MATHEMATICS:
IT'S MINE

PROCEEDINGS OF THE
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THE AUSTRALIAN ASSOCIATION OF
MATHEMATICS TEACHERS INC.

Edited by C. Hurst, M. Kemp, B. Kissane, L. Sparrow & T. Spencer

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OTHER PAPERS

PREFACE

The twenty-second biennial conference of The Australian Association of Mathematics Teachers is appropriately titled *Mathematics: It's Mine.*

This theme proclaims the importance of both students and teachers working confidently with mathematics across the range of settings represented in various ways at the conference. These settings include early childhood, primary and secondary schooling, as well as teacher education, professional development and the wider community. In suggesting that such a proclamation about mathematics ought be made, the theme aptly reminds us of the significance of mathematics in our society and our culture, and reflects our collective aspiration that Australians will come to share this perspective, with our help. The theme also suggests that it is important to encourage people to take ownership of their own mathematical learning. Indeed, many of the conference presentations are directed at student engagement, both inside and outside the classroom, that will help students be able to say: *Mathematics: It's mine!*

Mining is an important part of the economy of Western Australia, the site for this event. The 2009 conference has brought together teachers, mathematicians, teacher educators, researchers and other professionals from all states and territories of Australia and from other countries to demonstrate a variety of facets and perspectives of teaching and learning mathematics. In the best spirit of the new world of "data mining," the theme also suggests that the conference has been able to mine some of the extensive expertise in mathematics education of which the Australian Association of Mathematics Teachers is justly proud.

This publication, comprising papers presented at the conference as keynotes, major presentations, and seminar or workshop offerings, shows the range, nature and quality of the work which many of our colleagues have been prepared to share with us all. We are grateful to them for this.

Editors: Chris Hurst, Marian Kemp, Barry Kissane, Len Sparrow and Toby Spencer

REVIEW PROCESS

Presentations at AAMT 2009 were selected in a variety of ways. Keynotes and major presentations were invited to be part of the conference and to have papers published in these proceedings. A call was made for other presentations in the form of either a seminar or a workshop. Seminars and workshops were selected as suitable for the conference based on presenters' submission of a formal abstract and further explanation of the proposed presentation.

Authors of seminar and workshop proposals that were approved for presentation at the conference were also invited to submit a written paper to be included in these proceedings, with the possibility of the paper being subjected to peer review. Papers for which peer review was requested were scrutinised blind by at least two reviewers. Reviewers were chosen by the editors to reflect a range of professional settings. Papers that passed this review process have been identified in these proceedings as "accepted by peer review." Other papers that were submitted to the proceedings without peer review were accepted as suitable for publication by the editors.

The panel of people to whom papers were sent for peer review was extensive and the editors wish to thank them all:

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KEYNOTES

MATHEMATICS FOR TEACHING MATTERS

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In this paper, I illuminate the notion of mathematics for teaching (its matter) and argue that it matters (it is important), particularly for mathematics teacher education. Two examples from studies of mathematics classrooms in South Africa are described, and used to illustrate what mathematics teachers use, or need to use, and how they use it in their practice: in other words, the substance of their mathematical work. Similarities and differences across these examples, in turn, illuminate the notion of mathematics for teaching, enabling a return to, and critical reflection on, mathematics teacher education.

Introduction

This paper explores the notion of *mathematics for teaching*, and why it matters for the teaching and learning of mathematics in general, and mathematics teacher education in particular. This exploration builds on the seminal work of Lee Shulman. In the mid-1980s Shulman argued cogently for a shift in understanding, in research in particular, of the professional knowledge base of teaching. He highlighted the importance of content knowledge in and for teaching, criticising research that examined teaching activity without any concern for the content of that teaching. He described the various components of the knowledge base for teaching, arguing that content knowledge for teaching included subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curriculum knowledge (Shulman, 1986; 1987). Shulman's work set off a research agenda, with a great deal focused on mathematics. This paper draws from the mathematical elaboration of Shulman's work.

The profound insight of Shulman's work was that being able to reason mathematically, for example, was necessary but not sufficient for being able to teach others to reason mathematically. Being able to teach mathematical reasoning involves recognising mathematical reasoning in others' discourse, and at various curriculum levels, being able to design and adapt tasks towards purposes that support mathematical reasoning, and critically working with or mediating the development of such in others. We could say the same for being able to solve algebraic or numeric problems. This assertion is not news to most mathematics teachers or mathematics teacher educators. Yet, in the particular case of mathematical reasoning, its actuality in curricular texts, classroom practices and learner performances remains a challenge in many, if not most, classrooms (Stacey & Vincent, 2009). We could say the same for learner performance in many areas of mathematics, as well as algebra. Despite the longevity and consistency of elementary algebra in school mathematics curricula worldwide, large numbers of learners experience difficulty with this powerful symbolic system (Hodgen, Kuchemar, Brown & Coe, forthcoming).

In this paper I argue that strengthening our understanding of the mathematical work of teaching, what some refer to as *mathematics for teaching*, is a critical dimension of enhancing its teaching and learning. Mathematics for teaching matters, for all our learners, as do its implications for mathematics teacher education. I will develop this argument through examples from school mathematics classrooms, together with comment on developments in mathematics teacher education in South Africa. Ultimately, the argument in this paper poses considerable challenges for mathematics teacher education.

Teaching and learning mathematics in South Africa

The past fifteen years of post-apartheid South Africa can be categorised as a time of rapid and intense policy and curriculum change. New mathematics curricula are being implemented in schools across Grades 1–12, where there is greater emphasis than before on sense-making, problem-solving and mathematical processes, including mathematical reasoning, as well as on new topics related to data handling and financial mathematics. New education policy and curricula have strong equity goals, a function of the deep and racialised inequality produced under apartheid that affected teachers and learners alike. New policies and qualifications have been introduced into teacher education, with goals for improving the quality of teachers and teaching, and in the case of mathematics, addressing critical shortages of qualified secondary mathematics teachers that persist, and indeed have deepened over time. Tertiary institutions have responded, offering new degree and diploma programs for upgrading teachers in service, retraining into teaching, and preparing new teachers.

It is in moments of change that taken-for-granted practices are unsettled, in both inspiring and disconcerting ways. Moments of change thus provide education researchers and practitioners with challenging opportunities for learning and reflection. Of pertinence to this paper is that the challenge of new curricula in schools and thus new demands for learning and teaching, on top of redress, bring issues like the selection of knowledges for teacher education development and support to the fore. Mathematics teacher educators in all tertiary institutions have had the opportunity and challenge to make decisions on what knowledge(s) to include and exclude in their programs, and how these are to be taught/learnt. This has meant deliberate attention to what mathematics, mathematics education and teaching knowledge teachers need to know and be able to use to teach well. This is no simple task: in South Africa, teaching well encompasses the dual goals of equity and excellence. At the same time as strengthening the pool of mathematics school leavers entering the mathematical sciences and related professions, high quality teaching also entails catering for diverse learner populations, and inspiring school learners in a subject that all too often has been alienating.

Hence the question: what selections from mathematics, mathematics education and teaching[1] *are needed to provide the greatest benefit to prospective and in-service teachers?*

Shulman's categories provide a starting point to answering this question. Others, particularly Ball and her colleagues working on mathematical knowledge for teaching in Michigan USA, have argued that these categories need elaboration; and that elaboration requires a deeper understanding of mathematics *teaching*, and hence, of teachers' mathematical work. Ball, Thames and Phelps (2008) have elaborated Shulman's categories, distinguishing within subject matter knowledge, between *Common* and *Specialised Content Knowledge* where the latter is what teachers in particular need to know and be able to use. Within Pedagogical Content Knowledge, they distinguish *knowledge of mathematics and students*, and *knowledge of mathematics and teaching*. These latter are knowledge of mathematics embedded in (and so integrated with) tasks of teaching, that is, a set of practices teachers routinely engage in or need to engage in. In their more recent work where they examine case studies of teaching, Hill, Blunk, Charalambos, Lewis, Phelps, Sleep and Ball (2008) note that while their elaboration is robust, compelling and helpful, they underestimated the significance of what Shulman identified as Curriculum Knowledge. What this reflects is that all teaching always occurs in a context and set of practices, of which curricular discourses are critical elements. Ball et al.'s elaboration of Shulman's categories is useful, particularly as it has been derived from studies of mathematics classroom practice. They provide a framework with which to think about and make selections for teacher education. At immediate face value, they suggest that mathematical content in teacher education and for teaching requires considerable extension beyond knowing mathematics for oneself.

I go further to say we need to understand what and how such selections take shape in mathematics teacher education practice. As in school, teacher education occurs in a context and set of practices, and is shaped by these. In addition, as intimated above, in mathematics teacher education, mathematics as an "object" or "focus" of learning and becoming, is integrated with learning to teach. The research we have been doing in the QUANTUM[2] project in South Africa (that now has a small arm in the UK) has done most of its work in teacher education as an empirical site, complemented by studies of school mathematics classroom practice. The goal is to understand the substance of opportunities to learn mathematics for teaching in teacher education, and how this relates to the mathematical work teachers do in their school classrooms.

In this paper, I select two examples from studies of mathematics classrooms in South Africa. I use these to illustrate what mathematics teachers use, or need to use, and how they use it in their practice: in other words, the substance of their mathematical work. Similarities and differences across these examples, in turn, illuminate the notion of mathematics for teaching, enabling a return to, and critical reflection on, mathematics teacher education.

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[1] Mathematics education here refers to the field of research and other texts related to mathematics curricula; teaching refers to the professional practice.

[2] For details on QUANTUM**,** a project focused on Qualifications for Teachers Underqualified in Mathematics, see Adler & Davis (2006), Davis, Parker & Adler (2007); Adler & Huillet (2008), Adler (2009)

Designing and mediating productive mathematics tasks

Example 1: Angle properties of a triangle

The episode discussed below is described in detail in Adler (2001)[3], and takes place in a Grade 8 classroom. This teacher was particularly motivated by a participatory pedagogy, and developing her learners' broad mathematical proficiency (Kilpatrick, Swaffold & Flindell, 2001). She paid deliberate attention to supporting her learners' participation in mathematical discourse (Sfard, 2008), which in practice involved having them learn to reason mathematically, and verbalise this. It is interesting to note that the empirical data here date back to the early 1990s and long before curriculum reform as it appears today in South Africa was underway.

As part of a sequence of tasks related to properties of triangles, the teacher gave the activity in Figure 1 to her Grade 8 class. The questions I will address in relation to this task are: What mathematical work is entailed in designing this kind of task, and then mediating it in a class of diverse learners?

If any of these is impossible, explain why; otherwise, draw it.

- Draw a triangle with 3 acute angles.
- Draw a triangle with 1 obtuse angle.
- Draw a triangle with 2 obtuse angles.
- Draw a triangle with 1 reflex angle.
- Draw a triangle with 1 right angle.

Figure 1. A triangle task.

The task itself evidences different elements of important mathematical work entailed in teaching learners to reason mathematically. Firstly, this is not a "typical" task on the properties of triangles. A more usual task to be found in text books, particularly at the time of the research, would be to have learners recognise (identify, categorise, name) different types of triangles, defined by various sized angles in the triangle. What the teacher has done here is recast a "recognition" task based on angle properties of triangles into a "reasoning" task (reasoning about properties and so relationships). She has constructed the task so that learners are required to reason in order to proceed. In so doing, she sets up conditions for producing and supporting mathematical reasoning in the lesson and related proficiencies in her learners. Secondly, in constructing the task so that learners need to respond whether or not particular angle combinations are "impossible" in forming a triangle, the task demands proof-like justification—an argument or explanation that, for impossibility, will hold in all cases. In this task, content (properties of triangles) and processes (reasoning, justification, proof) are integrated. The question, of course, is what and how learners attend to these components of the task, and how the teacher then mediates their thinking.

[3] The focus of the study reported in Adler (2001) was on teaching and learning mathematics in multilingual classrooms. There I discuss in detail the learners' languages, and how and why talking to learn worked in this class. I have since revisited this data, reflecting on the teachers' mathematical work (see Adler, 2006).

In preparation for this lesson and task, the teacher would have had to think about the mathematical resources available to this classroom community with which they could construct a general answer (one that holds in all cases). For example, if as was the case, learners had worked with angle sum in a triangle, what else might come into play as learners go about this task? What is it about the triangle as a mathematical object that the teacher needs to have considered and that she needs to be alert to as her learners engage in reasoning about its properties?

Before engaging further with the details of the teachers' mathematical work, let us move to the actual classroom, where students worked on their responses in pairs. The teacher moved between groups, probing with questions like: "Explain to me what you have drawn/written here?", "Are you sure?", "Will this always be the case?" She thus pushed learners to verbalise their thinking, as well as justify their solutions or proofs. I foreground here learners' responses to the second item: Draw a triangle with two obtuse angles. Interestingly, three different responses were evident.

- Some said, "It is impossible to draw a triangle with two obtuse angles, because you will get a quadrilateral." They drew the shape shown in Figure 2.

Figure 2. Student drawing of a triangle with two obtuse angles

- Others reasoned as follows: "An obtuse angle is more than 90 degrees and so two obtuse angles give you more than 180 degrees, and so you won't have a triangle because the angles must add up to 180 degrees."
- One learner (Joe) and his partner reasoned in this way: "If you start with an angle say of 89 degrees, and you stretch it [to make it larger than 90 degrees], the other angles will shrink and so you won't be able to get another obtuse angle." Their drawing is shown in Figure 3.

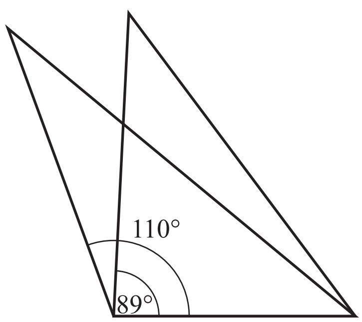


Figure 3. Joe and his partner's response.

On this part of the task, as with the one on a reflex angle, there was a range of learner responses—indicative of a further task-based teaching skill. It is designed with sufficient openness that diverse learner responses are possible. In addition, the third, unexpected, response produced much interest in the class, for the teacher, and for myself as researcher. The first two responses were common across learners and more easily predicted by the teacher.

Having elicited these responses, it is the teacher's task to mediate within and across these responses, and enable her learners to reason whether each of these responses is a general one, one that holds in all cases[4]. In the many contexts where I have presented the study and this particular episode, much discussion is generated both in relation to the mathematical status of the responses, and their levels of generality, as well as simultaneous arguments as to what can be expected of learners at a grade 8 level. What constitutes a generalised answer at this level? Are all three responses equally general? Is Joe's response a generalised one? How does the teacher value these three different responses, supporting and encouraging learners in their thinking, and at the same time judging/evaluating their mathematical worth?

These are mathematical questions, and the kind of work this teacher did on the spot as she worked to value and evaluate what the learners produced was also mathematical work. The point here is that this kind of mathematical work i.e. working to provoke, recognise and then mediate notions of proof and different kinds of justification, is critical to effective teaching of "big ideas" (like proof) in mathematics. In Ball et al.'s terms, this work entails knowledge of *mathematics and teaching* (designing productive tasks) and *mathematics and students* (and mediating between these and learners' mathematics).

We still need to ask questions about subject matter knowledge, or content in this example, and specifically questions about the angle properties of triangles? The insertion of a triangle with a reflex angle brought this to the fore in very interesting ways. Some learners drew the following, as justification for why a triangle with a reflex angle was possible; and so provoked a discussion of concavity, and interior and exterior angles.

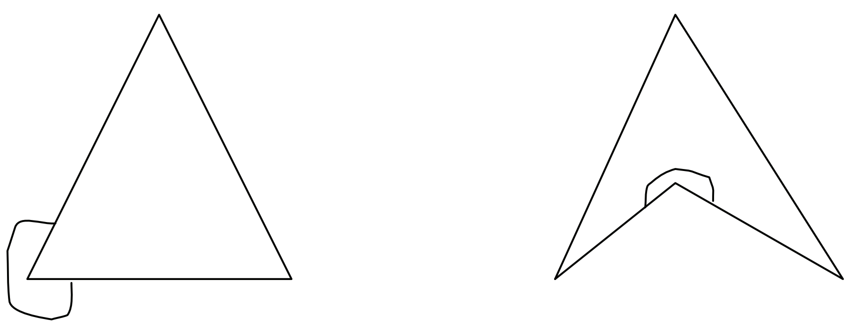


Figure 4. Learner drawings to justify triangles with reflex angles.

The tasks of teaching illuminated in this example are: task design where content (angle properties of triangles) and process (reasoning, justifying) are integrated; mediation of

[4] The interesting interactions that followed in the class are described and problematised in Adler (2001) and will not be focused on here.

both mathematical content and processes; and valuing and evaluating diverse learner productions. The mathematical entailments of this work are extensive, and are illustrative of both subject matter knowledge and pedagogical content knowledge. The teacher here reflects a deep understanding of mathematical proof, and in relation to a specific mathematical object and its properties. To effectively mediate Joe's response and the two above, she would also need to ask suitable questions or suggest productive ways forward for these learners, so that their notions of proof and of the mathematical triangle are strengthened. Indeed, as learners in the class engaged with the second triangle drawn above, their focus was that the answer was incorrect because there were three reflex angles not one, and the teacher had a difficult time shifting them from this focus and onto the interior angles.

In Adler (2001), I show that as the teacher mediated the three different responses to the triangle with two obtuse angles, she worked explicitly to value each contribution and probe learner thinking. However, her judgment of their relative mathematical worth was implicit. She accepted the first two responses above, but probed Joe's, with questions to Joe that implied she was not convinced of the generality of his argument. I argued there that if teacher judgment of the varying mathematical worth of learner responses offered is implicit, it is possible that only those learners who can themselves make such judgements, or who are able to read the implicit messages in the teacher's actions, will appreciate and so have access to what counts mathematically. Sociological theory and empirical research inform us that these kinds of practices favour students with school cultural capital, and so can reproduce inequality. In Bernstein's (1996) terms, implicit practices will connect with learners who already understand the criteria for what are most legitimate responses; and alienate or pass by those who are not "in" the criteria. Typically these will be already disadvantaged learners (Parker, 2009).

The example here is compelling in a number of ways, and provokes the question: Where, when and how does a mathematics teacher learn to do this kind of work, and in ways that are of benefit to all learners? Before attempting to answer this and so shift back into teacher education, we need to look at additional and different examples of the mathematical work of teaching.

Example 2: Polygons and diagonals — or a version of the "mystic rose"

The second example is taken from a Grade 10 class (see Naidoo, 2008), where the teacher posed the following task for learners to work on in groups: *How many diagonals are there in a 700-sided polygon?*

Here too, the teacher has designed or adapted a task and presented learners with an extended problem. They have to find the number of diagonals in a 700-sided polygon, a sufficiently large number to require generalising activity, and so mathematical reasoning. I pose the same questions here as for Example 1: What mathematical work is entailed in designing this kind of task, and mediating it in a class of diverse learners?
Many teachers will recognise the "mystic rose" investigation in this problem. The mathematical object here is a polygon and its properties related to diagonals. Yet the problem has been adapted from a well known (perhaps not to the teacher) mathematical investigation of points on a circle and connecting lines — a different, though related object. Here learners are not asked to investigate the relationship between the number of points on a circle and connecting lines, but instead to find an actual numerical solution

to a particular polygon, albeit with a large number of sides and so approaching a circle. I have discussed this case in detail in Adler (2009), where I point out that unlike triangles and their properties, the general polygon and its properties is not an explicit element of the secondary school curriculum. However, the processes and mathematical reasoning required for learners to solve the problem are desired mathematical processes in the new curriculum.

My concern in this paper is not with the merits of the problem and its adaptation in an isolated way. Rather, I wish to reflect on the mathematical work of the teacher in presenting the problem, mediating learner progress, valuing and evaluating their responses, and managing the integration of mathematical content and mathematical processes as foci in the lesson. I present selections from the transcript of the dialogue in the classroom to illuminate these four components of the teachers' mathematical work.

The teacher (Tr), standing in the front of the class, explained what the class had to do.

Tr: I want you to take out a single page quickly. Single page and for the next five minutes no discussion. I want you to think about how would you possibly solve this problem? (pointing to the projected problem: *How many diagonals are there in a 700-sided polygon?*

After seven minutes, the Teacher calls the class' attention. (Learners are referred to as Lr A, B, etc.)

Tr: Ok! Guys, time's up. Five minutes is over. Who of you thinks they solved the problem? One, two, three, four, five, six.
Lr A: I just divided 700 by 2.
Tr: You just divided 700 by 2. (Coughs).
Lr A: Sir, one of the side's have, like a corner. Yes… [inaudible], because of the diagonals. Therefore two of the sides makes like a corner. So I just divided by two… [Inaudible].
Tr: So you just divide the 700 by 2. And what do you base that on? …
[ ]
Tr: Let's hear somebody else's opinion.
Lr B: Sir what I've done sir is … First 700 is too many sides to draw. So if there is four sides how will I do that sir? Then I figure that the four sides must be divided by two. Four divided by two equals two diagonals. So take 700, divide by two will give you the answer. So that's the answer I got.
Tr: *So you say that, there's too many sides to draw. If I can just hear you clearly; … that 700 sides are too many sides, too big a polygon to draw. Let me get it clear. So you took a smaller polygon of four sides and drew the diagonals in there. So how many diagonals you get?*
Lr B: In a four-sided shape sir, I got two.
Tr: Two. *So you deduced from that one example that you should divide the 700 by two as well? So you only went as far as a 4 sided shape? You didn't test anything else.*
Lr B: Yes, I don't want to confuse myself.
Tr: So you don't want to confuse yourself. *So you're happy with that solution, having tested only one polygon?*
Lr B: [Inaudible response.]
Tr: Ok! You say that you have another solution. [Points to learner D] Let's hear.
[ ]
Lr A: I just think it's right… It makes sense.
Tr: What about you Lr D? You said you agree.
Lr D: *He makes sense… He proved it… He used a square.*

Tr: He used a square? *Are you convinced by using a square that he is right?*
Lr E: But sir, here on my page I also did the same thing. I made a 6-sided shape and saw the same thing. Because a six thing has six corners and has three diagonals.
Lr A: So what about a 5-sided shape, then sir?
Tr: *What about a 5-sided shape? You think it would have 5 corners? How many diagonals?*

I have underlined the various contributions by learners, and italicised the teachers' mediating comments and questions. These highlight the learners' reasoning and the teacher's probing for further mathematical justification.

