assessment tasks.

FRACTION TASKS	MEASUREMENT TASKS
1. *Give the child a blank piece of paper and pen*. Please draw a number line and mark two thirds on it. *If the child does not mark 0 or 1, ask:* Where does zero go? Where does one go? How did you work that out? (Clarke, Roche, & Mitchell, 2007). 2. If this is half, *point*, where would one and a half be on this number line?  3. Please mark where one quarter would go on this number line.  Adapted from Pearn and Stephens (2007). 4. *Point to arrow*, what number or fraction is that point on the number line?  5. *Point to arrow*, what number or fraction is that point on the number line?  6. *Point to arrow*, what number or fraction is that point on the number line?  Adapted from Pearn and Stephens (2007). 7. *Point to arrow*, what number or fraction is that point on the number line?  (Ministry of Education, 2007). 8. *Point to arrow*, what number or fraction is that point on the number line? 	9. Chocolate frog This centimetre ruler is broken. It is measuring a chocolate frog. How long is the frog?  How did you work that out? Adapted from Bragg and Outhred (2000). 10. Footy Card This ruler measures in centimetres but there are no numbers on it. How long is the footy card? How did you work that out?   Adapted from Bragg and Outhred (2000). 11. Wires These are two pieces of wire that can be bent and straightened. Between the dots is the same length. If the wires were straight, would they be the same length or would one be longer than the other? How did you work that out?  Adapted from Battista (2006).

from zero—is an example of a task requiring conceptual understanding of additivity and the zero point. On the other hand, using a ruler to measure an object 19 cm long is an example of a task that may only require a tools and procedures knowledge of additivity and the zero point.

Assessment interviews

The following are some of the assessment tasks that we used with 88 Grade 6 children in a one-to-one task-based interview to examine the children's performance across similar tasks in the different domains of measurement and fractions. When using this type of interview the children are not told whether their answers are correct or incorrect, but are always asked "and how did you work that out?" The benefits of this approach include being able to distinguish between students who get correct answers with a mathematically correct explanation and those who get tasks right for the wrong reasons. It also helps establish classroom norms of valuing students' own thinking, problem solving and explanations. The tasks are shown in Table 5.1.

Performance on number line tasks and broken ruler tasks

The order of difficulty of the number line tasks can be determined by the percentage of correct answers (see Table 5.2).

Table 5.2. Order of difficulty of number line questions.

Task	Q5	Q8	Q1	Q4	Q7	Q6	Q3	Q2
success	27.2%	31.8%	31.8%	38.6%	52.3%	55.7%	55.7%	71.6%

All children were asked every number line task and while some tasks were easier than others, there did not seem to be a predictable sequential order in which the children would be successful. Thus for teaching, the sequence is not important, but as our discussion of misconceptions later in this chapter will show, the use of a variety of tasks in relation to number lines can be beneficial. A number line score was assigned as the number of questions correct out of eight: 12.5% of students had a score of 8 (all correct) and 6.8% had a score of 0 (none correct).

For the measurement tasks, the Chocolate Frog task was a threshold task; if successful, the child was then asked the Footy Card task. If unsuccessful on the Chocolate Frog task, the child was asked the Wires task. Just over half of the students were successful on the Chocolate Frog broken ruler task. Breaking those results down further, the percentage of students correct at each level of the concept is shown in Table 5.3.

Table 5.3. Success on broken ruler tasks.

Tasks	Chocolate Frog and Wires incorrect	Chocolate Frog incorrect but Wires correct	Chocolate Frog correct and Footy Card incorrect	Chocolate Frog and Footy Card correct
score	0	1	2	3
success	11.4%	31.8%	19.3%	37.5%

Successful students on the Chocolate Frog broken ruler task:

- counted spaces;
- visualised a zero at 3 and then counted the lines successfully; or
- took 3 from 8.