At this point in the lesson, the teacher realises that some of the learners are confusing terms related to polygons, as well as some of the properties of a general polygon and so deflects from the problem for a while to examine with learners, various definitions (of a polygon, pentagon, a diagonal, etc.). In other words, at this point, the mathematical object in which the problem is embedded comes into focus. It is interesting to note here that at no point was there reflection on the polygons in use in developing responses to the problem. All were regular and concave. A little later in the lesson, another learner offers a third solution strategy. The three different solution representations are summarised in Figure 5, illustrating the varying orientations students adopted as they attempted to work towards the solution for a 700-sided polygon.

Three different representations and reasoning

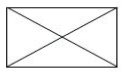
Learner A	***Learner B***	***Learner C***
700-sided polygon 700 / 2 = 350 diagonals	4-sided polygon 4 / 2 = 2 diagonals	7-sided polygon 14 diagonals 14 x 100 = 1400 diagonals
Representation: Verbal description	Representation: 	Representation: 
Reasoning: Because of sides – corners. 700/2 = 350 corners and 175 diagonals	Reasoning: Too big a number therefore use a quadrilateral. 4/2 = 2 diagonals therefore 700/2	Reasoning: 7-sided polygon has 14 diagonals therefore multiply by 100 which equals 1400.

Figure 5. Three different representations and reasoning.

As with Example 1, we see four tasks of teaching demanded of the teacher: task design or adaptation; mediation of learners' productions; valuing and evaluating their different responses; and managing mathematics content and processes opened up by the task.

The representations offered by learners give rise to interesting and challenging mathematical work for the teacher. All responses are mathematically flawed, though the approaches of Learners B and C show attempts at specialising and then generalising (Mason, 2002). While this is an appropriate mathematical practice, the move from the special case to the general case in both responses is problematic, though in different ways. Does the teacher move into discussion about specialising and generalising in mathematics (and if so, how)? Open-ended investigations and problem-solving as described above open up possibilities for this kind of mathematical work in class. Such opportunities were not taken up here. Each was negated empirically, and not elaborated more generally. Should they have been taken up by the teacher, and if so, how?

Tasks of teaching and their mathematical entailments

In selecting and presenting two different examples from different secondary school classrooms in South Africa, I have highlighted four inter-related tasks of teaching, each of which entails considerable mathematical skill and understanding over and above (or underpinning) the teaching moves that will ensue. The four tasks (two of which are discussed in each of the bulleted sections below) further illustrate categories of professional knowledge developed by Shulman and elaborated by Ball et al. in mathematics.

Designing, adapting or selecting tasks, and managing processes and objects

In the first example, the process of mathematical reasoning was in focus, as was the triangle and its angle properties. I will call this an *object-and-process-focused* task. Angle properties of triangles are the focus of reasoning activity. Learners engage with and consolidate knowledge of these properties through reasoning activity, and vice versa. Here the integration of learning content and process appears to keep them both in focus, and thus provides opportunities for learning both. Example 2 is also focused on mathematical reasoning. It is a *process-focused* task, having been adapted (what I would refer to as recontexualised) from an investigation and re-framed as a problem with a solution. The mathematical object of the activity, the polygon, is backgrounded. At a few points in the lessons, it comes into focus, when understanding polygons and their properties is required for learners to make progress with the problem: some learners make assumptions about what counts as a diagonal, perhaps a function of assuming regularity (and so finding three diagonals in a hexagon); some generalise from one specific case (a four-sided figure); while others over-generalise multiplicative processes from number, to polygon properties.

The intricate relationship between mathematical objects and processes has been an area of extensive empirical research in the field of mathematics education. It appears from studying two examples of teaching that selecting, adapting or designing tasks to optimise teaching and learning entails an understanding of mathematical objects and processes and how these interact within different kinds of tasks. The teaching of mathematical content and mathematical processes is very much in focus today. Reform curricula in many countries promote the appreciation of various mathematical objects, their properties and structure, conventions (how these are used and operated on in mathematical practice), as well what counts as a mathematical argument, and the mathematical processes that support such. In Example 1, we see opportunity for developing reasoning skills, and understanding of proof at the same time as consolidating knowledge about triangles. In Example 2, it is not apparent whether and how either proof or reasoning will flourish through this example and its mediation. The relevance of the mathematical object in use is unclear. Thus the question: *Do we need a* *mathematics for teaching curriculum that includes task interpretation, analysis and design with specific attention to intended mathematical objects and processes and their interaction?*

In other words, should a mathematics for teaching curriculum include attention to the mediation of mathematical content and processes as these unfold in and through engagement with varying tasks? If so, is this to be part of the mathematics curriculum, or part of the teaching curriculum? And hidden in this last question is a question of who teaches these components of the curriculum in teacher education? What competences and expertise would best support this teaching?

Valuing and evaluating diverse learner productions

Diverse learner productions are particularly evident in Examples 1 and 2, given their more open or extended nature. Thus, in each example, the teacher dealt with responses from learners that they predicted, and then those that were unexpected. In Example 1, the teacher needed to consider the mathematical validity of Joe's argument for the impossibility of a triangle with two obtuse angles, and then how to encourage him to think about this himself, and convince others in the class. Similarly, we can ask in Example 2: what might be the most productive question to ask Learner C and so challenge the reasoning that, since 700 can be factored into 7×100, finding the diagonals in a 7-sided figure is the route to the solution to a 700-sided figure? Such questioning in teaching needs to be mathematically informed.

Together these examples illuminate how teachers need to exercise mathematical judgement as they engage with what learners do (or do not do). This is particularly so if teachers are building a pedagogical approach and classroom environment that encourages mathematical practices where error, and partial meanings are understood as fundamental to learning mathematics. In earlier work I referred to this as a teaching dilemma, where managing both the valuing of learner participation and evaluation of the mathematical worth of their responses was important (Adler, 2001); and illuminated the equity concerns if and when evaluation of diverse responses—i.e., judgements as to which are mathematically more robust or worthwhile—are left implicit.

So, a further question needs to be asked of the curriculum in mathematics teacher education, and the notion of mathematics for teaching. Learner errors and misconceptions in mathematics are probably the most developed research areas in mathematics education. We know a great deal about persistent errors and misconceptions that are apparent in learners' mathematical productions across contexts. These provide crucial insight into the diverse responses that can be anticipated from learners. Yet, as Stacey (2004) argues, the development of this research into contents for teacher education has been slow. We have shown elsewhere that the importance of learner mathematical thinking in mathematics teacher education *is* evident in varying programs in South Africa (see Davis, Adler & Parker, 2007; Adler, forthcoming; Parker, 2009). Yet there are significant differences in the ways this is included in such programs, and so with potential effects on who is offered what in their teacher education. *How should a mathematics for teaching curriculum then include such content?*

Mathematics for teaching matters

I have argued that mathematics for teaching matters for teaching and also for opportunities to learn. I have suggested that what matters are task design and mediation, as well as attention to content, objects and processes within these. I have played on the word “matters” by suggesting firstly that these are the “matter” or the content of mathematics for teaching; and at the same time that they matter (have significance) in and for teacher education. Secondly, I have suggested that there are equity issues at stake.

I now return to the context of teacher education in South Africa where various innovative teacher education programs are grappling with a curriculum for mathematics teachers that appreciates the complexity of professional knowledge for teaching and its critical content or subject basis. I will focus here on what we have observed as objects of attention (and so meanings) shift from classrooms to teacher education and back again, observations that support the argument in this paper, that we need to embrace our deeper understanding of the complexities of teaching and so our task in teacher education.

In more activity-based, participative or discursively rich classroom mathematics practice, there is increased attention to mathematical processes as critical to developing mathematical proficiency and inducting learners into a breadth of mathematical practices. The examples in this paper illustrate how mathematical processes are always related to or based on some mathematical object. If the latter is not well understood, in the first instance by the teacher, in ways that enable her to notice when it goes out of focus or is completely missed by students, then their reasoning is likely to be flawed or mathematically empty. This phenomenon is apparent in classrooms in South Africa, and more so in historically disadvantaged settings, thus perpetuating rather then attacking inequality. Mathematical objects and processes and their interaction are the central “matter” of mathematics for teaching. The shift in new curricula to mathematical processes creates conditions for diminished attention to mathematical objects. Attention to *objects and processes* need to be embraced *in the context of teaching* if access to mathematics is to be possible for all learners.

Herein lies considerable challenge. In each of the two examples in this paper, a mathematical object was embedded in a task that worked varyingly to support mathematical reasoning processes. What the teacher in each case faced was different learner productions as responses to the task. These become the focus of the teachers’ work, requiring integrated and professional based knowledge of mathematics, teaching tasks and learner thinking. So what then, is or comes into focus in teacher education, and not only into teacher education, but into school curricula? What we have observed (and I have seen elements of this in elementary mathematics teacher education in the UK), is that learner thinking and the diversity of their responses become the focus, with the mathematical objects and tasks that give rise to these, out of focus. What one might see in the case of the triangle properties is a task that requires learners to produce three different arguments for why a triangle cannot have two obtuse angles. And there is a subtle but impacting shift of attention: from how to mediate diverse responses, to multiple answers or solutions being the required competence in learners; from teachers’ learning to appreciate diverse learner productions and their relative mathematical worth, and more generally, multiple representations, and how to enable learners to move flexibly between these, to these being the actual content of teaching. Simply, there are curricular texts that now require learners to produce multiple solutions to a problem. I leave this somewhat provocative assertion for discussion and further debate.

In conclusion, there is an assumption at work throughout this paper that teacher education is crucial to quality teaching. In South Africa, all pre-service and formal in-service teacher education has become the responsibility of universities. Tensions between theory and practice abound. I hope in this paper to have provided examples that illuminate the mathematical work of teaching, and through these opened up challenges for mathematics teacher education. Mathematics for teaching, and its place in mathematics teacher education, particularly in less resourced contexts, matters profoundly. There is much to do.

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GEOSTATISTICS: A MATHEMATICAL YOUNGSTER

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Geostatistics is concerned with the mathematical modelling of spatial data that arise in a variety of contexts. The initial applications were related to mining and the data under consideration geological. In the last 15 years the fields of application have broadened considerably, with geostatistics now being applied in such diverse areas as mining, oil and gas exploration, ecology, health and environment. We will discuss the key methods used by means of an example, describe some of the conceptual difficulties and give a brief overview of applications.

Introduction

At 60 years of age, geostatistics is a relatively young field of mathematics. Its origins lie in the need to estimate the size of an ore deposit as accurately as possible and the first applications were to gold deposits in the Witwatersrand in South Africa (Krige, 1951). An early limitation to its application was the data size and it is only with the emergence of fast computers that the development of geostatistical techniques has really taken off. From a mathematical point of view, while we are operating in a stochastic framework, the techniques that are drawn upon come from a variety of mathematical disciplines with linear algebra and numerical analysis of particular importance (For a comprehensive overview over the methods see Chilès and Delfiner, 1999).

These days we not only estimate, but also simulate spatial distributions. Moreover, we do so using personal computers and very sophisticated software. While mining is still one of the main areas in which geostatistics is used, the breadth of applications is breath-taking. They range from pollution studies through to the modelling of fishery data to lion populations, the spread of bushfires, all the way to health-related data.

Because of the nature of the data, dealing with them is typically messy and the user cannot simply rely on a set framework of algorithms to be applied each time. There is definitely a need to get one's fingers dirty. Throughout modelling decisions need to be taken and because of the potential impact, carefully justified. These decision concern the algorithm to be used and in particular the checking of the validity of model assumptions. In this paper I use a synthetic example to give an overview of the basic methodology and then discuss some applications.

A typical problem

In Figure 1 the histogram and spatial distribution of a synthetic sample are shown. The spatial map is colour-coded with warm colours representing high values and cold colours low values. The data are positively skewed with a long tail of high values, typical for mineral distributions, such as gold. From an inspection of the spatial map we see that pairs of samples that are separated by a short distance are more likely to have similar values than pairs of samples far apart. Moreover, there do not appear to be any features favouring particular directions in space, such as banding.

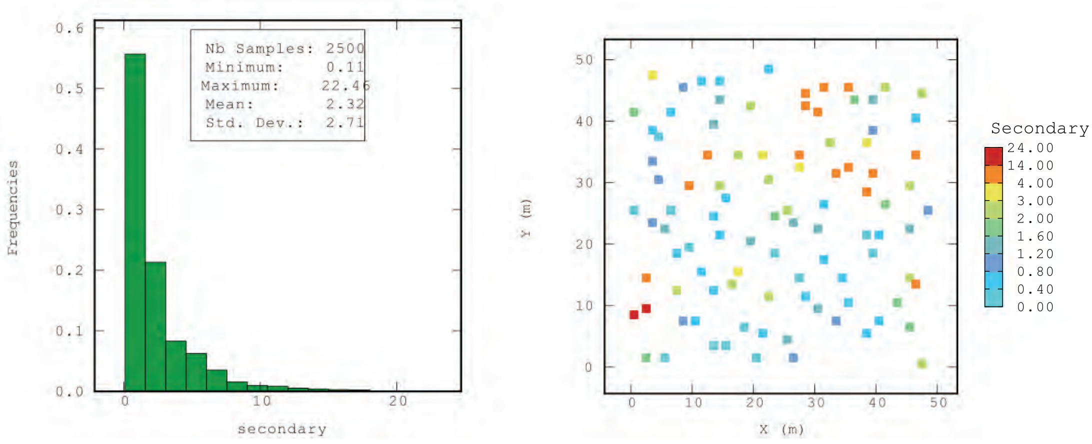

Figure 1. A gold sample: histogram (left) and spatial distribution with key statistics (right).

If we assume that the data do indeed represent a gold mineralisation, then there are several questions that would need to be answered:

- What is the exhaustive distribution of the mineral within the study region?
- What is the overall tonnage within the region and the average grade to be extracted?
- Do we have a deposit here that is worth mining, given the above results, and if so what are the optimal boundaries for the open pit that needs to be built and what mining schedule ought to be used?

The first of these questions requires “filling in the blanks.” Exhaustive drilling is evidently not the way to go because of the cost involved. An algorithm is needed to calculate estimates for the un-sampled locations. One requirement on the estimator is unbiasedness: the mean of the estimates is equal to the population mean. There are many different ways in which this filling in of the blanks can be done, including allocating the grade of the nearest neighbour, fitting a polynomial in the space coordinates to obtain an estimate and calculating a weighted average within a search window. The simplest method is to allocate the mean grade within the search window, giving equal weight to all samples it contains or else to take account of separation and possibly value. The latter is the approach taken in *ordinary kriging*, the most prevalent geostatistical estimation algorithm. This algorithm is named after one of the pioneers of geostatistics, Danie Krige who was one of the first to use windowed multilinear regression to calculate estimates.

The results for several “filling ins” are shown in Figure 2. Each one of these methods honours the data, in that the values at the sample locations are the actual sample values.

However, not all of the maps appear equally realistic. The spatial distributions obtained from nearest neighbour interpolation or moving window averaging appear patchy, while that from polynomial interpolation appears too smooth when compared with reality. Moreover, of the above approaches to the estimation problem, kriging is the only method that allows one to obtain a measure of the uncertainty in the form of an estimation error. Scrutiny of the histograms of the estimates and the exhaustive data (called “reality”[5]) in Figure 3 shows that only moving windows averages and ordinary kriging estimates have histograms that are close to the sample histogram. Moreover, polynomial interpolation results in negative estimates, which are not realistic in the given context.

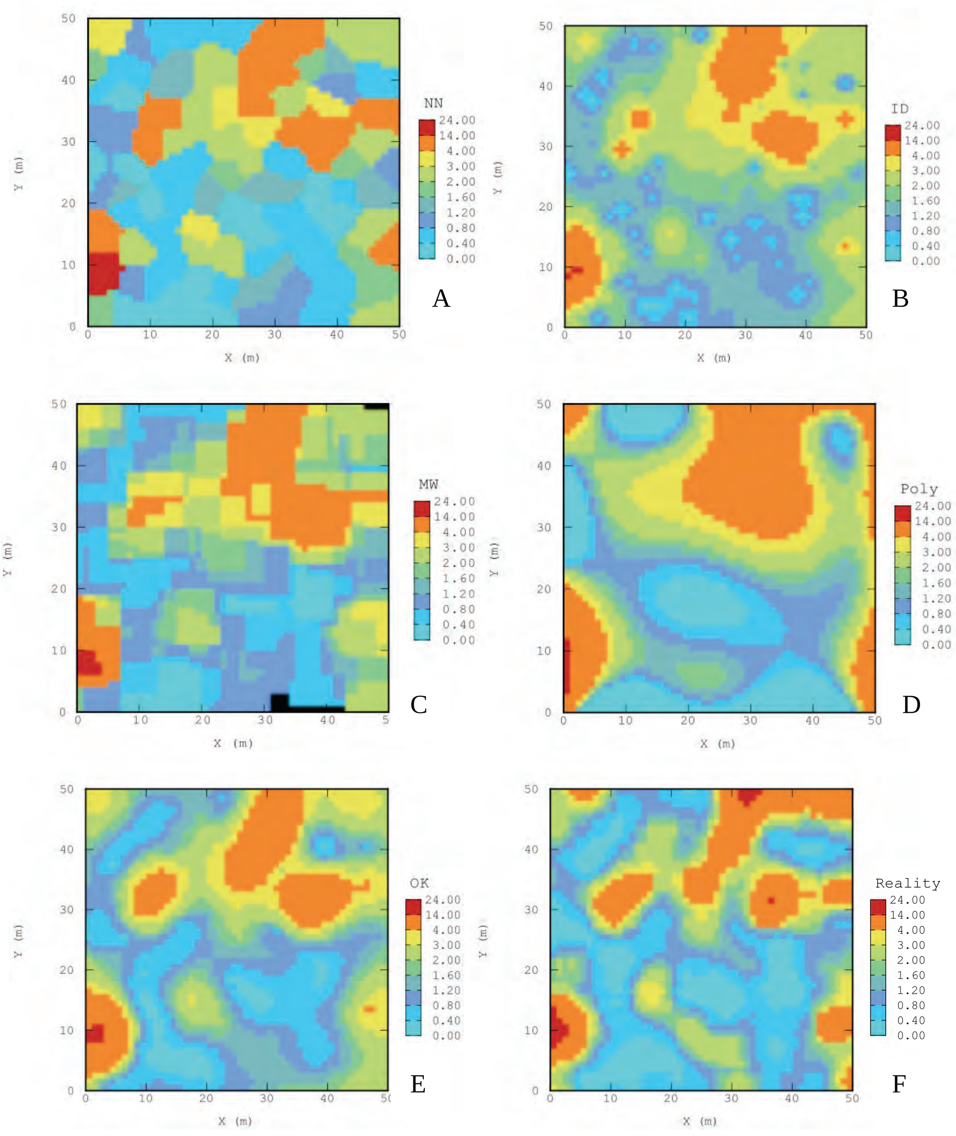

Figure 2. Estimated distributions resulting from different estimation methods: nearest neighbour estimation (A), inverse distance interpolation (B), moving window averages (C), polynomial of degree 6 in X and Y (D), ordinary kriging (E) and reality (F).

[5] While we know ‘reality’ in this case, of course this is not true in general.

If we accept that ordinary kriging provides a reasonable estimate for the spatial distribution of the gold mineralisation, then we can go ahead and use our estimates to tackle the second question and determine the average grade and calculate grade tonnage curves, the information required to decide if it is worthwhile to further develop the resource and maybe open a mine. Ultimately these decisions depend upon financial considerations, such as the type of gold mineralisation which in turn impacts on the milling process and the current gold price, as well as the possibility of forward sales.

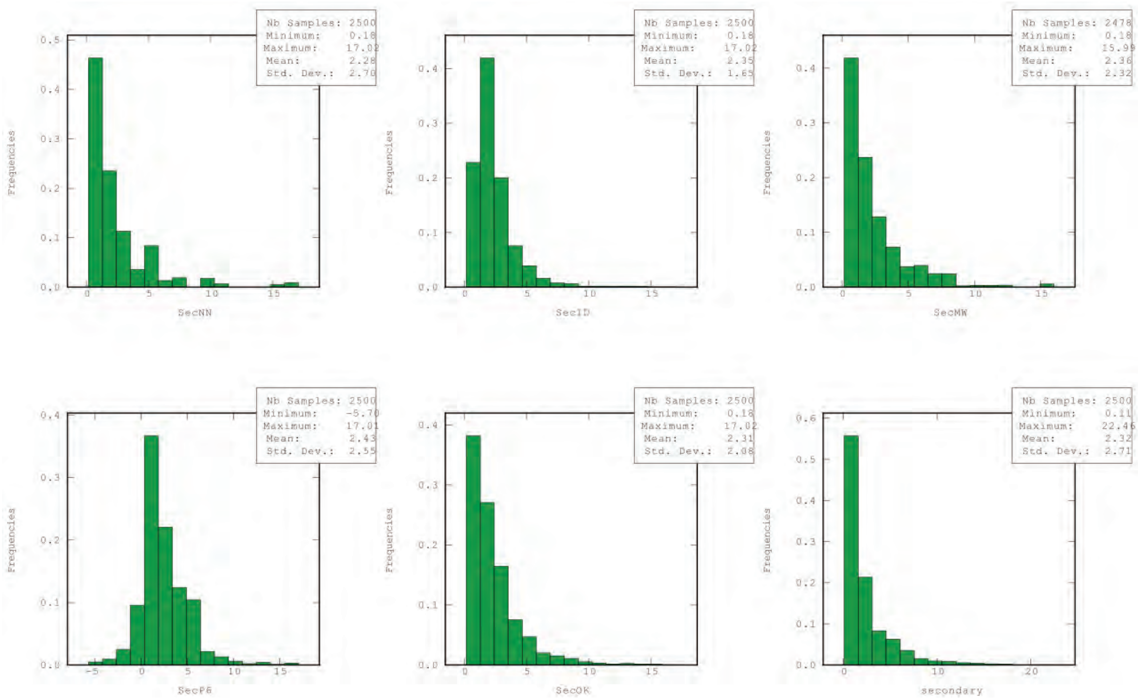

Figure 3. Histograms with key statistics for the estimates together with reality: nearest neighbour estimation (top left), inverse distance interpolation (top centre), moving window averages (top right), polynomial of degree 6 in X and Y (bottom left), ordinary kriging (bottom centre) and reality (bottom right).

To fully answer the question in relation to mine planning, potentially the last stage of the exercise, we need to use simulation. Based on strong model assumptions we generate equiprobable spatial distributions of the gold variable (see Figure 4) consistent with the sample data to assess the risk of choosing a particular pit design.

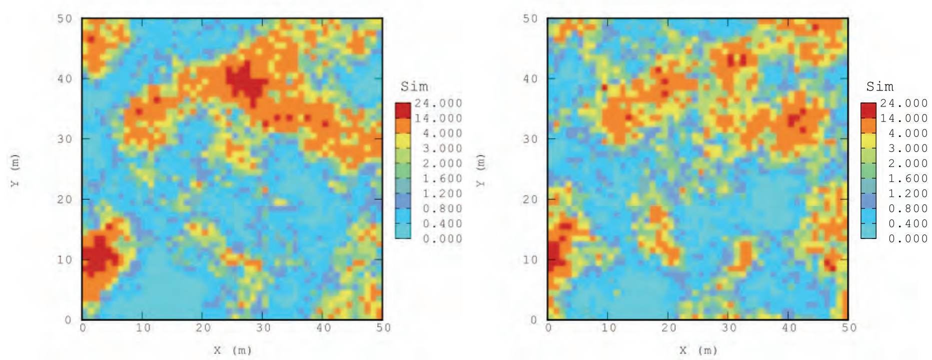

Figure 4. Two simulated spatial distributions of gold based on the sample in Figure 1.

As with the estimation, the simulations are consistent with the sample data, but each distribution represents a different possible reality, which in expected value approximates the map of ordinary kriging estimates. The simulations are used to generate histograms of the distributions at grid locations (see Figure 6) and maps that clearly indicate regions of high and low values as well as probability maps (see Figure 5) that allow one to visualise the probability of exceeding a threshold of interest, such as the minimum gold grade that will make the deposit economic.

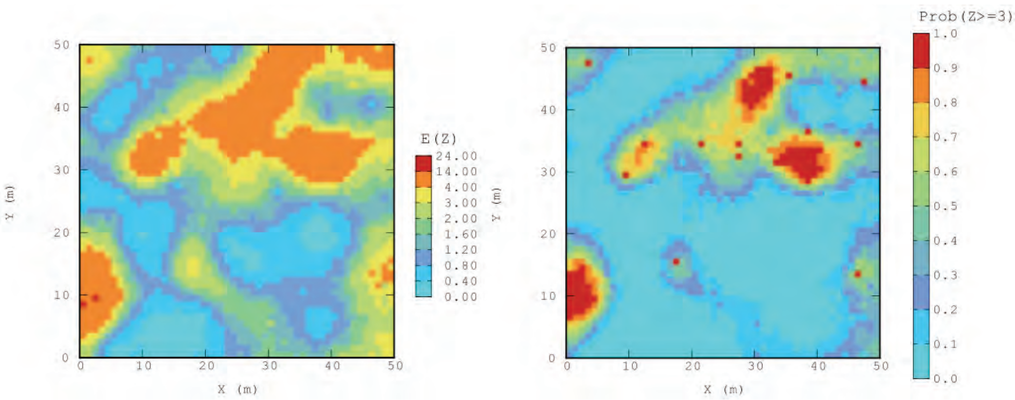

Figure 5. Average value of the simulations computed location by location (left) and spatial map of the probability of exceeding a grade of 3 ppm.

The spatial map of the mean grade calculated location by location shown in Figure 5 clearly highlights a region of high grades in the north eastern part of the region and if a cut-off grade of 3 ppm was applied, then only the north-east and the south-west would contain regions where the probability of exceeding this grade are high. From the map of the average values it is already apparent that the grade frequency distributions will differ from location to location. In fact, while the shape of the overall grade distribution of an individual simulation is not dissimilar from that of the exhaustive data (see Figure 4), the shapes of the distributions of the simulated grades at individual locations do not resemble the overall distribution of grades (see Figure 6). The grade distributions for locations (12.5,5.5) in the south-west and (25.5,36.5) in the north are both positively skewed, but they have very different ranges and kurtosis.

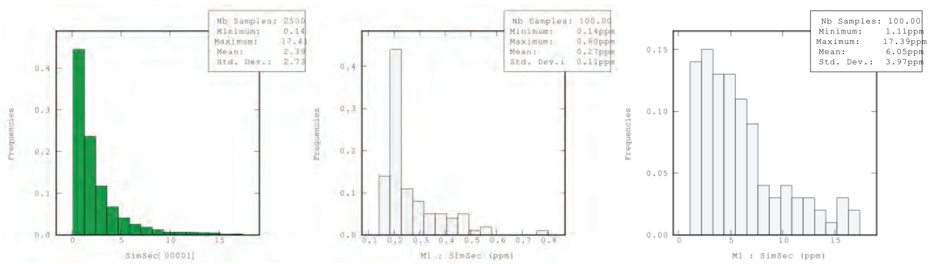

Figure 6. Histogram of simulation #1(left) and histograms for locations (12.5,5.5)(centre) and (25.5,36.5) (right).

The modelling approach

In the previous section I have given an overview over the typical workflow of a geostatistical study, moving from an exploratory data analysis to estimation and finally simulation taking into consideration the spatial continuity. I will now briefly consider the mathematical framework. The data with which geostatistics is concerned have the property that once sampled at the selected sites, there is no possibility of replication: once, say, a drill core has been pulled at a site, the action cannot be repeated. In essence, the process we are considering is deterministic. However, it is usually so complex that the construction of a deterministic model is not practical; just recall the poor job done by polynomial interpolation in Figure 1. It is for this reason that a stochastic framework is adopted.

This framework is known as the *Random Function Model*. We assume that at each location $\mathbf{u}$ in the study region there is a random variable $Z(\mathbf{u})$ and that the observed value $z(\mathbf{u}_\alpha)$ at a sample location $\mathbf{u}_\alpha$ is nothing but a realisation drawn from the random variable $Z(\mathbf{u}_\alpha)$ at the location. Because of this construct meaningful multivariate datasets may be constructed that enable the user to develop an insight into the spatial features of the attribute under study. Specifically the model allows us to calculate covariances that are functions of the separation distance. They are necessary for computing estimates and simulated values at un-sampled locations. The estimate at an un-sampled location is a weighted linear combination of the sample grades in the vicinity of the location. Determining the weights and hence the estimate ultimately comes down to solving a linear system, a procedure akin to standard linear regression.

A rigorous mathematical formulation of the estimation procedure was first given by Matheron who also coined the term *kriging*. The first formal courses in the subject were held at the Ecole des Mines de Paris in France and graduates from that school where responsible for the proliferation and dissemination of the techniques developed there worldwide. In this paper, rather than dwell on the technicalities of geostatistical techniques we will have a look at some of the applications.

Applications of geostatistics

This section includes an overview of some applications.

Mining

Mining applications are a stalwart of geostatistics. They cover all types of mining resources from metals to diamonds to coal and also petroleum. Here in Western Australia, geostatistical estimation is used regularly in the Pilbara iron ore mines for planning the mining, and in the WA goldfields for scheduling the extraction of gold. In the case of iron, it is not enough to analyse and model the distribution of iron, but in addition alumina and silica need to be modelled as their distribution impacts on the quality of the iron ore. Variables such as the grade of gold and the iron content are continuous variables, but diamonds, for which the use of geostatistics is well established, are discrete objects and so a model of the distribution of sizes is required. Their size distribution is highly positively skewed and as the interest is in large diamonds, extreme value modelling needs to be undertaken (Lantuéjoul 2008).

In the area of estimating the size of an oil reservoir hard data are scarce, as the drilling of oil wells is expensive. It is often the case that only a few oil wells are available, so denser secondary information such as seismic data are used to improve estimation.

Natural resources

One branch of natural resources modelling is concerned with fisheries. Since the early 1990s studies were conducted on survey data from the North Sea. The objective was an abundance assessment of commercially interesting fish species, such as herring and hake. An example from Western Australia concerns the catch and catch rate distribution of prawns and scallops in Shark Bay. One part of this study dealt with the ability of the annual scallop survey to adequately predict the subsequent scallop catch and an assessment as to whether or not trawling for prawns prior to the start of the scallop fishing season disturbed the settlement of the scallops. Given that scallops move very little, a distortion or shift in the distribution between the time of the survey and the distribution based on catch would have indicated an adverse effect of pre-season trawling. Our findings showed no clear evidence for a disturbance of scallop settlement (Mueller et al., 2008).

Environment

Applications in this field cover soil contamination, air quality in cities, water transport and the abundance of wildlife. The spatial distribution of whales within the Mediterranean whale sanctuary located in the waters between Spain and Italy has been studied extensively (Monestiez et al. (2006)). The raw data are the sightings of whales (by observers) over a period of ten years. The data are count data and a specifically tailored kriging algorithm was used to construct a map of the spatial distribution of whales. A complication in the modelling of wildlife data is the use of enthusiastic volunteers for data collection. They tend to frequent areas where observation of the animal of interest is more likely. An attempt to deal with this obstacle was presented in a paper on the spatial modelling of bird distributions in Croatia (Hengl et al., 2008).

Reforestation is another area of environmental application. The variable of interest is the number of plants surviving. To assess the survival rate a sampling design is necessary that locally allows the prediction of the rate with a prescribed maximum error. To be cost-effective, the sampling design needs to contain as few sites as possible and still be sufficiently accurate. In a case study from Chile (Emery et al., 2008) an initial distribution of sampling sites was available and sites for in-fill samples needed to be found to provide reliable estimates of the survival rate. In this case study two approaches to determine such an in-fill pattern for a plantation in Chile are discussed. The interest in the design of an in-fill pattern arose because of financial incentives by the Chilean government for the establishment of new plantations, but in order to qualify for the payment, there is a requirement of a 75% plant survival rate.

Health

Applications to health geography are fairly recent and are often a mixture of Geographic Information Systems and geostatistics. One of the human diseases of interest is cancer and there have been several studies concerned with the analysis of the spatial distribution of the incidence of various cancers (e.g.,, Goovarets, 2005). The objective of such studies is to obtain good estimates of the incidence risk and factors influencing the level of risk. The availability of reliable maps of incidence risk is important for public health campaigns and eradication programs. The applications are not restricted to cancer or human diseases. Contagious diseases like cholera and dysentery have also been investigated, for example for the Matlab region of Bangladesh (Ali et al., 2006). The mapping revealed a patchier cholera risk map than dysentery risk map, and also identified higher risk in the more urban areas for both diseases. An example of an animal disease studied in this way is foot and mouth disease (Perez et al., 2006). One of the common features of these studies is the need to use imperfect data. There are usually reporting inaccuracies, and in the case of human disease the need to use census data that are only recorded in selected years, contributes to the imperfection.

Concluding remark: Geostatistics in the secondary classroom

Geostatistics is a fascinating discipline with a wide variety of applications. Its interdisciplinarity and the nature of its applications make it a good candidate for highlighting the importance of mathematics in many different disciplines and its relevance in the modern world. Consideration of spatial data can provide students with interesting applications of some of the statistical techniques they learn in high school. While the construction of a semivariogram and its evaluation for kriging or simulation may be too difficult, some of the basic ideas, such as the construction of an abundance map, are accessible at the upper secondary level and would make interesting extension exercises. The construction of a spatial sample map requires the use of a meaningful colour scale so that the user can glean relevant information from it readily. This relies on calculating descriptive statistics and possibly deciles of the sample distribution. Going on from there, an exploration of characteristics of the sample is possible and the construction of estimates at unsampled locations using either moving windows or another weighted linear combination of the data, allowing an exploration of the impact of the sampling support and of the importance of a good estimator.

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GETTING BETTER: WHAT THEY NEED TO KNOW

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The use of the *First Steps in Mathematics* materials in the *Getting it Right* Numeracy Strategy has strongly impacted on the approach taken to mathematics teaching and learning in many Western Australian primary classrooms since 2002. The trial of the *Getting it Right* Numeracy Strategy in secondary mathematics classrooms since 2006 has revealed the need to re-assess the content and approach taken in programs designed to assist underachieving students. The Key Understandings and the Diagnostic Maps in the *First Steps in Mathematics* resources have been instrumental in clarifying the learning needs of these students and making explicit what it required to accelerate their progress.

Background

In Western Australia, the *Getting it Right* Literacy and Numeracy strategy (GiR-LNS) has been operating in several hundred identified government primary schools since 2002 and as a pilot in a selected few secondary schools since 2006.

The GiR-LNS professional learning model was based on research showing that, to be highly effective, professional learning for teachers should be school-based and integrated into the day-to-day work of teaching (Hawley & Valli, 1999). In the GiR-LNS strategy, Specialist Teachers (STs) participate in extensive professional learning, collaborate in planning sessions and work "shoulder to shoulder" with their peers in mainstream classrooms. Their work in schools is supported by a Central Office team through school visits and network meetings. Through working closely with teachers and students in classrooms the STs enhance their colleagues' professional judgements about what mathematics students currently know, what they need to know next, and how best to support their progress. As described in the Final Report of the evaluation of the strategy conducted by Australian Council for Educational Research (ACER) in 2003 and 2004, "the role of Specialist Teacher combines the roles of coach, mentor, expert consultant, teaching partner and locates significant expertise in schools" (Meiers et al., 2006, V.2, p. 134).

After completion of the extensive evaluation, involving surveys and case studies, the ACER researchers concluded that:

> …the concept of working shoulder to shoulder in classrooms, and in collaboratively identifying students' learning needs and planning activities that will move them forward,

> is central to the GiR-LNS. This collaborative work has enhanced the understandings, confidence and teaching skills of the Specialist Teachers and their colleagues. It has made a definite impact on the capacity of teachers to select, apply and develop diagnostic, formative and summative student assessment strategies and instruments so that they are now better able to focus on individual learning needs. (Meiers et al., 2006, vol. 1, p.124)

The numeracy stream of GiR-LNS is strongly focused on the *First Steps in Mathematics* (FSiM) research base, resources and professional learning program, which was described by Ingvarson (2005) as a high quality, research-based curriculum resource that teachers used “collaboratively to plan the school’s mathematics curriculum, to plan learning activities tailored to students in their classroom and to map development in their mathematical thinking” (p. 64).

The research and development phase of FSiM was funded by the WA Department of Education and Training and conducted by a team of primary classroom practitioners under the leadership of Professor Sue Willis at Murdoch University. The project team conducted an extensive review of national and international research literature and found gaps in the field of knowledge about the learning of mathematics. Tasks were designed to replicate and build on those in the research literature and were then used to interview students from diverse locations across the state. Through analysis of these data, the team identified a series of key mathematical understandings that were considered essential for the successful learning of mathematics throughout the primary years. These FSiM Key Understandings are intended to form the basis for classroom planning. They are written from a developmental perspective and provide advice about the emphasis needed at different stages of schooling. Included also are Sample Learning Activities and Lessons, as well as advice about the kinds of difficulties many students typically encounter (Willis & Treacy, 2004, p. iv). Another major outcome of the FSiM research was the production of Diagnostic Maps for Number, Measurement, Space and Chance and Data in which characteristic phases in the development of students’ mathematical thinking are described.

The development of the FSiM resources and the GiR-LNS numeracy professional learning program are strongly based on the view that successful and sustainable progress in the learning of mathematics requires a greater emphasis on depth of understanding and an ongoing insistence on students making sense of mathematics. There is a strong belief that “a focus on short-term performance and procedural knowledge at the expense of robust knowledge places students “at risk” of not continuing to progress throughout the years of schooling” (Willis & Treacy, 2004, p. 5). During the implementation of the GiR-LNS numeracy strategy, many Specialist Teachers and their classroom colleagues have stated that they have been challenged to re-examine their practices in the context of this belief.

A secondary perspective

Some years ago, when the first author was teaching in a secondary classroom prior to her engagement with FSiM and GiR-LNS, she worked hard to ensure that students who underachieved in mathematics were provided with the help she thought they needed to achieve success. For example, she successfully enabled most students to carry out conversions between metric units by displaying a wall chart showing the diagram shown in Figure 1.

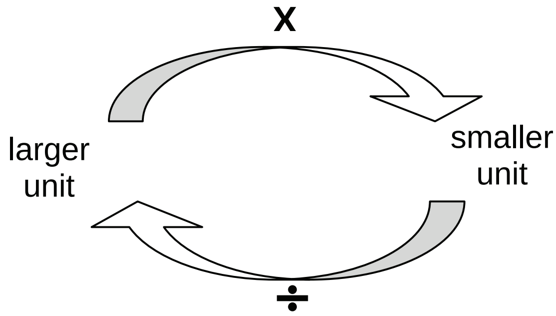


Figure 1. Diagram of unit relationships.

This author believed she was providing a useful summary and memory prompt for students who understood how to convert metric units and a helpful crutch for those she knew did not understand the mathematics of the procedure. Thus the author believed she was catering to the full range of abilities in her classroom by enabling all students to produce correct answers to relevant questions — whether or not they understood the relationships involved.

Following her work in the FSiM research and later use of the resources, the first author realised that by providing this diagram for her underachieving students, she was only supporting short-term success. She was not providing opportunities for many of the students to achieve the depth of understanding needed for ongoing progress. Furthermore, by providing what she thought was a supportive diagram she managed to mask the difference between those with a full understanding and those without.

To fully comprehend the conversion of metric units, students need a wide range of understandings. These include knowledge about the nature of units and part units, multiplicative relationships in the decimal number system, the connection between metric prefixes and decimal place value, as well as an operational understanding of multiplication and division. It became clear to the author that missing any one of these complex ideas would result in the students failing to make sense of metric conversions.

By making it easy for students to obtain a correct answer “for the wrong reasons,” that is, without making sense of the mathematics, many students are excluded from the opportunity to access mathematics in the future. Specialist teachers working in the GiR-LNS strategy consistently find that, by secondary school, underachieving students have often lost connection with the meaning of the mathematics they are expected to work with. The first author has observed that many secondary teachers attempt to explain new ideas to their students using language associated with earlier, seemingly simpler concepts. However, underachieving students typically do not understand these earlier concepts, so cannot make sense of their teacher’s explanations of the new ideas. As a result of working in the GiR-LNS numeracy strategy, many secondary teachers have recognised the necessity of changing their thinking about what is required in order to accelerate the learning of their underachieving students.

Getting at the source of difficulties

Fluent users of mathematics often find it difficult to deconstruct their mathematical ideas in order to make concepts accessible to struggling students. For example, when solving the following problem, secondary mathematics teachers working with the first author immediately recognised that addition was required.

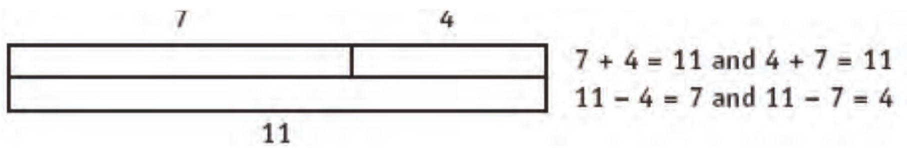
> *James went shopping. He spent $68.35, and was left with $181.65. How much did he have to begin with?*

However, many of their underachieving students were influenced by the "take away" actions described in the situation itself and so chose to subtract. The teachers recognised that the students who chose to subtract instead of add in such contexts were getting their cues from words such as "spent" and "left." However, the same teachers often had difficulty explaining the reason for choosing addition or subtraction to students who persistently made such errors.

Primary teachers in *Getting it Right* schools, or those who have accessed the *First Steps in Mathematics* professional learning, will often say that they thought they had been teaching operational understandings in the past, but now realise they had inadvertently focused only on calculating. The Key Understandings for Understand Operations in the FSiM resources are explicit about the important operational concepts students need in order to ensure future success. The descriptions are designed to help teachers to know when and how concepts should be introduced, as well as to anticipate the difficulties students might encounter.

Key Understanding number Two in the Understand Operations chapter of the Number books states: "Partitioning numbers into part-part-whole helps us relate addition and subtraction and understand their properties" (Willis et al., 2004, p.11). It is this information that most adults unconsciously use to decide whether solving a problem requires addition or subtraction. The following information included with this Key Understanding expands on these important ideas.

> A quantity, while being thought of as a whole, can also be thought of as composed of parts. That is:



> The part-part-whole relationship shows how addition and subtraction are related, with subtraction being the inverse of addition. If the whole quantity is unknown, addition is required. If one of the other quantities is unknown, subtraction is required. This enables students to see why a problem that they think of as about adding, but with one of the addends unknown, could be solved by subtracting or vice-versa… Linking the joining and separating of the parts that make the whole to a variety of situations also helps students to see why subtraction can be used to solve a take-away problem and also a comparison problem…
>
> The part-part-whole relationship is also the key to students seeing why addition is commutative and why subtraction is not. The commutativity of addition is of obvious practical use in calculating, but knowing that, and understanding why, addition is commutative and subtraction is not, helps students represent word problems with appropriate addition and subtraction sentences. (Willis et al., 2004, Chapter 3, p.20).