Relationships between measurement and fraction tasks

A strong relationship was found between student performances on the broken ruler tasks (taken as a score from 0 to 3) and their performances on the number line tasks (taken as a score from 0 to 8). The graphs in Figure 5.1 show the students' performance on the broken ruler tasks and then, in the same order, their corresponding number line score. The highest performing students on number line tasks, (those who were correct on 6, 7 or 8 number line tasks) all come from two top performing groups in the broken ruler tasks. These were students who were successful on at least one broken ruler task. The lowest performing students on the broken ruler tasks scored between 0 and 5 on the number line tasks, although the majority of them scored between 0 and 3. In the top half of the cohort (by broken ruler tasks) only three students scored a 0 or 1 on the number line tasks. In contrast, students' performances on the tools and procedures task of using a ruler to measure an object 19 cm long, did not correlate with their performance on number line tasks.

There was one misconception demonstrated by students on both the measurement and the number line tasks—the incorrect counting of the zero point. A line on a ruler or a number line represents *both* the end of one unit and the beginning of the next. Of course many people count lines, *as markers of the ends of units*, or through rote procedure, but they do not count the line at the zero-point. Just over two thirds of the incorrect responses to the Chocolate Frog task and nearly 90% of the incorrect responses to the Footy Card task were due to students counting the zero point or first mark and giving an answer one higher than the actual length: 6 instead of 5 for the Chocolate Frog, 8 instead of 7 for the Footy card.

Counting the zero point was the least frequent of the identified errors in the number lines tasks, with 12.5% of students demonstrating it one or more times. However, while this specific misconception occurs in both the number line measure sub-construct and the measurement additivity context, students who demonstrated it in one context did not neccessarily do so in the other. So it would appear that common misconceptions are not responsible for the correlation between fraction and measurement tasks.

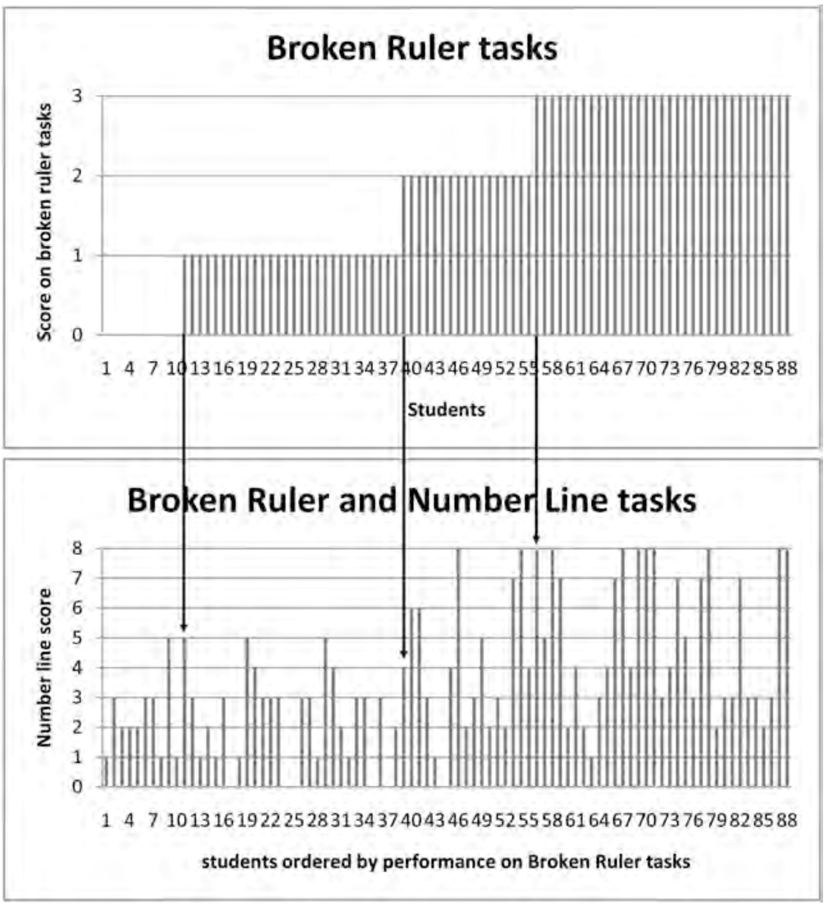


Figure 5.1a and 5.1b. Relationship between number line scores and broken ruler performance.