Students who have developed these ideas effectively in the early years are able to distinguish the whole from the parts in problems such as James and his shopping, and in other additive situations where different parts of the problem are unknown. They find it obvious that addition is required when the whole is unknown, while subtraction is required when a part is unknown. For example, after careful development of the ideas over time, Year 3 students in the second author's classroom were given "problems without numbers" and blank part–part–whole diagrams on which to represent the structure of the problems in words. They were also asked to write a "number sentence" that would enable the problem to be solved if numbers had been included. The following example is typical of the results and shows how some students even began to use letters to represent the unknowns.

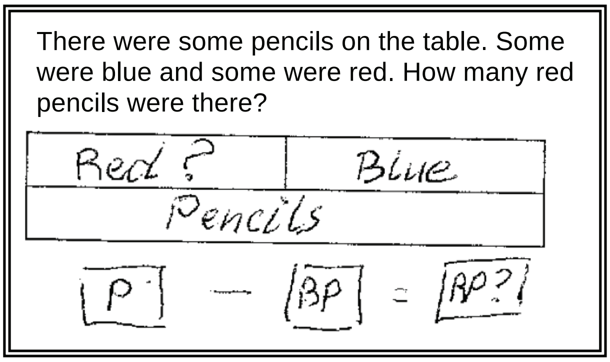


Figure 2. Example of a Year 3 student's response to "Problems without numbers."

This Key Understanding is fundamental to making sense of the inverse relationship between addition and subtraction and its later use to solve more abstract algebraic problems. While the ideas can be developed very early, numeracy Specialist Teachers in GiR-LNS schools found that students needed a great deal of experience interpreting and representing the full range of additive problem types. They first needed to use drawings, materials and physical actions, before representing problems semantically using non-standard equations (e.g., $\square$ – 65.35 = 181.65 is a semantic representation of James' shopping problem mentioned at the beginning of this section). Teachers found that students initially needed to use grid paper to construct part-part-whole diagrams before they could understand the nature of the model itself and realise that length is used to represent the equality of the combined parts with the whole. Teachers also found that because the careful and explicit development of operational understandings at this deeper level had typically been overlooked throughout schooling, students at every year level need to go through a similar process when first introduced to these operational ideas.

Experience in the GiR-LNS secondary numeracy pilot has confirmed that secondary students who had not developed these ideas for themselves also needed careful introduction and development of the concepts. What is particularly powerful for secondary teachers using the FSiM Key Understandings is that the language used to describe the concepts in the books provides a model for explaining the ideas to students in Year Eight and beyond. The ideas themselves are sophisticated and powerful, but the language used to describe the core concepts is straightforward and does not assume specialised pre-knowledge. In the first author's experience, the learning of many underachieving secondary students can be accelerated using these careful, simple language explanations and demonstrations that do not assume levels of understanding they do not yet possess.

Digging deeper

When working with some of the lowest achieving secondary students, it became clear that some students could not make sense of the explanations and demonstrations of part-part-whole relationships when larger numbers were involved. Again, the FSiM resources provide some insight into why this is so.

The FSiM Diagnostic Maps describe the major progressions in thinking that underpin a diverse range of competencies at each phase of development. There are typical age ranges attached to the phases, which enable teachers to consider whether or not their students are progressing alongside their peers at an acceptable rate, whether or not their students are in a position to benefit from planned learning experiences, and also what interventions might be needed and when. Thus students who are through the *Quantifying* phase (typically between six and nine years of age) can manipulate small numbers competently when the numbers refer to quantities or collections they can visualise and reason about. However, students are through the *Partitioning* phase (typically between nine and eleven years of age) before they fully understand that numbers can be thought about as mathematical objects with their own meanings that can be partitioned and manipulated out of the context of any real world quantity or collection. To quote from the Diagnostic Map: Number, during the *Partitioning* phase:

> Students come to see the significance of whole numbers having their own meaning independent of particular countable objects. They learn to use part-whole reasoning without needing to see or visualise physical collections.
> As a result, students see that numbers have magnitudes in relation to each other, can interpret any whole number as composed of two or more other numbers and see the relationship between different types of addition and subtraction situations (Willis et al., 2004, Diagnostic Map: Number, foldout)

The *First Steps in Mathematics* research revealed that by the age of eleven the majority of students were through the *Partitioning* phase, and in the *Factoring* phase, during which students "extend their additive ideas about whole numbers to include the coordination of two factors needed for multiplicative thinking" (Willis et al., 2004, Diagnostic Map: Number, foldout). However, it is clear from interviews with students in GiR-LNS schools that some secondary students still cannot make sense of numbers unless they are attached to physical collections. In the GiR-LNS professional learning program, a great deal of attention is given to developing the ability of teachers to ask high quality focus questions that reveal the depth of students' understanding of numbers.

Many students appear to understand the meaning of numbers because they have learned to "say the right words," but a well-placed question can reveal that this assumption is often flawed. For example, the second author asked a year four student to explain how four MAB tens rods meant 40. The student pointed to each ten rod and demonstrated saying, "It's ten, twenty, thirty, forty, it's four tens and you write it four zero." When she was then asked how many small cubes there would be if the four tens rods were sawn along the lines to make them into small cubes, she replied, “I don’t know, I’d have to do it then count them.” When asked to pretend they were cut up and told to count them, she counted by ones going backwards and forwards across the four rods. She was most surprised when she reached 40 and said, “Oh, it’s the same number!”

Older students can have similar disconnected ideas about the meaning of numbers, but these may not be exposed during day to day classroom lessons unless teachers know to ask appropriate probing questions. For example, a year seven student had just completed an exercise requiring her to record the value of underlined digits in six-digit numbers. One item was 4<u>6</u>5 271 and she had correctly written 60 000. When the second author asked her to explain why that number was chosen, the student confidently explained that the six was in the “tens of thousands” column, and “six tens are sixty” so it had to be sixty thousand. Many teachers would accept this as evidence that the student understood place value into the hundreds of thousands, However, the student was then asked, “What if you had four hundred and sixty five thousand, two hundred and seventy one dollars in the bank and you withdrew sixty thousand dollars to buy a very expensive car, how much would you have left?” The student replied without hesitation, “Forty five thousand, two hundred and seventy one dollars.”

By the secondary years, it is usually assumed that students are making sense of the numbers they are working with. This assumption can be misplaced. For example, a group of Year 8 students correctly completed a task using a rounding rule to round numbers to a given number of decimal places. However, by asking a simple question, the first author demonstrated that while the students were able to successfully complete the rounding task, they did not understand the meaning of the decimal numbers. When asked how they would give her exactly “seven point four apples,” given a sharp knife and a pile of apples, students’ responses included:

“Seven apples and four pieces. There’s four pieces in an apple so that’s another apple, so that’s eight apples.”

“I’d get an apple and chop it into seven bits and give you four.”

“I’d give you seven and a quarter apples.”

“That’s seven and four quarters.”

Feedback from teachers in the GiR-LNS project has confirmed that it is easy to incorrectly assume that teacher and student share the same meanings for numbers and mathematical terminology used in the classroom. Consequently, misconceptions and faulty premises can go unchallenged and important learning opportunities can be lost. In contrast, teachers in the project have since reported that once they began to probe students’ mathematical ideas more deeply and reveal the gaps, it was relatively easy to move them forward.

Multiplicative reasoning

It is through the use of the FSiM resources that the authors’ have come to appreciate the complexity involved in developing students’ additive reasoning that connects their understanding of numbers as independent representations of quantity with part–part–whole modelling of situations and the inverse relationship between addition and subtraction. While many more students are being provided with opportunities to develop deeper understanding of these ideas during the primary years, the learning of these core concepts is ongoing and some students will continue to need help to develop additive reasoning into the secondary years.

Given that the development of additive reasoning is complex, requires explicit teaching, and is dependent on students developing sound foundational ideas about number and quantity, the development of multiplicative reasoning is, not surprisingly, even more problematic.

In the *First Steps in Mathematics* Diagnostic Map: Number, the *Factoring* phase describes the ideas students need in order to reason multiplicatively. It explicitly refers to the use of an array structure as part of this development.

> [Students] learn to construct and coordinate groups of equal size, numbers of groups and a total amount. Students also learn to visualise multiplicative situations in terms of a quantity arranged in rows and columns (an array).

Just as the part–part–whole model can be used to develop additive reasoning, so can the array model be used to illustrate the connections between different multiplicative situations as well as the inverse relationship between multiplication and division. However, as with additive thinking, students must first connect the numbers with the quantities the numbers represent if they are to make sense of the mathematics. In the GiR-LNS numeracy project, grid paper is often used for this purpose. Finer grid paper lends itself to the representation of very large numbers. For example, a square metre created from millimetre grid paper can be used to highlight multiplicative place value relationships (Willis et al., 2004, pp. 82–84).

Grid paper models can also be used to help students make sense of mental calculation strategies which involve the manipulation of factors to make calculations easier. Arrays can be partitioned and rearranged to illustrate how some numbers can be factorised and the factors rearranged without altering the total quantity (see Figure 3).

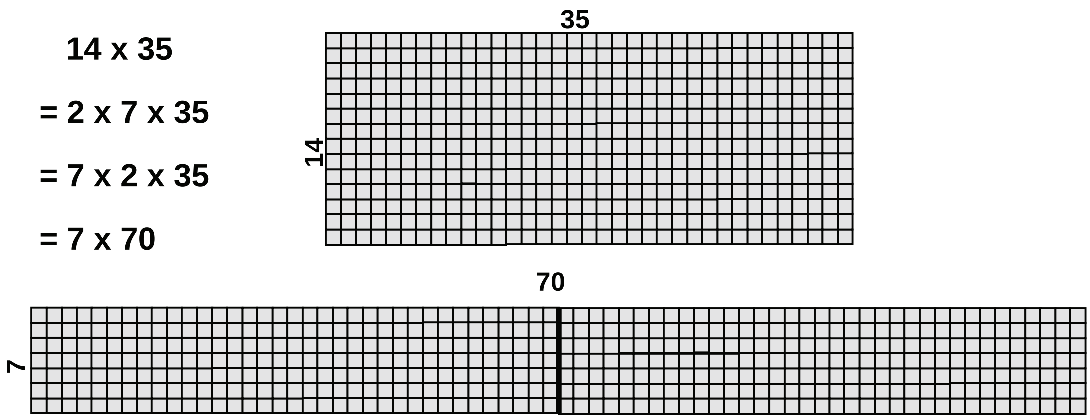


Figure 3. Doubling and halving factors.

When presenting a lesson in which students needed to multiply two two-digit numbers, the first author gave students the choice of using an abstract diagram on blank paper that showed how the numbers could be helpfully partitioned (see Figure 4) or grid paper that illustrated the physical quantities involved as an array (see Figure 5).

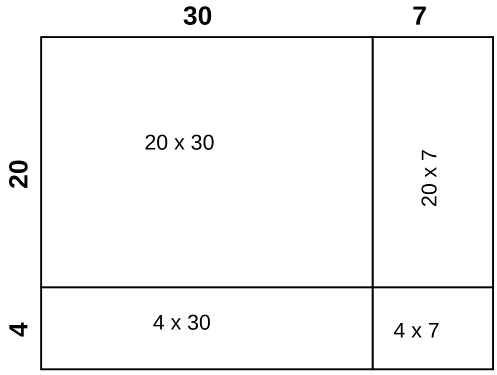


Figure 4. Abstract multiplication diagram.

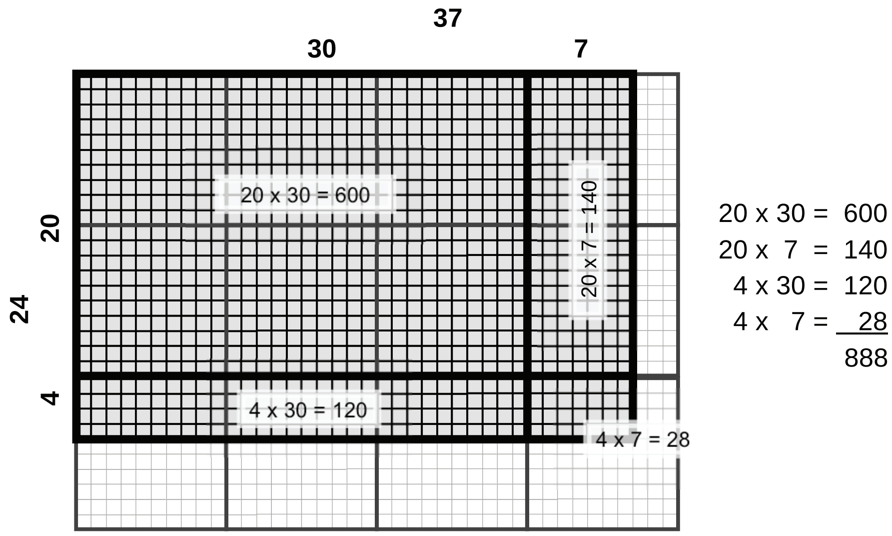


Figure 5. Grid paper array model showing physical quantities.

For many students the grid paper array enabled them to make sense of the quantities involved in the calculations. This had such an impact on one student that he sought out the first author in his lunch hour and insisted she teach him everything she could about mathematics in the time available. He was openly annoyed that no one had ever helped him multiply by talking to him about this or showing him how to "break up" numbers in this way.

Conclusion

The *First Steps in Mathematics* resources, while originally written for generalist primary teachers, are highly valued by secondary mathematics teachers in the *Getting it Right* secondary numeracy project. The Key Understandings provide explicit advice about the development of the important concepts required by students to be successful learners of mathematics. To exemplify the usefulness of the Key Understandings, this paper has focused on the development of operational understanding and, in particular, the inverse relationship between addition and subtraction.

The *First Steps in Mathematics* Diagnostic Maps identify the major phases of development that students pass through as they learn mathematics. Teachers in the *Getting it Right* strategy often admitted that they had not previously realised that their upper primary and lower secondary students may not have understood the meaning of larger numbers when they were not attached to a real world quantities or collections. The *Partitioning* phase in the Diagnostic Map: Number describes the ability to make sense of de-contextualised numbers as a significant milestone in students' mathematical development. The *Getting it Right* numeracy strategy and professional learning program supports teachers to recognise the importance of developing a strong number/quantity sense in their students. Developing this understanding is fundamental to the successful development of students' additive and multiplicative reasoning.

Teachers using the *First Steps in Mathematics* resources have found that underachieving students who would otherwise be excluded from making further progress are able to engage with the mathematics and regain ownership of their developing ideas.

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FOSTERING MATHEMATICAL CREATIVITY AND UNDERSTANDING

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On one hand, teachers hope to build robust connected conceptual understanding, and on the other hand develop procedural fluency especially with computation. This presentation draws on examples from practice to illustrate how these two dimensions of learning can be mutually supporting, and not be seen as exclusive options. A particular focus will be on ways that teachers can ensure the appropriate mathematics development of the best students, while assisting low achieving students to achieve their mathematical goals.

Introduction

The following discussion seeks to align considerations associated with the forthcoming national mathematics curriculum with approaches to teaching mathematics effectively to heterogeneous groups of students. It first presents some of the challenges included in the framing paper (National Curriculum Board, 2009), introduces some terms and emphases associated with planning and teaching mathematics, and then discusses the way that tasks can be used to engage students at all levels of readiness.

Challenges within the national mathematics curriculum

The national mathematics curriculum that is currently in development poses a number of challenges not only for the curriculum writers but also for teachers. In the overview of the framing paper, among other things, the following proposals are put:

- some students are currently excluded from effective mathematics study, and the curriculum and school structure should seek to overcome this;
- all aspects of the curriculum will be clearly and succinctly described;
- advanced students can be appropriately extended using challenging problems within current topics;
- specific aspects within three content strands will be described for each phase of the curriculum, and for each of those expectations for four proficiency strands will be delineated, along with the details of numeracy.

The following discussion presents a way that each of these can be addressed. The examples are chosen from a range of levels, and each address the Number and Algebra strand of the curriculum. The discussion also incorporates some of the terms that are used in the framing paper that describe the actions associated with mathematics learning. These actions, adapted from Kilpatrick, Swafford and Findell (2001), are presented in the framing paper as:

Understanding includes building robust, adaptable and transferable mathematical concepts, the making of connections between related concepts, the confidence to use the familiar to develop new ideas, and the "why" as well as the "how" of mathematics;

Fluency includes skill in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently, and appropriately, and, in addition, recalling factual knowledge and concepts readily;

Problem solving includes the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively; and

Reasoning includes the capacity for logical thought and actions such as analysing, proving, evaluating, explaining, inferring, justifying, and generalising.

A model for teaching in mixed ability classes

It is relevant for teachers to consider the implications of the challenges and the specification of the proficiency expectations of the curriculum. One possible model that could be used for planning and teaching was developed by Sullivan, Mousley and Zevenbergen (2004) after conducting research on ways to overcome barriers to learning mathematics experienced by some students. They described elements that they claim can be part of effective whole class heterogeneous mathematics teaching. They argued that teaching should include five elements, three of which are directly relevant here. These are:

- *Tasks and their sequence*
  Sullivan et al. (2004) argued that engaging tasks that are effectively incorporated into a sequential development of learning resulted in improved students' engagement, as evidenced by time on task, participation in discussions, and increase in successful completion of the teaching and learning activities focusing on mathematical problems.
- *Enabling prompts*
  Sullivan et al. (2004) suggested that enabling prompts can allow students experiencing difficulty to engage in active experiences related to the initial problem. Such prompts can involve slightly lowering an aspect of the task demand, such as the form of representation, the size of the number, or the number of steps, so that a student experiencing difficult can proceed at that new level, and then if successful can proceed with the original task. This approach can be contrasted with the more common requirement that such students (a) listen to additional explanations; or (b) pursue goals substantially different from the rest of the class, or (c) use materials in ways that reduce learning opportunities.
- *Extending prompts*
  It was also proposed that students who complete planned tasks quickly can be posed supplementary tasks or questions that extend their thinking and activity. Extending prompts have proved effective in keeping higher-achieving students profitably engaged and supporting their development of higher-level, generalisable understandings.

These three aspects connect to the above curriculum goals in some important ways. In particular, they emphasise the importance of teachers posing tasks that are engaging for all students, that are accessible by most students, and that have potential for meaningful extension for those students who may be ready for it. It goes without saying that this engagement should be with the important topics of the curriculum, and that the tasks should foster each of the aspects of the proficiency expectations. The following presents some illustrative activities and explains the way that the tasks can foster the desired actions.

Some illustrative activities

The following four examples of tasks are drawn from the proposed Number and Algebra strand, and are chosen to exemplify the ways that teachers can address some of the challenges posed in the national curriculum framing paper.

Example 1: A counting task (suitable for school entry students)

Compare these two tasks:

(a) Count the number of letters in the word "Peter."
(b) What is there in this room that there is exactly five of?

Essentially both tasks address the same content. To answer task (a), students use 1–1 correspondence to match the numbers with the letters, and then state the final counting number. While this task does give a guide to the level of fluency, and perhaps of understanding, it does not address problem solving and reasoning. In the case of task (b), to answer the question students need to scour the room for objects that can be described, they need not only to count the objects one to one, but also classify and describe the objects. The check on fluency is present as in the other question, but the responses give insights into understanding (since there is more chance of error), the students have to solve the problem of finding objects, and then have to explain and justify their answers. It is engaging for the students in that they have to make choices. For students who experience difficulty, it may be possible to give prompts such as, "How many windows are there?" "Go to the tub and count out five of something." Students who give some responses quickly might be asked to find other possible answers, or to find something of which there are six. As an aside, a lesson based on task (b) is likely to be more confusing for students and slightly more chaotic for the teacher. Yet this classroom chaos and student confusion may be good things. At a class in the Kimberleys, at the first try the students were confused and gathered a range of inappropriate responses. The next time, though, even without any direction, students came back with responses like "five senses," "five days in the school week," and "five fingers."

Example 2: An open-ended number exploration that can be used to introduce the idea of equations with unknowns

Often students can be posed the following task:

> For the equation $3a + 2b = 70$, what might be the values of a and b?

Some introductory equation exercises are quite routine and even confusing for students in that they can see the answer without any manipulation. In this case, there are many possibilities for a and b, and in searching for those possibilities the students not only explore that nature of a variable, but also the patterns that arise in the responses. There is a problem solving dimension in that students must find a way of guessing and checking, and a reasoning element in that students must record responses systematically, and explain the processes by which they found the solution, including how they might move from one solution to another. The choice in approach and in the numbers is engaging for students. Students experiencing difficulty can be posed tasks like: "Make up some sums with at least two numbers, one of which is 3 and one of which is 2, that have the result 70;" or, "What might be the missing numbers in $3 \times c + 2 \times c = 70$?"

Students who finish this task quickly can be asked questions like "For the equation $3a + 2b + 4c = 70$, what might be the values of a, b, and c?"

Example 3: A game for introducing like terms — Race to $5x + 5y$

Players take turns to add either 1 or 2 to the previous total to race to 10. Assuming that the players start at zero, a possible sequence could be as follows:

Player One	1		4		6		8	
Player Two		*3*		*5*		*7*		*10*

The interesting part of this game is that there is a winning strategy and students quickly see that there is. This leads on to a range of other variations, one of which might be Race to $5x + 5y$ counting on x, y, or $x + y$. A possible sequence could be as follows:

Player One	x		$2x + 2y$		$3x + 4y$		$4x + 5y$	
Player Two		$x + y$		$2x + 3y$		$3x + 5y$		$5x + 5y$

There is problem solving in the search for the winning strategy, reasoning in the extension of the strategy to other contexts, and in the explication of the strategy, and the game is developing fluency with adding like terms. The low risk nature of the game is engaging, and it can be easily adapted for students experiencing difficulty by having them play simpler games such as Race to $10x$. It can be extended by playing Race to $5x + 5y + 5$, as before except with the extra option of adding 1 or 2.

Example 4: Selecting the right operations between expressions

The following is an example of a mathematical puzzle, adapted from a suggestion by Swan (no date). The puzzle involves a set of rectangular cards on which there are mathematical expressions and a set of arrow cards on which there are operations. Figure 1 shows examples of the types of mathematical expressions that might form the basis of the puzzle:

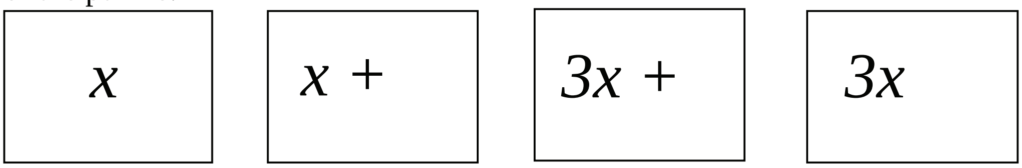

Figure 1. Examples of expression cards.

The following are examples of the operation cards.

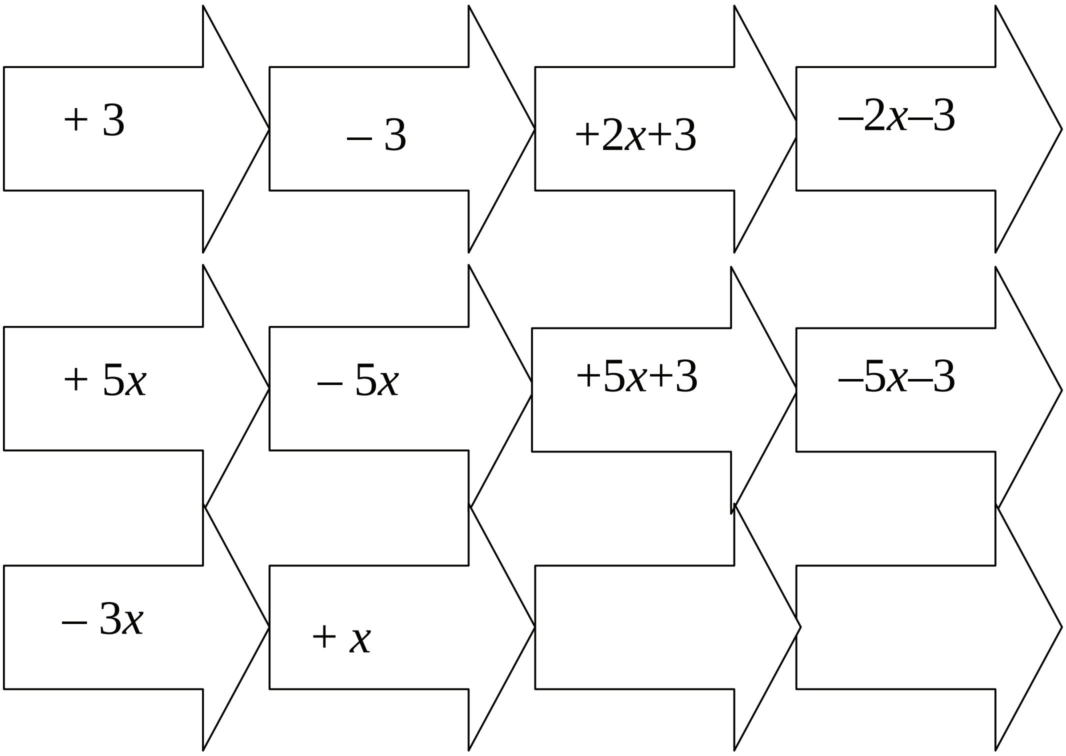

Figure 2. Examples of operation cards.

The goal is to use all the operation cards to represent the relationships between the expression cards. So, for example, to connect the x and $3x + 3$ expression cards, we have to hunt for the relevant operation card ($+ 2x + 3$). Once all the operation cards are placed, students write on the blank operation cards the relationships that are not already shown.

The activity builds understanding because it requires students to search from among a range of possibilities for the correct operation, and especially if this is done in groups, there is a requirement to communicate about the choice of operation which again requires reasoning. The game type format is engaging for students, and it can be easily adapted for students experiencing difficulty by reducing the number of cards involved, and those who complete it can be given a more demanding set of cards. There is potential for further engaging the students by asking them to create their own set of expression and operation cards.

Some pedagogical considerations

Each of these examples require pedagogies which are different from conventional approaches to mathematics teaching to maximise their effectiveness. For example, the following might be the key actions that a teacher takes for Example 2 above:

- Teacher poses and clarifies the purpose and goals of the task. If necessary, the possibility of multiple responses can be discussed.
- Students work individually, initially, with the possibility of some group work. Based on students' responses to the task, the teacher poses variations.
  - The variations may have been anticipated and planned, or they might be created during the lesson in response to a particular identified need.
  - The variations might be a further challenge for some, with some additional scaffolding for students finding the initial task difficult.
- The teacher leads a discussion of the responses to the initial task. Students, chosen because of their potential to elaborate key mathematical issues, can be invited to report the outcomes of their own additional explorations.
- The teacher finally summarises the main mathematical ideas.

This highlights that the challenges posed by the curriculum are not just for the writers and resources developers, but also for teachers. Some of these challenges may require changes in practice for some.

Conclusion

The challenges in the national curriculum can be described briefly as the need to include more students in mathematics study for longer, the need to extend the best students without rushing from topic to topic, and the need to broaden the range of actions in which students engage as part of their mathematics learning. It is hoped that the curriculum writers will find ways to accommodate those challenges. It is also hoped that teachers can act to increase the chances of successful participation by posing tasks that students find interesting, that allow ready adaptation for students experiencing difficulty and for those who finish quickly, and which allow a range of mathematical actions by students. It is noted that one of the more controversial aspects of the framing paper is the issue of ensuring the best students are extended appropriately. While merely rushing from one topic to another does engage the bright students it does not extend them. The task examples presented above have the potential to do this, especially if accompanied by the associated pedagogical actions by teachers.

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PEER-REVIEWED PAPERS

USING NAPLAN ITEMS TO DEVELOP STUDENTS' THINKING SKILLS AND BUILD CONFIDENCE

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National testing programs provide challenges and opportunities for mathematics teachers. One challenge is to focus on the diverse learning needs of students while preparing them for national testing early in the school year. An opportunity arises if we use test items to assist students who have difficulty reading and interpreting mathematical text, to further develop students' thinking skills, and to analyse common errors and misconceptions, frequently presented as alternative solutions in multiple-choice items. One approach to "teaching to the test" is to use NAPLAN items as discussion starters so that students develop number sense, adopt new problem-solving strategies, and build confidence and resilience.

Background information

In Australia, the debate surrounding mathematics and numeracy achievement has been similar to that experienced elsewhere. There is a growing recognition of the need for greater proficiency and that early intervention provides the best chance of success for children at risk of failure. Until recently, each state and territory in Australia collected student achievement data for the federal government. Concern about the proportion of students not meeting the minimum national benchmark standards (Curriculum Corporation, 2000) has continued with large investments by governments to address the needs of students at risk.

To better monitor student achievement across Australia the *National Assessment Program in Literacy and Numeracy* (NAPLAN) was introduced in 2008. The same tests in literacy and numeracy are now administered throughout Australia to all students in Years 3, 5, 7 and 9. In Years 7 and 9 students complete two 32-item test papers for numeracy, one without the use of a calculator. Testing early in the school year provides diagnostic information to teachers about their students' performance in mathematics topics common to all states and territories as outlined in the *Statements of Learning for Mathematics* (Curriculum Corporation, 2006). The results from the assessments are reported in several forms including individual student reports to parents, school and aggregate reports. For more information about NAPLAN see the website http://www.naplan.edu.au.

The school reports enable teachers to analyse the results for each year group to determine which items appear to be understood and which are problematic. In addition, school data can be compared to the Australian student data. The information is useful to assist addressing common errors and misconceptions as well as to aid planning and programming of future learning.

Whether we approve of a national testing regime or not, this level of accountability to the federal government is in place for the foreseeable future with pressure on school principals and teachers to improve results. While the information may be useful after the results are released, teachers of Years 3, 5, 7 and 9 are experiencing increased pressure early in the school year to prepare students for the test. Principals, school systems personnel and parents are scrutinising the results to determine whether schools and their teachers are "measuring up." They want students to be well prepared and to achieve above minimum standards.

Rather than abandon good pedagogical practices and have students individually practise released test items under timed conditions, NAPLAN provides an opportunity to use quality-teaching approaches for test preparation. In this paper, strategies are presented to assist reading and interpreting mathematical text, to promote thinking strategies and number sense, and to raise awareness of common errors and misconceptions. Examples from NAPLAN tests are used for illustrative purposes with reference to relevant research.

Reading and interpreting test items

Teachers may assume that incorrect answers are the result of errors in applying mathematical procedures or lack of understanding. However, many students experience difficulty reading and comprehending test items before they begin to apply mathematical skills and processes. Presenting questions in context adds considerably to the information students need to read and interpret. Many items on NAPLAN require careful reading – Figure 1 provides one example from the sample items for Year 9 available on the NAPLAN website (http://www.naplan.edu.au).

> Jane, Meg and Oliver scored 1400 points altogether on a computer game.
>
> Jane has 50 more points than Meg and Meg has 4 times as many points as Oliver.
>
> How many points does Oliver have?

Figure 1. From Year 9 Numeracy calculator-allowed sample test (www.naplan.edu.au/verve/_resources/NAP08_num_y9_calc_web_v10.pdf).

To identify errors in answering word problems, Newman (1983) interviewed and analysed students' solution attempts. From this, she developed *Newman's Error Analysis* of five levels of difficulty (Table 1). Most errors occurred in the second and third levels of "comprehending" and "transforming" the text into an appropriate mathematical strategy. Since Newman's research, the results have been replicated in other studies with Clements and Ellerton (1992) reporting 70% of students having difficulty at some stage in the first three levels. If teachers continue to prepare students for NAPLAN by providing practise using de-contextualised questions requiring the application of a procedure, they will have missed an opportunity to support the development of skills associated with reading and comprehending mathematical text.

Table 1. Levels in Newman's Error Analysis

1. Reading the question	Reading
2. Comprehending what is read	Comprehending
3. Transforming the words into an appropriate mathematical strategy	Transforming
4. Applying the mathematical process skills	Processing
5. Encoding the answer into an acceptable form	Encoding

By translating each of the levels from Table 1 into a question for students, teachers are able to determine the first level of difficulty (White, 2005). A possible set of questions is:

- Read the question to me. If you don't know a word, tell me.
- In your own words, tell me what the question has asked you to do.
- Now tell me what method you will use to find your answer (and work out your answer on your paper.)
- Go over each step in your working and tell me what you were thinking.
- What is the answer to the question? Are you able to use another strategy to check this answer is correct?

Using this protocol in an interview situation provides valuable information about individual students although it can be adapted for whole class discussion or students could use the protocol in pairs.

Mathematical text is lexically dense so each word needs to be read carefully and analysed for meaning. It may be a challenge to convince students to do this if they are anxious or worried about completing the test in the allocated time. However, to interpret and make meaning of mathematical text, it is usually necessary to focus on prepositions as well as the order of words and their relation to one another (Dawe, 1995; MacGregor & Moore, 1991). One approach to developing knowledge of the structure of mathematical text is to have students write in mathematics lessons using "impromptu writing prompts" (Miller, 1992), create their own word problems (English, 1997), or develop self-constructed test items (Clarke, 1997). This writer's experience of using self-constructed items is that students enjoyed creating challenging questions particularly when it is suggested they might be used to compile a class test paper. Additional suggestions for reading and writing tasks are described in Wood's (1992) article.

Using thinking strategies

Teachers frequently witness instances in which students write down the first answer they obtain, or record an answer that appears on a calculator without considering whether it makes sense. Encouraging students to think about questions and apply reasoning about number to evaluate answers can be a challenge. One way to support the development of students' thinking strategies is to use test items that focus on mental computation, estimation and number sense (McIntosh, Reys & Reys, 1997). McIntosh et al. (1997, p. 322) describe number sense as: "a person's general understanding of numbers and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful and efficient strategies for dealing with numbers and operations."

Many NAPLAN items require an understanding of operations and place value as well as knowledge about rational number (e.g., Figure 2).

What is the missing number?

$15.0 \times 0.4 =$ ____?____

A) 60 B) 6 C) 0.6 D) 0.06

Figure 2. From Year 9 Numeracy non-calculator sample test (www.naplan.edu.au/verve/_resources/NAP08_num_y9_calc_web_v10.pdf).

While students are frequently reluctant to estimate, this is an important first step. Options in multiple-choice items may be eliminated after considering whether the solutions are reasonable. After students have estimated the answer, pose questions such as the following:

- What strategies could you use to check the solution?
- What would the question need to be to obtain each of the alternative answers?
- What happens when you multiply a whole number by a number less than one?

Many errors occur in items about percentages (see Figure 3). Multiple-choice items typically include common errors and misconceptions as alternative solutions. Pointing this out to students highlights the need for then to take more time to think about their solutions and to consider ways to verify their approach. Questions for discussion about the item in Figure 3 could be:

- How much is the decrease in electricity bill?
- Is the decrease less than or greater than 50% of the first electricity bill?
- What are the fraction equivalents for 25% and 33%? How much would the discount be for each of these?

Hugo's electricity bill was $180 last month. This month it is $135.

What percentage decrease is this?

A) 25% B) 33% C) 45% D) 55%

Figure 3. From 2008 Year 7 Numeracy calculator allowed test (http://www.naplan.edu.au/verve/_resources/nap08_num_y7_calc_netversion.pdf)

After estimating the answer for the NAPLAN item in Figure 4, students should be able to eliminate several options. Teacher and students can discuss the errors students would have to make to obtain the incorrect alternatives offered. Further discussion about errors and misconceptions is presented in the next section of this paper.

5427 ÷ 9 = _?_

A) 63 B) 603 C) 630 D) 6003

Figure 4. From 2008 Year 5 Numeracy test (http://www.naplan.edu.au/verve/_resources/NAP08_num_y5_netversion.pdf).

Errors and misconceptions

Students' errors in mathematics may be caused by many factors. Poor comprehension, language difficulties, anxiety, rushing and carelessness can all lead to errors in completing tasks (Anderson, 1996). However systematic errors are usually a consequence of misconceptions. Ryan and Williams (2007) surveyed 15 000 students aged 4 to 15 years in the United Kingdom to identify errors and misconceptions in key mathematical topics. While there were errors due to "slips" and other uncertainties, the researchers identified four main development errors caused from "modelling," "prototyping," "overgeneralising," and "process-product linking." The researchers argue these types of errors "should be valued by learners and teachers alike" (p. 13) since they provide focused learning opportunities. A summary of each type is presented here with examples of NAPLAN items that could be used as a stimulus for class discussion.

Modelling

The first error type involves students attempting to connect school mathematics questions to their experiences in the "real" world. If questions are de-contextualised and abstract, students may attempt to relate them to an everyday context and make an incorrect response. For example, when students are confronted with the question $6 \div 1/2$ and they write 3, they are probably thinking "divide 6 in half" rather than "divide 6 by a half." When teachers rephrase the question as "how many halves are there in 6?" students can usually answer correctly.

Teachers frequently use models to support students' learning but all models have limitations, as do the models used by children to make sense of school mathematics. Ryan and Williams (2007) suggest errors in students' use of models (or contexts, or metaphors) provide learning opportunities when identified by teachers. For the example provided above, teachers are encouraged to ask students:

- What did you think about when you read and attempted the question?
- How would you rephrase the question to help someone who could not do it?
- Rewrite this question so that the answer is 3.

Prototyping

The second error type arises when students use "typical examples" frequently presented to them by teachers or in textbooks. Two examples are presented here for shape identification and reading scales.

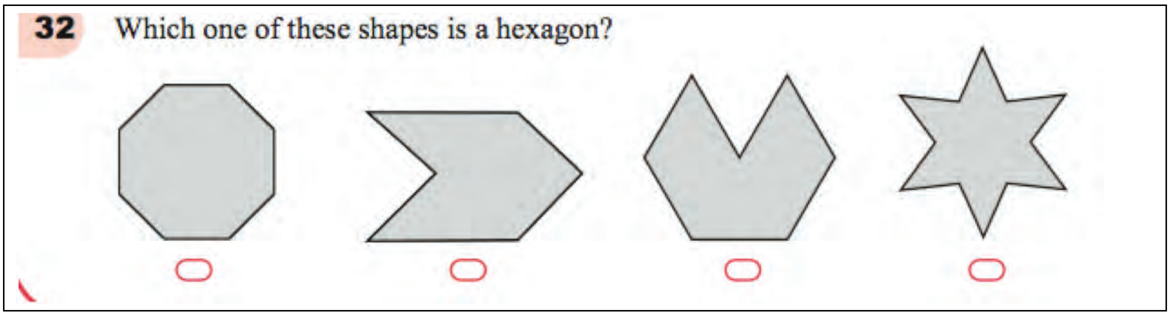

Figure 5. *From 2008 Year 5 Numeracy test (http://www.naplan.edu.au/verve/_resources/NAP08_num_y5_netversion.pdf).*

Figure 5 includes one prototypical shape as the first alternative that could lead to students nominating this alternative rather than checking the number of sides. Teachers are encouraged to present shapes in a range of orientations as well as present counterexamples for student discussion.

The second example arises when students practise reading scales marked in units, tens or multiples of ten more often than other types of scales such as those with divisions of 2, 250 or 0.25. For the NAPLAN example in Figure 6, many students misread the scale.

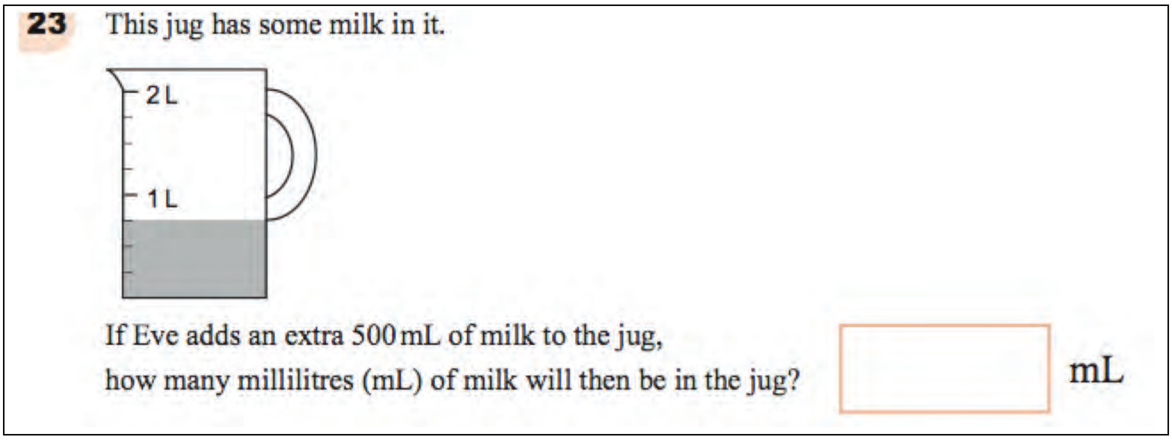

Figure 6. *From 2008 Year 7 Numeracy non-calculator test (http://www.naplan.edu.au/verve/_resources/nap08_num_y7_calc_netversion.pdf).*

Overgeneralising

Ryan and Williams (2007, p. 23) use the term "intelligent overgeneralisation" to refer to students' predisposition to create rules based on experiences. At a particular stage in number learning, children are inclined to use the generalisation "multiplication makes bigger" since this fits the types of questions they are doing. However, this generalisation breaks down when numbers less than one are introduced. Other common generalisations requiring discussion include:

- division makes smaller;
- division is necessarily of a bigger number by a smaller number;
- longer numbers are always greater in value.

Figure 7 presents a typical test item where students could achieve the correct answer from estimation. However, if they are relying on the "division makes smaller" generalisation, they will not choose the correct alternative.

What is the answer to 6.6 ÷ 0.3? A) 0.022 B) 0.22 C) 2.2 D) 22

Figure 7. From 2008 Year 7 Numeracy non-calculator test (http://www.naplan.edu.au/verve/_resources/nap08_num_y7_calc_netversion.pdf).

Process-product

The final error type involves understanding the connection between a mathematical process and the product, which is the outcome of the process. For example, 3 + 5 = ? is interpreted as carrying out the process of adding 3 and 5. However the number sentence 3 + 5 = 8 is a number sentence requiring an understanding that "3 + 5 is the same as 8" or "8 is the same as 3 + 5" (Ryan & Williams, 2007, p. 25). Other examples of this error involve students interpreting travel graphs as pictures of walking up or down a hill rather than as indicating a relationship between distance and time.

Cautionary ending

The focus of this paper is the use of NAPLAN items to develop students' competence in reading mathematical text, to promote thinking strategies including estimation, and to evaluate alternative solutions for errors and misconceptions. Showing students test items and discussing strategies for thinking about questions and responses promotes student confidence and resilience, and enables a greater sense of student control over their learning (Martin, 2003). This paper is not advocating national testing as the most desirable approach to assessing students' knowledge, skills and understanding. Teachers best carry out assessment as they talk to and observe their students — see the AAMT position paper *The Practice of Assessing Mathematics Learning* (AAMT, 2008). For teachers who feel the pressure to prepare their students for the tests, I am recommending the use of NAPLAN items as discussion starters to promote thinking.

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REPRESENTING PROPORTIONAL RELATIONSHIPS ALGEBRAICALLY

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This paper describes the responses of a group of middle school students to two tasks in a sequence of activities aimed at helping them to represent proportional relationships as ratios, tables of values, graphs, and algebraic expressions. Possible reasons for difficulties students experienced with using algebraic expressions in this context are discussed along with suggestions for ways forward.