Correlation studies do not prove cause and effect, but they do point us in the direction of thinking about what was it that successful students in both domains understand. It is worth including broken rulers in our measurement repertoire if that helps children grapple with the representational and conceptual context of number line fraction tasks. If we are teaching and assessing more conceptually focussed measurement tasks than just tools and procedures tasks, then we may be able to help our students make explicit connections between two different representations of scales: number lines and rulers.

To summarise the relationships between students' performances on number line fraction tasks and broken ruler tasks, it seems that:

- there is a strong relationship between students' successful performance on fraction number line tasks and their understanding of additivity in the length measurement context;
- conceptual measurement tasks have a stronger link to number line success than tools and procedures measurement tasks; and
- if students have a misconception it does not mean that they will demonstrate it on every task.

Using knowledge of number line misconceptions in the classroom

We do not know what it is that makes students successful on both broken ruler and number line tasks, but we suspect that it is worthwhile helping students make connections between conceptual understandings of measurement and fractions tasks. Categorising and examining misconceptions is one way to approach making these connections in the classroom. As misconceptions are not generalised into every context, we have an opportunity to work with our students from the known to the unknown. Using our knowledge gained from interviewing, we can present common misconceptions to students. Further classroom examples of broken ruler tasks and number line tasks give students the opportunity to identify specifically where they demonstrate the misconception. Students are then able to evaluate the tasks where they *did not* demonstrate the misconception (working from the known) in order to identify what mathematical understandings they successfully drew on and compare this to the tasks where they were unsuccessful.

Types of misconceptions

From their explanations, several misconceptions were identified in children's strategies for solving number line tasks, including:

- a limited part-whole understanding;
- assuming a decimal number line; and
- counting the zero point.

Table 5.4 shows the spread of these misconceptions across our sample of students in relation to the questions asked.

Table 5.4. Frequencies: success and errors on number line tasks.

Type of task	Task	Success	Limited part–whole understanding	Assuming decimal number lines	Counting the zero point
Making own partitions	1	31.8%	30.7%		
	2	71.6%			
	3	55.7%	29.5%		
Reading pre-marked partitions	4	38.6%		13.6%	3.4%
	5	27.2%		17.0%	10.2%
	6	55.7%		4.5%	2.3%
	7	52.3%		will be correct	2.3%
	8	31.8%		4.5%	3.4%

The most common misconception, affecting just over a third of the students on one or more tasks, was a *limited part-whole understanding*. This misconception took a variety of forms. When drawing a number line and marking $\frac{2}{3}$ (Q1), some students used a ratio, for example labelling a number line from 0 to 9 and marking $\frac{2}{3}$ at 6. This type of error was made by 21.6% of the students on Q1. Other ratios used included 2 out of 3, 4 out of 6, 6.6 out of 10, 8 out of 12, and 16 out of 24. These students could partition *a line* into two thirds, but did not understand all the conventions of a number line, where $\frac{2}{3}$ is between 0 and 1. A second variation also occurred on Q1 in which students again demonstrated an ability to partition *a line* into two thirds but then mislabelled where 1 would go. They could not use the conventions of a number line to show that $\frac{2}{3}$ was two thirds of the way between 0 and 1, even though they could partition a line into two thirds. A third related misconception was evident in Q3 in which some students marked $\frac{1}{4}$ on a number line from 0 to 2 at $\frac{1}{2}$, because it was a quarter of *the line*. Our students seem to have difficulty distinguishing between a line as a part-whole model and a number line model. In Q1, 62.5% could partition the line into two thirds, but only half of them could reconcile this with the specific conventions of number lines, by labelling $\frac{2}{3}$ between 0 and 1.