Introduction

The development of students' proportional reasoning has been recognised as a key aim of middle school mathematics (Lamon, 1993) and as a frequent cause of difficulty (Lamon, 2007). It underpins understandings of equivalent ratios (including fractions), as well as other mathematical topics such as similarity, scaling, and trigonometry; it is essential to sophisticated statistical reasoning (Watson & Callingham, 2003); and it is inherent in science topics such as density, speed and molarity (Dole, Clarke, Wright, Hilton & Roche, 2008).

The transition from arithmetic to formal algebra is also recognised as both a crucial and problematic aspect of middle school mathematics. This is in spite of evidence that even very young children are capable of functional thinking (Blanton & Kaput, 2004). Stacey and MacGregor (2001) noted a shift in Australian curriculum documents from an approach to algebra in which pronumerals represent specific unknown numbers to one focussed on generalising patterns in which they represent variable quantities. They pointed to a lack of research evidence to guide a choice of one approach over the other and described findings concerning the ability of students in Grades 7-10 to identify and express patterns in function tables. The patterns approach has the advantage of lending itself well to linking early formal algebra with proportional reasoning.

This paper briefly describes activities that were used with a group of middle school students to engage them with situations involving proportional reasoning. It focuses on their responses to two tasks: one involving moving between multiple representations of a contextualised proportional function, and another that involved sorting a set of cards describing proportional situations in a range of ways. The latter activity proved especially challenging. Reasons for this are postulated along with implications for the teaching of proportional reasoning and algebra more generally.

The study

The activities reported here were undertaken as part of a case study conducted in one of several schools participating in a larger research project focussed on improving middle school mathematics teaching and learning. The focus and design of the case study were determined collaboratively between the author and the school's middle school leader. It was aimed at designing and trialling challenging activities for a group of relatively high achieving Grade 5–8 students in a rural K–10 school.

The students

A group of approximately 14 middle school students (attendance varied) in a rural K–10 school participated over a period of several months in a series of weekly lessons of one to one and half hours aimed at extending their understandings of fractions (including decimal representations), ratio, proportion, and functions. Two students were from Grade 5/6 classes while the others were from Grade 7/8 classes. All had been identified by their teachers as mathematically capable and likely to benefit from challenging work.

The tasks

Thunder and Lightning

You can calculate how far away, in kilometres, a lightning strike is by counting the number of seconds from the moment you see the lightning until you hear the thunder, and then dividing by three. If you count 12 seconds from the lightning flash until you hear the thunder, the lightning is approximately 4 km away.

a. Make a table showing the time (in seconds) between the lightning and thunder, and the estimated distance (in kilometres) from you to the lightning.
b. What is the ratio of time:distance?
c. Draw a graph showing the relationship between the time and the distance.
d. Does it make sense to join the points on the graph? Why?
e. How many points did you need to plot to know what the graph would look like? How could you be sure this was enough?
f. Use algebra to write the coordinates of the point on the graph that corresponds to lightning being x kilometres away.
g. Write an algebraic rule (function) that describes the relationship between the time between the lightning flash and the thunder and the distance to the lightning strike.
h. How does the ratio you wrote down in (b) show up on the graph?
i. How does the ratio show up in the function?

Figure 1. Thunder and Lightning task.

A range of tasks focussed on developing understandings of fractions, decimals, ratio and functions was used. The two reported here are the *Thunder and Lightning* task shown in Figure 1, and a *Card Sorting* task. The *Card Sorting* task involved sorting into two groups a collection of cards that included ten scenarios, each involving quantities related in the ratio 6:1, as in the famous students and professors problem (e.g., Hubbard, 2004). The cards included a range of ratio, algebraic and graphical representations corresponding to the scenarios. An example of a matching group of cards is shown in Figure 2[6]. Several other scenarios would also have been included in this group. Students initially worked on the *Card Sorting* task in pairs and later an A4 set was used to work with the whole group.

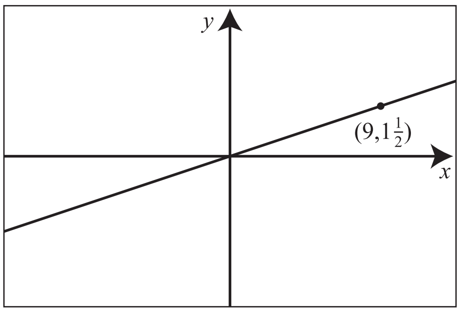
There are six times as many cats as there are dogs in our street. x = the number of cats y = the number of dogs	$y{:}x = 1{:}6$
$6y = x$	$y = \dfrac{x}{6}$
$x = 6y$	$x{:}y = 6{:}1$
<table><tr><td>x</td><td>6</td><td>12</td><td>18</td><td>24</td></tr><tr><td>y</td><td>1</td><td>2</td><td>3</td><td>4</td></tr></table>	

Figure 2. Example of a set of matching cards.

[6] The complete set of cards is provided in the Appendix.

Student responses and discussion

Thunder and Lightning

Parts (a) to (d) of this task presented few difficulties but relatively few students responded to later parts. All eleven of the students who attempted this task were able to make an appropriate table of values. None of the tables showed iterative patterns throughout the values of each variable, but rather the students appeared to choose values for one variable and either divided or multiplied by three to calculate the corresponding value. The fact that a table or partial table of values was not provided may have prevented students from being distracted by iterative patterns as described by Stacey and MacGregor (2001). Two students included values for the number of seconds that yielded non-integral results when divided by three. One of these is shown in Figure 3.

Figure 3. Non-integral value.

Eight students accurately recorded the ratio of the number of seconds to the number of kilometres as 3:1. The remaining three students did not record an answer to this question. Several students struggled to create a graph showing the relationship. The students had not experienced plotting coordinate pairs before and had little experience with line graphs. Links were made between the axes of the graphs and number lines, with which they had worked extensively in previous lessons. The two Grade 5/6 students were two of the three students who labelled intervals of the axes rather than points. Five students chose to put time on the horizontal axis while six put time on the vertical axis. Establishing that the two types of graphs that resulted were mathematically equivalent was unproblematic: all students connected the points that they plotted and were able to explain why this was reasonable in terms of the line giving all of the possible pairs of times and distances.

The proportional nature of the relationship in this task means that the graph passes through the origin. The graphs produced by the three students who labelled axis intervals rather than points did not show this clearly (see, for example, Figure 4.)

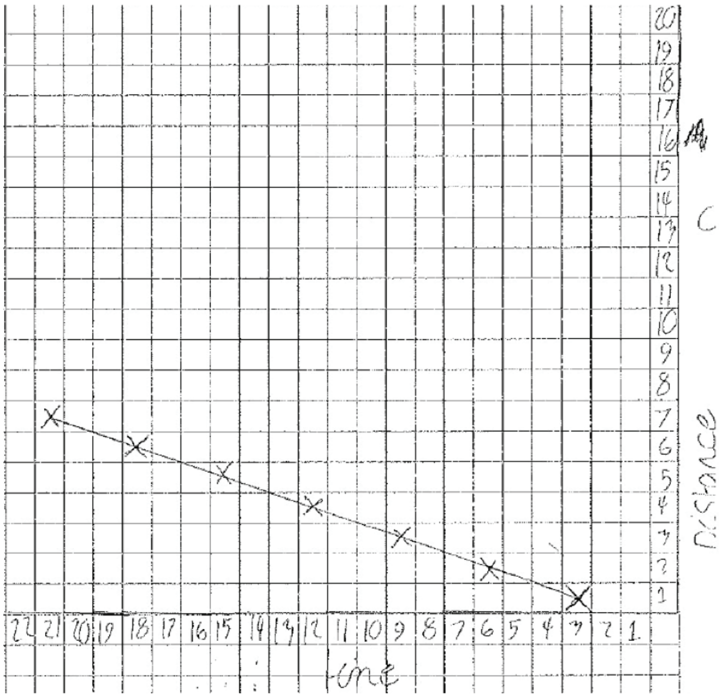


Figure 4. *Graph with labelled axis intervals and horizontal axis reversed (Grade 6).*

Figure 4 is also interesting in that the horizontal axis is labelled from right to left. All of the remaining graphs showed the line passing through the origin with four graphs having this point labelled clearly. Examples of these graphs are shown in Figure 5 and Figure 6.

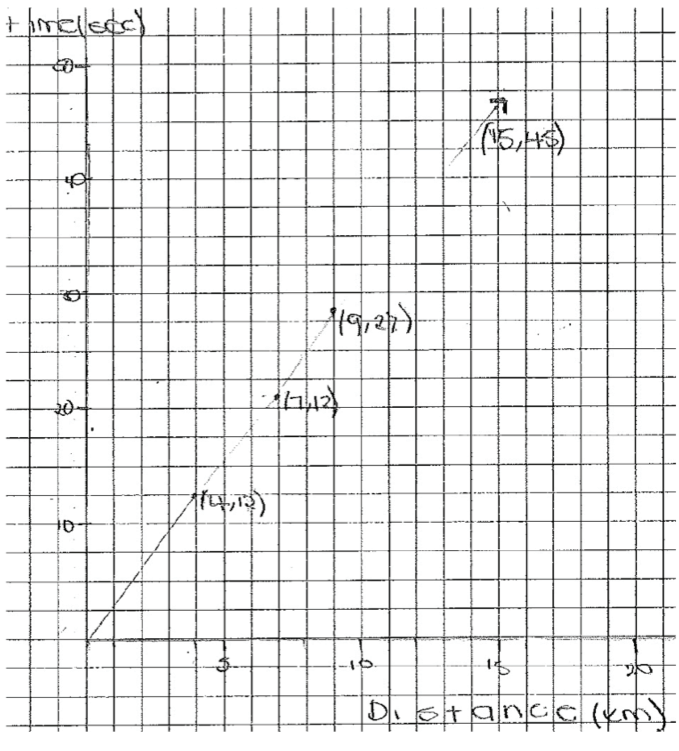


Figure 5. *Graph showing line through unlabelled origin (Grade 8).*

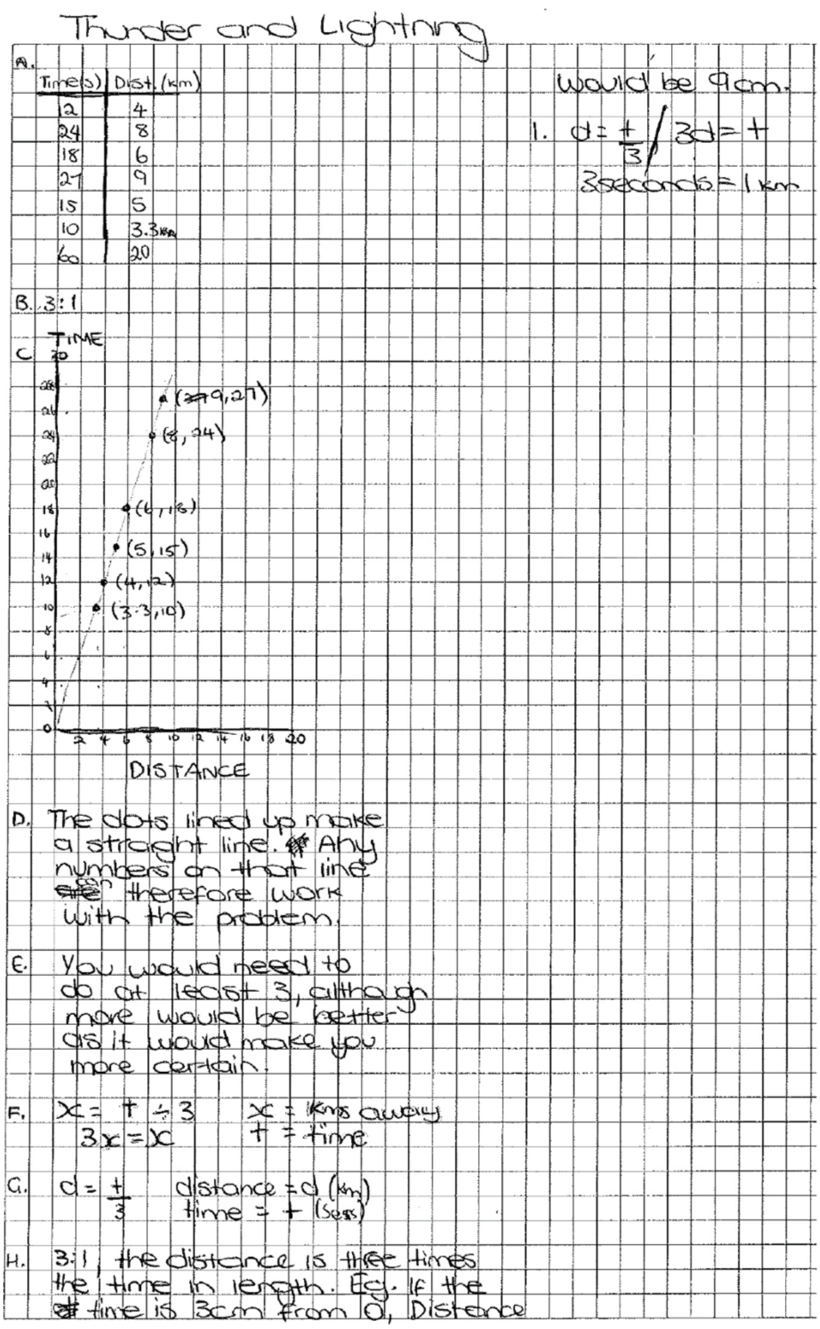


Figure 6. A response from a Grade 8 student.

The three students, all in Grade 8, who answered part (e) expressed the opinion that three points was the minimum number required but that more points were better in order to be sure. These three students were also among the four who responded to part (f). Just one student wrote down the ordered pair $(3x,x)$. Other responses were $3x = x$, and $3x$, and a Grade 6 student added the values $3x$ and x to the appropriate columns of his table. The same three Grade 8 students were also able to describe the relationship between the time and distance using algebraic symbols, with one offering both $3d = t$ and $d = \frac{t}{3}$.

These three students also offered a correct response to part (h) and/or part (i). Their responses followed discussion among the whole group about $3t$ corresponding to $1d$, which they expressed ("correctly") as $d = 3t$. However, they could see from the table of values that this was incorrect and that the equation should be $t = 3d$. Eventually a few were able to articulate that time:distance = 3:1 meant that the number of seconds was three times the number of kilometres ($t = 3d$) and that distance:time = 1:3 related to $d = \frac{t}{3}$ because the distance was one third of the time. The complete response of one of the students who was able to make a written response to parts (h) and (i) is shown in Figure 6.

As Stacey and MacGregor (2001) observed, being able to see the functional relationship as demonstrated in this task, by correctly producing a table of values, does not necessarily mean that students can describe that relationship using algebraic symbols.

Card Sorting

Difficulties with the "students and professors" problem have been documented among some first year university engineering students (Hubbard, 2004) as well as among school students. Unsurprisingly, therefore, the *Card Sorting* task proved difficult even for the students who had been successful with the latter parts of the *Thunder and Lightning* task. The task was in fact simplified by removing the cards showing tables and graphs, in order to assist the students to focus on the relationship between the worded cards and those containing algebraic equations and ratios.

Students struggled with the correct order of the x and y in their equations. For example for the scenario shown in Figure 2, there were lengthy and animated discussions about whether the correct equation was $6x = y$ or $6y = x$, with students frequently changing their minds. Once there was agreement as to which of these options applied, identifying equivalent equations was relatively straightforward.

Errors in which the pronumerals are reversed could have arisen from directly translating the words into symbols as follows:

There are six times as many cats as there are dogs in our street
$6 \quad x \qquad\qquad c \qquad\qquad = \qquad d$

and then transposing the pronumerals c and d for x and y. However, Stacey and MacGregor (1993) found that in a similar problem, worded so that a direct translation of this nature would result in a correct equation, only 37% of Grade 9 students produced a correct response. They concluded that this was not the principal cause of the error. Similarly, through using decontextualised problems, they demonstrated that confusion between the pronumerals and the first letters of the nouns involved did not explain the error. They theorised that students form mental representations of such situations in terms of a larger and smaller quantity, and that the numeral involved (6 in this case) is associated with the larger quantity.

In the context of this study, students were not helped by the fact that the thinking required to write an algebraic expression is "backwards" compared with the ratio that expresses the relationship. For example, if there are six times as many cats as dogs then the ratio of cats:dogs is 6:1, but the number of cats is six times the number of dogs. That is, if x = the number of cats and y = the number of dogs, then $x:y = 6:1$, but $(1)x = 6y$. In the *Thunder and Lightning* task, most students struggled to connect the ratio to the graph. Of course the ratio is the gradient of the line, and later students learn that this is

$$\frac{\text{rise}}{\text{run}}$$

but perhaps less often make a connection between this fraction and the ratio relating the variables in a proportional relationship.

Since the students had studied little if any formal algebra outside of these lessons, it was thought that they might not be confident about the meaning of expressions such as $6y$ or, more generally, with how to read an equation like $x = 6y$. However, reminders of the omitted multiplication sign and suggestions that students consider particular numbers when they considered which of $x = 6y$ or $y = 6x$ applied in a given situation were not helpful in overcoming these difficulties.

Implications

Several of the students involved in this small study were able to express successfully using algebra the functional relationship between variables such as the number of seconds between the lightning and the thunder, and the number of kilometres from the storm, but struggled with the same sorts of relationships in the *Card Sorting* context. It may be that working through the process of devising a table to describe the relationship and/or representing the relationship with a graph provided a vital bridge between the words that described the context and the algebraic equation that expressed it.

Consistent with Stacey and MacGregor's (1993) suggestion that students construct cognitive models of these sorts of relationships in terms of a larger and smaller quantity, it may be helpful to use a visual approach to modelling the situations. For example, if there are six times as many cats as there are dogs in a street with x = the number of cats and y = the number of dogs, then rectangles or lines sketched approximately in proportion and labelled appropriately as in Figure 7 might be helpful in showing that $x = 6y$. The larger and smaller quantities are clearly shown and could assist students to attach the numeral (6) to the smaller quantity to form an equality.

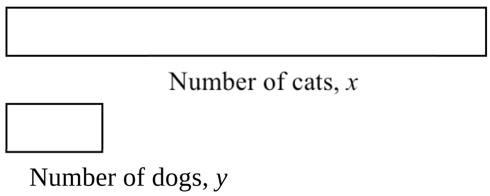


Figure 7. Diagrammatic representation of the relative sizes of quantities involved.

Stacey and MacGregor (2001) noted a lack of relationship between students' ability to identify functional relationships and their ability to express such relationships algebraically or to articulate them verbally. However, students who could accurately verbalise the relationship were much more likely than other students to be able to write a correct algebraic equation (Stacey & MacGregor, 2001). It is worth considering ways to articulate relationships that may be useful in formulating equations, and also helpful ways of verbalising equations. For example, $x = 6y$ could be expressed as, "If we want to find x, we multiply y by 6," could be more helpful than, say, "x is six times as big as y" (which expresses the "backwards" nature of the algebraic expression). "Six lots of y are the same as x" or, "x is the same as 6 times y" are clear readings of the equation but perhaps less helpful in working from the words to an equation. It could be that constructing tables and graphs to represent relationships fills a similar kind of role in building an accurate mental model of the relationship involved.

Stacey and MacGregor (1993, 2001) used decontextualised problems in order to gain a clearer idea of the factors affecting students' ability to express relationships algebraically. In terms of teaching, however, it may be that providing contexts is another way to assist students to build richer mental representations of relationships. It could be argued that students might rely on their existing understandings of the variables in the context, rather than the mathematical description of the relationship in formulating correct equations. Such reliance on existing knowledge in preference to data was noted by Beswick (2008). However, the well-documented difficulties with the students and professors problem, when presumably the respondents would be familiar with the fact that the number of students would likely be greater than the number of professors, suggests that this is not the case. Even if contexts are not helpful in terms of thinking about the mathematics, any contribution that they make to engaging students with the relevant mathematics could make them worthwhile.

Acknowledgements

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Appendix: Cards for *Card Sorting* task

There are six times as many cats as there are dogs in our street. x = the number of cats y = the number of dogs	On an excursion, one adult is needed for every six children. x = the number of adults y = the number of children
The ratio of students to teachers was 6:1. x = the number of teachers y = the number of students	Each shirt requires six buttons. x = the number of shirts y = the number of buttons
The school timetable has six lessons every day. x = the number of lessons y = the number of days	In an aeroplane, there were six seats in each row. x = the number of seats y = the number of rows
Six people can fit into each life raft.. x = the number of life rafts y = the number of people	On average, each female rabbit raises six young in a year. x = the number of female rabbits y = the number of young rabbits raised
$y = 6x$	$x = \frac{y}{6}$

The school expects students to do six hours of homework each week. x = the number of hours of homework y = the number of weeks	It takes one litre of paint to paint 6 square metres of wall. x = the number of litres of paint y = the number of metres squared of wall
$6x = y$	$6y = x$
$\frac{y}{6} = x$	$\frac{x}{6} = y$
$x = 6y$	$y = \frac{x}{6}$
$x:y = 6:1$	$x:y = 1:6$

$x{:}y = 6{:}1$

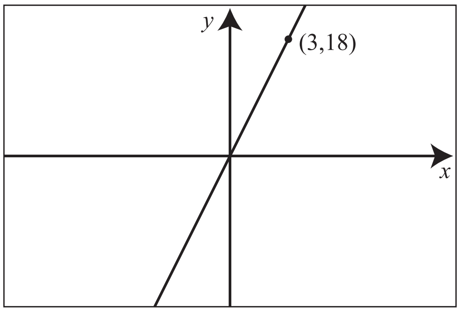
$y{:}x = 1{:}6$

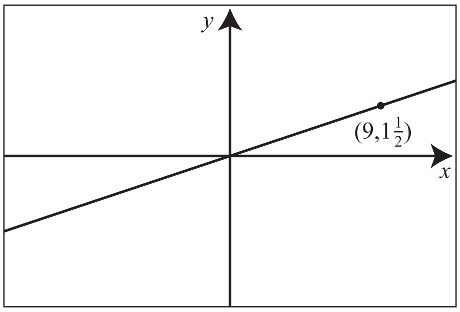
x	1	2	3	4
y	6	12	18	24

x	1	2	3	4
y	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$

x	6	12	18	24
y	1	2	3	4

x	0.5	1.0	1.5	2.0
y	3	6	9	12





THE CHANGING FACE OF NUMBER

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Our everyday experience of number as a taken-for-granted practical means of recording, counting and calculating belies the fact that conceptions, uses and representations of number have been changing over many centuries and continue to change. This paper illustrates some of those changes, showing how number has been constructed in a variety of ways to meet changing human needs. Do we sufficiently acknowledge the continuing evolution of number? In view of the changing face of number in the 21st century, how can the curriculum assist students to demystify and better cope with using number in today's number-dominated world?

Introduction

Over 2500 years ago, the famous Greek mathematician Pythagoras claimed that: "The essence of all things is numbers" (Johnson, 1994). Do we believe this to be true today? We cannot deny that numbers these days seem to dominate our lives, even defining our very essence: *birth registration number, birth date, birth weight* (or mass), *body weight, height, house number, postcode, telephone number, fax number, Tax File Number, Medicare number, credit card, bank account* and *personal identification numbers*, a *passport number*, *client* or *customer numbers*, *computer password number*, *Australian Business Number, salary, bank account*, and the list goes on. Our identity is so bound up in numbers that there is now a crime known as "identity theft." Who steals our numbers, steals us. Without control over those numbers, we are powerless, and to all intents and purposes, non-existent.

Given that numbers are so important to us, it would be helpful to know something of where they came from and how they have developed to become so dominant in our lives. The following thumbnail historical survey of number is derived from a number of sources listed at the end of the paper. These sources are well worth consulting.

Numbers as labels

There are several distinct kinds and roles of numbers. Most of the numbers mentioned already are really *labels* or *identifiers*. We could equally well use letters of the alphabet for the purpose of labelling or identifying, as we do with computer passwords, or we could use a combined alphanumeric code (a mixture of letters and numbers), as in the case of car registration numbers. But numbers seem somehow more familiar and easier to handle. (This explains why the activity called *Sudoku* looks to us like a number puzzle, which it is not.)

However, numbers used as labels are limited in what they can do; they cannot tell us *how many*, or *how big* or *in what order* things are. A *postcode* for example cannot tell us anything except perhaps roughly where a location is, and it makes no sense to add two postcodes. Nor is the location identified as 6458 necessarily bigger than the location 5052. A postcode is simply a device to expedite the sorting of mail. This may seem obvious, but sometimes people confuse "number as label" with "number as quantity:" when medicine chests were first distributed by the Royal Flying Doctor Service to remote outback stations, every item was numbered and could be prescribed by number over the radio. This made for very efficient communication especially when used in conjunction with numbered body charts included in the chests. One station manager was recorded as having run out of the prescribed No. 9 tablets and gave his wife a No. 4 and a No. 5. He reported that the combination worked just fine (BRI, 2003).

The common use of numbers used as labels is relatively recent. Take the case of *telephone numbers*. After the telephone was invented in 1876, telephone exchanges simply listed their subscribers by name, and operators had to memorise them in order to connect one with another. The use of numbers was resisted because it was felt to be undignified and would lead to loss of personal identity. However, in 1880, a measles epidemic hit the town of Lowell, Massachusetts, and the local doctor, Dr Parker, was very concerned that if the four operators took sick with the measles, then the whole town's telephone system would be paralysed. If numbers were used instead of subscribers' names, he argued, then it would be much easier to train substitute operators. The suggestion was taken up, and nobody objected. So successful was the innovation that by 1895, the adoption of telephone numbers was universal (BRI, 2003).

Another example of number as label is *house numbers*. Up until the nineteenth century, it was common practice to identify houses by name or location — or both. However the practice of numbering houses was taken up in most countries with the advent of postal services and national censuses, because both require some precision in location in order to make their implementation more efficient, especially in settlements with a fairly regular pattern of streets. House numbers were originally implemented, and probably for taxation purposes, in Paris in 1463.

Numbers as mystic symbols

While the use of numbers as labels has become widespread only recently in human history (and largely as a result of the availability and power of electronic and digital technologies), the use of numbers as *mystic symbols* goes back thousands of years (Eves, 1969). The practice of substituting numerical values for the letters in a name arose from the fact that many of the numeral systems in ancient times (such as Syrian, Hebrew and Greek) were alphabetical systems. In the Greek system, for example, α had the value 1, β the value of 2, ι the value 10 and ρ, 100. Thus it was possible to give a numerical value to any word or name, as in Scrabble. From this arose a mystic pseudoscience known as *gematria*, *arithmography* or *numerology*, whereby it was suggested that words with similar values were related. According to Baumgart et al., 1989), this practice was well known among the ancient Hebrews and Greeks, particularly the Pythagoreans, who imbued it with religious and philosophical significance.

For the followers of Pythagoras, every number had a sacred meaning. The number one stood for God, the divine being, the root of all other numbers and existence. Even numbers were feminine, odd numbers were masculine. Four stood for justice, five for marriage (the combination of the first two masculine and feminine numbers) and six for perfection.

Many numerological systems were devised, which either totalled the numerical value of a word or name or reduced the numerical value to a single digit from 0 to 9 by successive addition of digits. The result of converting words to numbers was used for all sorts of purposes from divination and fortune telling to textual interpretation. Nowadays the practice of numerology has found a regular place in the pages of popular magazines and in publications of occult bookshops. Are we still superstitious about number? Just think about the issue of gambling!

Numbers for arithmetic

Besides numbers as labels and numbers as mystic symbols, the most common use of numbers is *arithmetic***,** for the purposes of counting, recording, estimating, calculating and measuring. We use arithmetic numbers, that is, *counting numbers, fractions*, and *negative numbers***,** every day of our lives, almost without thinking.

Studies of hunter-gatherer peoples suggest that early humans expressed numbers with reference to specific parts of the body: eyes, nose, ears, hands, feet, fingers and toes. Some peoples had specific names only for numbers one and two, and sometimes, three, yet could count up to six through combinations of numbers. In some indigenous languages, typically from Australia, New Guinea and Melanesia, Eves (1969) claimed that speakers tend to refer to everything beyond six as “many,” “much” or “plenty.”

At first, we may think of this as ‘primitive’ or at least ‘odd’, but a similar thing happens in contemporary English. When we talk of trees we count them: one, two, three, four, … and so on, until the individual trees become unimportant, and we talk then of a clump of trees, a copse, coppice or spinney, a grove, a wood and eventually a forest. We also express an undefined number of (usually animate objects) by means of a collective noun (Henderson, 2003). Thus we speak of a flock of sheep or a murder of crows. Less well-known examples of collective nouns are: a business of flies, a bed of snakes, a knot of toads, a paddling of ducks (if they are swimming), a raft of ducks (if they are floating in the water) and a team of ducks (if they are flying). If we were creative, we could invent some more, such as a flush of plumbers, a rash of dermatologists, a column of accountants, a shush of librarians, a peck of kisses and a magazine of editors.

Today we consider it natural to count on our *fingers*, and this was certainly the case for many people for thousands of years. This of course explains why we have a numeration system based on grouping in tens, that is, a *decimal system* (Baumgart et al., 1989). To understand how this might have evolved, consider the slightly more novel system of counting on fingers *and* toes, which is used even today by indigenous

Greenlanders, according to Gullberg (1996). This is a mixture of quinary (or base five) and vigesimal (or base twenty) systems.

We begin by counting fingers on one hand: one (1); two (2); three (3); four (4); five (5) then onto the other hand: second-hand (6); second-hand, two (7); second-hand, three (8); second-hand, four (9); second-hand five (10) and onto the toes of the first foot: first-foot (11); first-foot, two (12); first-foot, three (13); first-foot, four (14); first-foot, five (15) and the toes of the second foot: second-foot (16); second-foot, two (17); second-foot, three (18); second-foot, four (19); second-foot, five (20). An alternative name for twenty (second-foot, five) is “person counted out.” The first person, having been counted out, stands for a group of twenty. This indicates that we have run out of fingers and toes from the first person, and we move onto the fingers and toes of a second person, and so on. The number 21 is thus described as “second-person, one” and 32 would be “second-person, first-foot, two” and so on.

Whereas the Greenlanders count and group in twenties, we in our decimal system count and group in tens. Lest you think that Greenlanders are idiosyncratic, you should know that there is evidence of counting in twenties in Celtic languages, such as Gaelic, Manx, Welsh, Cornish and Breton, and that this formed the basis of number in the American civilisations such as the Mayan and Aztec, according to Eves (1969). In English, there are vestiges of this practice in the use of the word “score,” while in French, there is the rather strange rendering of the number 80, which is *quatre-vingt* or “four twenties.” These are probably legacies of the Ancient Britons and Gauls.

The earliest decimal system of recording number that we know of emerged in the fourteenth century BC in China. From here the system moved to Japan and South-East Asia and was adopted in the Indian sub-continent. Richardson (1997) has reported that the first evidence of the use of a zero can be found in Cambodian and Sumatran inscriptions from the seventh century AD.

Between the third century BC and the eighth century AD, the Hindu mathematicians of India developed their own way of recording numerals, which evolved over time to become Arabic numerals as a result of the spread of Islam to India, which in turn allowed Islamic traders and scholars to carry Hindu learning to the Middle East (Eves, 1969). In 825 AD the Persian mathematician al-Khowarismi published the first fully developed base ten system of numeration, together with many of the methods of setting out and calculation invented by the Hindus, and still in use today. Baumgart et al. (1989) note that the great achievement of the Hindu mathematicians was to devise a numeration system, which was most efficient for calculating, whether in writing or in the head. Al-Khowarismi’s name is celebrated in the use of his Latinised name “algorism” or “algorithm” which now means “a procedure or set of instructions.”

Al-Khowarizmi’s text was translated into Latin in the twelfth century and soon became known throughout Europe. Mathematical scholars such as Fibonacci popularised the use of Hindu methods of calculation in Europe and the adoption of Hindu-Arabic numerals in the thirteenth century, according to Johnson (1994). At that time the numerals began to look very much like the numerals we use today. At first there was a great deal of opposition to these new-fangled written methods of calculation using Hindu-Arabic methods. The major concern about writing numbers down was that they could be easily falsified. A 1 could easily be changed to a 4 or a 7, and a 0 could be changed to a 6 or a 9. Obviously this system could be misused by unscrupulous merchants. In the end, written calculation triumphed because it was portable and it was possible to check one's working for errors.

The names of numbers

The words we use for our numbers are mainly Anglo-Saxon in origin. Because Romans occupied Britain for around 400 years before them, we might have expected Latin names to endure. But Latin, and for that matter, Greek number names only occur in relation to specific nouns, although these abound in English. Greek number names appear in such words as monograph, diphthong, triglyph, tetrahedron, pentagram, hexameter, heptane, octopus, decathlon, hecatomb, kilogram. Latin number names are found in words such as unicorn, biceps, tricycle, quartet, quintessence, sextant, Septuagint, octave, nonagenarian, decade, centennial, millipede (Gullberg, 1996).

The invasions of Britain by the Anglo-Saxons in the fifth and sixth centuries and the Norse Vikings in the eighth to the eleventh centuries left a legacy of words from their languages, Old English and Old Norse, which were both Germanic and quite similar. For some reason, these Anglo-Saxon number names persisted, despite yet another invasion of Britain, from the Normans, who spoke French. Another contender for number names was Latin, which was the main language of science, trade and diplomacy during the Middle Ages. Most of the language of arithmetic in fact is Latin in origin, and most mathematical and arithmetic books in Britain from Roman times until the seventeenth century were written in Latin.

One non-Anglo-Saxon number name that we have inherited is the word *zero*, which derived from the Hindu word *sunya*, via the Arabic *sifr*, the Latin *cephirum* and the Italian *zevero*. As one might guess, it relates also to the word "cipher" (Eves, 1969). Another is the word for *million***,** which came from Medieval Latin, meaning "great thousand." There is also one word of French origin that slipped through: *dozen* from the French *douzaine* meaning twelve. The tricky Anglo-Saxons had their comeback by inventing the "baker's dozen."

By the seventeenth century, the names and symbols for numbers were pretty much as they are today, and the methods of arithmetic calculation that we learned at school were well known, as Eves (1969) has noted. But the demand for more efficient methods of calculating and measuring, particularly using large numbers, led to the invention of new calculating devices and a new and standard system of measuring: the metric system.

Naming of very large (and very small) numbers

A pioneer in naming and calculating with very large numbers was Archimedes. In an essay *The Sand Reckoner* he proposed a unit called the *myriad* equal to 10 000, and used this as the basis for naming even larger numbers. More recently, the nephew of an American mathematician Edward Kasner coined the word *googol* in the late 1930s as the name for the number written as a 1 followed by 100 zeros. (Gullberg, 1996) Now the invention of such names as myriad and googol could be thought of as just fanciful, were it not for the inventions of the metric system and the electronic computer and the demands of science and technology. These have required us to extend our number names, so that we are all beginning to become familiar with such words as “megabyte” and “gigalitre,” which were virtually unknown a decade ago.

Today we need a new number vocabulary to cope with the very large and the very small measures required today for astronomy and nanotechnology, for computing and other emerging technologies. The metric system devised by the French in the late eighteenth century introduced a basic list of number words to help us measure all aspects of our universe: the metre, kilometre, millimetre, the litre, decilitre and centilitre etc. Gullberg (1996) notes that these were standardised and updated with the adoption of further words and prefixes at Metric Conventions in 1875, 1960 and 1991 which extended the language of measurement, as part of the International System of Units. The latest additions are the prefixes *yotta* and *zetta* at the large end of the scale, and *zepto* and *yocto* at the small end of the scale of measurement. Fortunately most of us will have no need to use such words, but the capacity of our home computers has inexorably grown along with our need to be familiar with the ever increasing units, from kilobytes, to megabytes, to gigabytes and now to terabytes. The original Greek roots of these prefixes say it all: “kilo” comes from *chilioi* = thousand, “mega” from *mégas* = big, “giga” from *gigas* = giant or gigantic, and “tera” from *téras* = monster. Clearly the history of number continues to be written, for we still need to find ways of numerically addressing the broadening spectrum between the infinite and the infinitesimal (Gullberg, 1996).

We also need to decode many new uses and applications of number which constantly confront us in everyday life. (Consider for example what the following mean: barcode, BSB, A380, 75 MB, 103.9 FM, 110%, 9/11, etc.) The challenge for teachers and the curriculum is to find ways to help students to do this.

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CUTE IDEA: WHERE'S THE MATHS?

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This paper is based on a personal reflection following the teaching of a mathematics session with young children 5–6 years old. It raises issues about the use of teaching materials, or "manipulatives" as they are often called, in the teaching and learning of mathematics. Reasons for using manipulatives are addressed briefly, as is advice to teachers from research as to how manipulatives can be used without increasing the cognitive load for children. Finally the paper considers the empty number line as an example of a model that develops children's thinking and in doing so becomes redundant.

One of the very difficult parts of teaching young children mathematics is finding new ways of treating fairly limited content. Trying to come up with different contexts for problems and ways to present ideas is a challenge.

Recently I came up with what I thought was a good idea. I would make edible dominoes with Preps (first year of school in Victoria)! We would use cracker biscuits that are made in pairs to be the base of a domino and use peanut butter to stick on sultanas as dots (Figure 1).

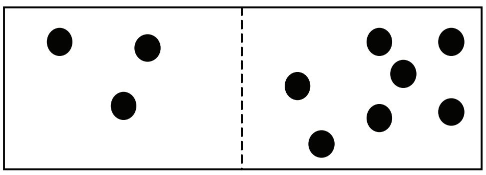

Figure 1. Diagram of the sultanas on a cracker.

It was very appealing and I knew that the children would be very excited by the idea, but where was the mathematics? Usually when I am planning sessions I begin with the mathematical idea that is to be the focus of the lesson, then I think of ways of presenting it. So this reversal caught me by surprise and I realised how seductive it is, for a teacher, to come up with what is a "cute idea" that will be intrinsically motivating to children and to lose sight temporarily of the mathematical focus.

I needed to stop and ask myself a few questions. What was the potential of the materials as a mathematical manipulative? What mathematics did I want the children to learn? Could I somehow marry the two?

The main feature of the manipulative was that it provided a stimulus to count objects (sultanas) and to look at part–part–whole relationships (how many sultanas on each end of the cracker). I had been thinking about the skills involved in developing place value concepts and the importance of the number ten for young children. Perhaps instead of the usual set of dominoes we would make "Biscuit Dominoes of Ten." The focus would then be:

- counting to ten;
- making collections of ten;
- comparing collections of ten;
- noticing two parts of a collection of ten (part-part-whole relationships of ten);
- sorting and categorising biscuit dominoes; and
- making a graph of our biscuit dominoes.

I was very conscious of getting some rich mathematics out of the activity so that the novelty of the materials did not override the mathematics. A teacher-colleague offered to try out the ideas with me in her Prep classroom. We began by focusing on ten, singing *Ten Little Indians* with finger actions several times. The children were reminded of the regular dominoes they had used many times. The task was demonstrated and the children were reminded to count out exactly ten sultanas, put peanut butter on the cracker and then arrange the ten sultanas. They were asked to think about, "How many sultanas are on this end?" "How many are on the other end?" and "How many sultanas are there altogether?" Children were challenged to make different "ten dominoes" and to describe them. Many children made several combinations to ten (Figure 2).

Figure 2. Children making their biscuit dominoes.

Children then selected their favourite domino, and slipped it into a small plastic bag for handling. We looked at all of the different combinations of numbers of sultanas that people had made. The children noticed that some numbers were the same (Figure 3).

We gathered together biscuit dominoes that were the same. One of the children observed that a 2 and an 8 was the same as an 8 and a 2 if he turned the bag around.

This was quite a nice recognition of the commutativity of addition which I had not necessarily expected of the children.

Figure 3. Children displayed their favourite biscuit domino.

I then explained that the groups of bags could be seen more easily if they were lined up and we graphed the results (See Figure 3). Having interpreted the results, we counted the total number of sultanas on crackers in bags on the floor by counting in tens. Finally we ate the mathematics of the day.

Figure 4. Biscuits with the same part-part-whole relationships formed a graph.

Why am I telling this story? There are several reasons. As I said at the outset I was surprised by how easy it is to lose focus of the mathematics learning when an appealing activity comes along. This is something you no doubt already know. I also thought that this was quite a good context for raising some questions about the use of manipulative to teach mathematics to young children. The questions are these: Why use manipulatives? What are the best to use? When and how should they be used? A further complication is the question of when not to use manipulatives or materials at all.

Why use manipulatives?

First let us look at the question of why we use manipulatives and models for developing children's mathematical thinking. Certainly they have come to assume a critical role in helping students learn mathematics (Suydam, 1986). Mathematics is abstract, so we like to use physical objects to try to help embody a mathematical idea, illustrate a concept, or show a relationship. Current theories of how children learn also contribute to the development of an environment that builds knowledge through the child's *actions*. "This action may be physical and involve materials but it must always have a mental dimension" (Perry & Conroy, 1994, p. 52). Learners participate in the construction and negotiation of mathematical meaning in the classroom (Cobb, Stephan, McClain & Gravemeijer, 2001) and the learning resources utilised in this process are important.

The research of Sowell (1989) indicated that lessons using manipulatives had a higher probability of producing greater mathematical achievement than did lessons without such materials. Handling materials appears to help children construct mathematical ideas and retain them. However, handling materials does not automatically carry meaning for children; the focus must be on understanding (Thompson, 1994).

Sometimes using a range of manipulatives to demonstrate the one concept is a good strategy. This is called multiple embodiment and the use of several models is intended to encourage children to consider the common mathematical idea represented by the manipulatives. We expect that children will see the "same idea" whether they are bundling icy-pole sticks into tens and hundreds, or joining Unifix cubes together in tens and hundreds. The use of different materials also prevents one manipulative always being associated with one mathematical idea. I think we should be explicit about the mathematics that we intend children to see in the materials.

In addition the use of non-examples is very powerful in clarifying the features of an example of a mathematical concept. Research shows that children learn more when they are presented with both examples and non-examples (Hiebert & Carpenter, 1992). For instance, in developing the understanding of the concept of circle, children might be encouraged to trace around dinner plates, saucers and lids but they might also be asked whether a curtain ring (unjoined) and an oval platter are circular.

What are the best to use? When and how should they be used?

The work of Boulton-Lewis (1999) showed that manipulative need to be used selectively, strategically and knowledgeably in order to be helpful to children without increasing the cognitive load of tasks. We need to make sure that the handling of the materials does not add to the difficulty of the mathematics but rather helps the child to understand what is being modelled.

Ultimately we want children to be able to use mental imagery to conjure mathematical abstractions. One mathematical model that is a particular favourite of mine is the empty number line because it eventually makes itself redundant.

An excellent model: The empty number line

There are no marks or numbers on this number line. Students only mark the numbers they need for their calculation. The marks on the empty number line are not intended to be proportional and they are used as a way of recording thinking as it happens. The empty number line is a sort of sketch of the steps of thinking.

An example of the use of the empty number line is shown in Figure 5. It shows one possible solution strategy for 65 – 38. First 65 is marked then, because 38 is almost 40, 40 is subtracted to make the question easier. The resulting 25 is marked. However the problem was to subtract 38 so too much has been taken away. To compensate, 2 is added and the final answer is marked at 27.