Another misconception was *assuming a decimal number line* and counting parts as tenths. One quarter of the students did this on one or more tasks. The basic form of the error was to count by tenths, 0.1, 0.2, 0.3, 0.4, etc. from zero or the closest whole number. In Q4, $3\frac{3}{4}$ became 3.3; in Q5, $\frac{5}{6}$ became 0.5; in Q6, after imagining the missing line, $\frac{3}{4}$ became 0.3; and in Q8, 1 and $\frac{2}{5}$ became 1.2. Sometimes the students gave their answer in tenths, for example $\frac{3}{10}$, but they were still assuming a decimal number line. A variation of this was to count backwards from the closest whole number on the right hand side. In Q4, $3\frac{3}{4}$ became 3.9 because it was one tenth back from 4; in Q5, $\frac{5}{6}$ became 0.9; and in Q6, $\frac{3}{4}$ became $\frac{9}{10}$. The importance of the interview as an assessment tool comes through here, as it was the students' explicit explanations that enabled this misconception to be identified and separated from other answers given in decimal form (estimating for example).

The final misconception was *counting the zero point* when counting the lines not the spaces. When counting the number of parts, some students counted the mark at 0 as "one" and ended up thinking the whole had one more part than it actually did. Only 12.5% of students did this on one or more tasks. Reading $\frac{5}{6}$ as $\frac{6}{7}$ was the most common error (Q5). In Q4, $3\frac{3}{4}$ was read as $3\frac{4}{5}$. In Q6, students imagined the missing line and then called $\frac{3}{4}$, $\frac{4}{5}$. In Q7, students called 6.8 either 6.9 or $6\frac{9}{11}$. In Q8, some students counted the lines between 0 and 1, including 0 and described the parts as sixths. If they also counted the zero point at 1, they gave an answer of $1\frac{3}{6}$, and if not, they gave an answer of $1\frac{2}{6}$. The correct answer was $1\frac{2}{5}$. Every student who only made one error of this nature did so on Q5. There were two students who consistently made this type of error on three or four of the number line tasks. Students had to give both an answer consistent with this method and an explanation that made their counting explicit in order to be categorised in this way.

Working from the known to the unknown and back again

We can see that not every number line generates each type of error. The counting the zero point error, while not common, was present in all five tasks in which the students had to read partitions, but not where they had to make the partitions themselves. Also, the assuming tenths misconception occurred in four of the five reading partitioning tasks, was undetectable in the fifth because the number line was marked in tenths, and was absent in the three tasks in which students had to make their own partitions. A limited part-whole understanding was more evident in the tasks requiring the students to make their own partitions. However, other research has shown that if asked to read a mark at $\frac{1}{2}$ on a number line similar to our Q3, children will read it as a quarter. The types of errors on different tasks suggest that in children's understanding of number lines, there is a difference between reading partitions and making partitions.

In classrooms, we would like to empower our students by helping them articulate the factors that may prompt them to draw on correct mathematical thinking rather than misconceptions. A one-to-one task-based formative assessment interview supports classroom norms valuing students' reflective thinking about the strategies they use to attempt mathematics tasks. Teacher and student examination of misconceptions enables further discussion about possible strategies. Teacher and student knowledge of the three different interpretations of the measure sub-construct enables further evaluation of the misconceptions. Classroom tasks provide the opportunity for students to work from the known to the unknown to identify contexts in which misconceptions are more likely to occur, for example:

- "I don't assume number lines are marked in tenths when I draw my own, but sometimes I do by mistake when reading pre-made partitions."
- "When I am looking at a number line I can read improper fractions like three and three quarters, but sometimes when I draw my own I think of it as just a line and forget that two thirds of a line and two thirds on a number line don't always look the same."
- "I don't count the zero-point when I am reading a number line marked in tenths so I wonder why I make that mistake on a broken ruler task."

Having gone from the known to the unknown, we would like our students to set about correcting the misconception. Once they have an explanation that satisfies them on a question where they had previously demonstrated a misconception, we would like them to reflect on their correct mathematical thinking in the question where they had been successful and confirm that these strategies align. Tasks from different domains, for example broken rulers and fraction number lines, can also be explored together, in order to help children make connections between mathematical ideas in different contexts. While eliminating misconceptions will help children's performance, so too will identifying what makes students successful in both broken ruler tasks and fraction number lines.

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