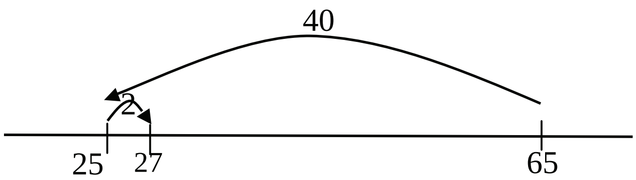

Figure 5. Diagram showing a solution strategy for 65 – 38

Features of the empty number line

According to (Gravemeijer, 1994) there are three principal reasons to choose the empty number line as a model. These reasons are:

1. The empty number line is a model that fits the counting aspect of number. (Freudenthal, 1973) described four aspects of number: counting number, numerosity number ("manyness"), measurement number, and reckoning number. Models such as multibase arithmetic blocks (MAB 10) and other set-type materials represent numerosity number. Counting needs a linear representation. When applying number to "real-life" contexts or problems, students will encounter both set-type situations that deal with quantities and linear-type situations. For example, in a situation involving travelling distances an empty number line would seem a more appropriate model to use.
2. An empty number line supports informal solution procedures, in that it is not restrictive, allowing students to express and communicate their own solutions in a variety of ways. Marking numbers on the number line scaffolds a way of thinking. It shows partial results, the way an operation has been carried out, and what remains to be done.
3. The empty number line fosters the development of more sophisticated thinking strategies. The most basic strategy for many examples would involve counting by ones. Next, structured counting sequences may be used by counting in groups of tens and ones. The empty number line also supports strategies for skilful calculating such as "compensating." For example, in solving 76 – 49, first subtract 50 then add 1. Eventually students' strategies become so sophisticated that they no longer need empty number lines to scaffold mental computation.

This last feature is the reason that I mention empty number lines here. They support children's early thinking and eventually become unnecessary when children's thinking is sufficiently abstract.

Linking manipulatives and abstraction

In fact the step to abstraction from a physical model or manipulative can be a giant leap for some children. It is important to build a bridge with transition activities and materials that require increasing levels of visualisation of the mathematics. For example, creating rectangular arrays to represent repeated addition of numbers that connect to early multiplicative thinking can be enhanced by covering sections of the array to prompt children to imagine the objects that are hidden (Cheeseman, 2004).

A further complication is the question of when not to use manipulatives or materials at all. This is a very vexed issue. Perhaps in some situations children can decide for themselves when not to use manipulatives. During a problem solving session with young children I recently saw a teacher present the problem and ask the children to work on a solution, "using any materials they might need to help them". The option not to use manipulatives was provided.

In conclusion

The use of manipulative materials is perhaps best summed up by the following statement:

> By their very nature, mathematical thoughts are abstract, so any model that embodies them is imperfect and has limitations. The model is not the mathematics — at best it illustrates the mathematical concept under consideration. Helping children to link, connect, or establish meaningful bridges from the model to the mathematics is the challenge, but a rewarding struggle (Reys, Suydam, Lindquist & Smith, 1999, p. 25).

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ERRORS: WINDOWS INTO THE MIND

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Students' errors should be celebrated. Errors provide insight into students' thinking, often to a greater extent than correct solutions. This paper discusses common errors identified from verbal responses made by students explaining their thinking as they dealt with introductory algebra questions. The errors, and students' explanations of their thinking, reveal sources of underlying misconceptions as students struggle to connect their previous understandings with new ideas of algebra. The errors also serve to explain the identified relative difficulties of different algebra questions. Implications for classroom practice that could avoid, or mitigate, these errors are suggested.

Introduction

Students' errors usually have a disconcerting, compelling logic to them. Linchevski and Herscovics (1994) suggested that students' errors indicated a *cognitive gap* — a conceptual hurdle to be cleared before students could progress further in their algebra learning. Common errors identified include those of: inappropriate conjoining of terms (Falle, 2007; Linchevski & Herscovics, 1994; Matz, 1979); the *generalised distribution* of exponents in expressions such as $(x + y)^2$ (Matz, 1979); the perception that a pronumeral has the value of "1" unless otherwise stated; a tendency to dissociate the negative sign from the succeeding number, (Linchevski & Herscovics, 1994); problems with syntactical understanding (MacGregor & Stacey, 1996); and, overgeneralisation, or false generalisations, from arithmetic (and other) contexts (MacGregor & Stacey, 1997). Understanding arithmetic signs, and the equal sign, only as operators (Gray & Tall, 1994; Horne, 2005), has also been identified as contributing to misunderstandings and consequent errors made by novice algebra students.

This paper identifies and discusses errors made by students in junior secondary school in responses to algebra items. It was found that different items attracted different types of errors, and different frequencies of particular errors. It is suggested that analysis of response types, in association with identification of particular errors, can serve as bases for planning learning experiences.

The study

The data discussed in this paper is derived from a larger study, the scope of which is outside the present focus. Students (N = 222) from Years 8 and 9, from three regional secondary schools completed a 40-item algebra survey (test). Items included expressions to be manipulated and equations to be solved. These were taken from the *NSW 7–10 Mathematics Syllabus* (Board of Studies NSW, 2002), and textbooks used by the participating schools. Responses were marked as correct or incorrect. Incorrect responses included those for which students gave no response. Rasch-modelling[7] (Bond & Fox, 2001) of the scores revealed clusters of items centred around three mean difficulty estimates (using *Quest* software, Adams & Khoo, 1996). Errors by every student for each item were counted. These data, together with interviews with students (n = 31), were used to define characteristics of item clusters. For some items, particular errors predominated, whilst in other cases, error patterns were not so clearly marked. Examples of items for which clear error patterns emerged are discussed in this paper.

Items for which typical errors could be identified

The occurrence of a most frequent (typical) error suggests that there is a dominant conceptual pattern — that a single factor is most likely to be operating to produce that error. These items, their difficulty estimates, and the typical errors they illustrate are summarised in Table 1, and discussed under the headings of each error type.

Items 3, 15, 30, 35, 36: Fractions

Items categorised as fractions included expressions or equations with at least one term written as a fraction. All items requiring manipulation of fractions were at a difficulty level approximately equal to the mean ability level of the participants (-0.64 logits)[8], or greater (mean difficulty of fraction items: 0.928 logits). The relatively high level of difficulty of the items involving fractions may be attributed to the difficulties and misconceptions students have with operating on arithmetic fractions (e.g., Stephens & Pearne, 2003).

Item 3 [$a/5$ + $a/10$] attracted 153 incorrect responses, of which there were 27 varieties. Of all the incorrect responses, 42 stated that the answer was "$a/15$", and 33 gave the result as "$2a/15$". In response to being asked what he would do with Item 3 [$a/5 + a/10$], one student responded with the following:

> Well, firstly what I'd do is put it into one fraction. So a plus a is just $2a$, I think. That's right. Then I'd go to five plus ten.

Although this was a common response, more students simply added the denominators to give the answer as "$a/15$". The reasons for this are not clear, except to suggest that students mentally detached the letter from the expression and replaced it once they added the denominator of the fractions. Other students seem to confound adding $1 + 1$ with multiplying 1×1.

Another student used a similar strategy, of treating the numerator and denominator as unconnected numbers when responding to Item 15 [$3p/4 - p/8$], at the same time,

[7] Rasch modelling calculates probabilistic estimates of item difficulty and student ability (success on the test items).
[8] The Rasch estimates are expressed in *logits*; higher values denote more difficult items and more able students.

detaching the *p* from the numerals: “You just go three over four because *p* minus *p* is nothing, so it would probably be three over minus four for the answer.”

Table 1. Items for which typical errors were identified

Item number	Item	Difficulty (logits)	“Typical” error(s)	Error type
3	Simplify: $a/5 + a/10$	1.59	$a/15$ or $2a/15$	
15	Simplify: $3p/4 - p/8$	1.18	$2p/4$	Fraction (mis)conceptions
30	Solve: $x/4 = 12$	-0.93	$x = 3$	
35	Solve: $x + x/3 = 4$	2.17	$x = 6$	
36	If $63/x = 180$, find x.	2.64	$x = 11340$	Integer focus Multiplication makes larger Trial-and-error
6	Simplify: $5a - 2b + 3a + 3b$	0.33	$8a - 5b$	Minus sign
11	Simplify: $8p - 2(p + 5)$	2.28	$6p + 10$	
2	Simplify: $5p - p + 1$	-2.6	$(5 + 1) = 6$	Letter is equal to 1
25	Take n away from $3n + 1$	0.2	$(3 + 1) = 4$	
10	Simplify: $(x + y)^2$	3.56	$x^2 + y^2$	Generalised distribution of exponent
17	Simplify: $(6xy)^2$	0.46	36xy	
13	Simplify: $4r \times 5t \times 3$	-0.34	$4r \times 5t \times 3$	
5	Simplify: $2ab + 3b + ab$	-1.98	$3a5b$ or $6ab$	Two pronumerals Like terms
4	Simplify: $4 \times 5b$	-3.27	$5b \times 4$ or $4 \times 5b$	
9	Simplify: $2ab \times a$	-1.08	$2ab \times a$ or $3ab$	

Other students were confused about the correct notation; so “$3a/10$” became “$3\frac{a}{10}$” (three and one tenth *a*), whilst others seemed to generalise, incorrectly, the notion of *like terms*.

S: Because, um, five and ten are like terms, ’cause you know what they are and the two *a*-s they’re… [the student’s explanation ceased].

Responses to Item 15 [$3p/4 - p/8$] demonstrated similar thinking on the part of participants. However, dealing with quarters and eighths appears, from difficulty estimates, to have been easier than dealing with fifths and tenths, although both Item 3 and Item 15 only required students to find common denominators by doubling strategies. This particular procedure could not be generalised to a situation that required students to use multiples other than doubles.

S: the first one I get, um, 3a over 10, the next one I get 5p over 8 […]
S: Because that is half of that, I need to double [3] that to make it simpler, but if you double the bottom one you have to double the top one as well, so I double the five to make it ten and I double the *a* to make it 2*a*, which means 2*a* plus *a* over ten, which equals 3*a* over ten… [student continues with an explanation of Item 15]
I: So what happens if I give you something like [writes] *t* over seven plus *t* over three?
S: I don't know. I'm not quite sure, um. I get a bit confused. I'm not really sure. I can't really remember how to do it. I think I've done it before, but I can't remember.

Others could add fractions, as in Item 3, albeit incorrectly, but not subtract fractions as in Item 15: "I haven't done minus fractions. I just think you just do 3*p* minus *p* and then four minus eight."

Incorrect responses to Item 30 [$x/4 = 12$] also suggested that students *reacted* to the numbers "4" and "12", rather than read the equation with meaning. The most common incorrect response was "$x = 3$" given by 32 students out of the 45 who answered incorrectly. On the other hand, incorrect responses to Item 35 [$x + x/3 = 4$] indicated that students tended to follow the procedure of "doing the opposite"; multiplying by three the right hand side of the equation only, rather than finding a common denominator first. This procedure resulted in the most common incorrect response of "$x = 6$", given by 42 students out of the 96 students who answered incorrectly. The incorrect response "$x = 1$" was given by 15 students, suggesting that students relied on numeric substitution to find a solution – but failed to keep all the necessary data in working memory. Students simply "doing the opposite", together with a perception that multiplication invariably gives a larger result and a reluctance to deal with fractions also led to many errors for Item 36 [$63/x = 180$].

Items 6 and 11: Dissociation of the minus sign

Incorrect responses to Item 6 [$5a – 2b + 3a + 3b$] indicated that students most commonly tended to "lose track of the signs", as one student confessed during the interview. Of 136 incorrect responses, 38 responses were "$8a – 5b$", and 29 responses were "$2a + 5b$". This difficulty was also evident in the responses to Item 11 [$8p – 2(p + 5)$] where there were 158 errors, 40 of which were the response "$6p + 10$". Losing track of the signs suggests that students fail to understand that arithmetic signs can act as relational symbols, and are meaningfully attached to the succeeding number. Such a misunderstanding can also be seen when students rearrange terms in an expression such as Item 6, changing the position of the terms, but leaving the signs in the same place. The difficulty does not become apparent when expressions have only positively signed terms.

Items 2 and 25: Pronumeral has a value of one unless otherwise stated

Item 2 [$5p – p + 1$] and Item 25 [Take *n* away from $3n + 1$] appear to require the same mathematical understanding. The difficulty estimates suggest that students found Item 25 more difficult. One student only did not attempt Item 2 for which there were 13 different responses out of 49 incorrect responses. Of these, the most frequent error (20 out of 49 responses) was to remove the letter *p* to give the answer as either "5 + 1" or

"6". For Item 25, there were 118 incorrect responses (including 23 non-attempts) with 99 of those offering the answer of either "3 + 1", or "4".

Interview data also suggest that students understand a single letter (e.g., *p*) as having a value of "1" (see also Linchevski & Herscovics, 1994). This might also explain the apparent *conjoining* of terms where an expression such as "5*p* + 1" is simplified to "6*p*". By understanding *p* as equal to 1 the expression "5*p* – *p* + 1" could be changed to "5 + 1" or "6".

Items 10 and 17: *Generalised distribution* of the exponent (Matz, 1979)

Responses to these items indicated that students understood that the brackets grouped the terms. One student interviewed went so far as to explain that $(x + y)^2$ would be $(x + y)(x + y)$ but did not continue the expansion. Meaning can be given to this idea by situating the algebra in the context of area (Norton & Irvin, 2007) and hence developing a generalisable procedure for the expansion of binomials. A similar illustration might also help students understand the distribution of the exponent in the case of $(6xy)^2$.

Pictorial representations however, must eventually be abstracted so that students can develop a more general rule for dealing with expressions with exponents greater than three.

Items 13, 5, 4, 9: Meaning of *like terms*

For Item 4, there were 26 incorrect responses, 12 of which were either those that left the expression as stated, or switched the digits, 4 and 5. Similar incorrect responses were made most frequently to Item 9 [$2ab \times a$] and Item 13 [$4r \times 5t \times 3$]. Students seemed confused about what to "do" with unlike or like terms. Item 13 also challenged students' thinking because few seemed to have encountered examples where more than two terms were to be multiplied, or had two different pronumerals (also Item 39). Students also seemed confused when Item 9 introduced two pronumerals.

Although structurally similar to Item 6 [$5a – 2b + 3a + 3b$], Item 5 [$2ab + 3b + ab$] prompted a great variety of errors, particularly by more able students. Out of 65 incorrect responses, there were 33 different responses. The two most common errors were to give the simplification as "3*a*5*b*" (6 instances out of 65) or "6*ab*" (6 instances out of 65). The introduction of the terms containing *ab* seems to have prompted multiplication responses.

Item 5 (difficulty: -1.98 logits) shares many characteristics with others in this group, but requires a more subtle appreciation of surface similarity, or dissimilarity, between the terms *ab* and *b*. This item was often answered correctly by students who, given the difficulty of the item, would not be expected to have correctly answered it. Interviews suggest that students who used superficial (visual) procedures to manipulate the expression were more likely to answer Item 5 correctly.

One student who gave the correct answer to the item explained: "I'd circle the ones with the same pronumerals and then I'd do a different circle for the ones that were different to the one's I've already circled."

On being asked why the student would circle the pronumerals, the reply was: "Because that's what I was taught in maths."

This student also answered Item 6 [$5a – 2b + 3a + 3b$] correctly, by manipulating it in the same way. The naivety of the strategy seems to afford some protection from confusion about like terms for students.

Discussion and implications for teaching

The errors described represent classes of misconceptions held by students. What does appear to be the clue to many of the errors are aspects of the learning experiences encountered by students. These include: (a) a limited range of examples provided by text books (Vincent & Stacey, 2007) and an associated emphasis on exercises; (b) limited opportunities for students to explain their thinking and hence, limited opportunities for teachers to get inside the minds of their students to discover erroneous thinking; and (c) little focus on careful language use by teachers, and students.

Simple examples that comprise many textbook exercises often avoid having students engage with messy arithmetic, relying on integer solutions (often positive). Few examples have large or fractional coefficients, fractional or negative solutions, or use more than one pronumeral. Many equations to be solved thus rely on students using trial-and-error strategies that do not encourage students to find “an easier (i.e., algebraic) way.”

By having students struggle with more difficult examples of algebra, either as word problems or as symbolic manipulations, the need for algebraic processes becomes apparent. When students articulate their thinking they clarify their understanding and provide teachers with insights into that understanding.

Teacher-language, and its adoption by students, needs to make the mathematics explicit. Expressions such as “simplify” can be misleading — perhaps encouraging the inappropriate conjoining of terms as students try to simplify expressions to a unique term, as their arithmetic experience has required. Students also confuse “simplify” with “make simpler” with “make easier”, thus missing the mathematical importance of rewriting expressions in different ways that might be more useful in particular contexts. “Cancelling” is another term that is confusing, and misunderstood by novices (Falle, 2007).

Interference from learning (MacGregor & Stacey, 1997), where students try to access fragments of partly remembered, loosely associated topics, can explain many errors that appear to be idiosyncratic. Students talking about their thinking, and teachers taking time to analyse errors can reveal the nature of that fragmentary learning.

But talk takes time. More talk might mean fewer exercises completed, and less of the curriculum covered. Understanding the deep nature of many student errors and being able to address these can promote more effective learning, and in the end save time. Teachers accepting errors, and investigating the thinking behind errors can identify teaching moments, on which to build further understanding. Without this foundation, much of the mathematics taught becomes “sound and fury, signifying nothing.”

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TEACHING MATHEMATICAL METHODS CAS IN VICTORIA

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In 2010 all Victorian Certificate of Education (VCE) Mathematics students will be required to sit external examinations in Mathematics that assume the use of a CAS (Computer Algebra System). This presentation will describe the journey of one school in Victoria that has been using CAS since 2000. Examples of student work and anecdotes will be given. An emphasis will be on the change in teaching style that comes about when teaching with CAS.

Two students

Jack: I don't know. We just are pleased that we are allowed to use CAS.
Jai: … 'Cause we are very familiar with it 'cause we have been using it since Year 9. I think our school has been using it for ages longer than other schools.
Jack: … And yeah, we learned very quickly how to use it.
Jai: At Year 9 the teachers taught us how to do stuff, but by Year 10 we knew more than the teachers did.
Jack: …And after we got over putting all these games on it, we started to use it for Maths.
Jai: We got over the novelty value of playing snake in class.
Jack: I don't use my CAS all the time. I think the weaker students do use it all the time.
Jai: … Yeah, 'cause they don't know what else to do.
Jack: … But often it just isn't worth typing it all in. Easier just to do it by hand.
Jai: I use it in Methods a lot.
Jack: … But in Spec I only use it about half the time, but — *wow!* — when it is useful in Spec it is *really* useful!
Jai: I don't know if we will ever know all that it can do. I keep finding new things.
Jack: That doesn't worry me though.

Context

Jack and Jai are Year 12 students in a Victorian regional independent school. The above conversation occurred when they were answering questions from a university group of students about the use of computer algebra systems (CAS) in their mathematics classes. Jai is studying Mathematical Methods (CAS), the middle subject in terms of difficulty in Victoria. Jack is studying Mathematical Methods (CAS) as well as the more difficult subject Specialist Mathematics. In 2010, all VCE classes in Victoria will take externally set assessments where CAS is assumed. There is some ambivalence amongst the teaching community in Victoria about the use of CAS in teaching, learning and assessment. Just last year I asked a group of pre-service secondary mathematics teachers to explore their school's readiness for the introduction of CAS while they were on their first mathematics teaching rounds. Most students came back relaying comments from teachers that they were not prepared for CAS because they thought it was coming on too fast, or that it was all too hard. One student from this group of pre-service teachers found a sealed bag containing a class set of 20 Casio ClassPad 300 CAS calculators under a desk in the staffroom and no-one there seemed to know why it was there or when it arrived.

The Victorian Curriculum and Assessment Authority (VCAA) CAS Pilot Study has been exploring the use of CAS in the senior secondary classroom since 2000. Three schools in Victoria started using CAS using the idea of "congruence" (see Leigh-Lancaster, 2000) of the three aspects of pedagogy, curriculum and assessment, each using a different brand of CAS handheld technology. The change in this approach, compared with what was permitted previously, was the use of CAS in high stakes Year 12 final examinations. This is where it became serious for the students and staff involved. At times this felt like a perilous path to be on, but the initial pilot schools reported that they could only have managed with the strong support shown by the relevant school's governing bodies and the general school community.

> A particular feature for us in a regional country town is the sense of community and knowledge that exists about what our school is doing. It is not uncommon for discussions about the introduction of CAS into the curriculum to emerge at the local shopping centre or at social events where school parents are present… the concern of articulate parents about the possible loss of algebraic skills has been expressed to me in social situations (Garner, 2002, p. 399).

The first external examinations in the new subject, Mathematical Methods (CAS), were held in November 2002 with 78 students enrolled from the three schools. Since then, the number of enrolments has steadily increased, with 2009 being the final year of Mathematical Methods (CAS) being a separate subject (www.vcaa.vic.edu.au).

Computer algebra systems

Articles from both the tertiary and secondary sectors and from Australia and overseas, are increasingly being written focussing on computer algebra systems in the classroom. These articles present both the challenges and possibilities that CAS affords. CAS, either computer based or handheld, has been available for use in Victorian mathematics classrooms for over a decade. The Mathematical Association of Victoria's annual conference proceedings (accessible at www.mav.vic.edu.au) have included articles on the use of CAS in the mathematics classroom since 1986. It is, however, its use in all aspects of the VCE classroom, including examinations, that has promoted CAS to the forefront of mathematics educators' minds. The concerns of teachers about this change have been expressed at various forums across Victoria. Fears that are commonly voiced include the lack of time to learn the new technology, equity issues, potential loss of "by-hand" skills, and fear of devaluing what a teacher does and does well. While some teachers describe working with CAS as "just button pushing," others see the power of

CAS to connect the continuum of the multiple representations of a function and thereby provide a powerful teaching and learning tool. A 2009 education student studying mathematics curriculum reported that she wanted to present the issues around CAS as an assessment because she initially felt that this new tool, for her, devalued her own hard won mathematical prowess.

Every year since 2003, education researchers have reported to the annual conferences of the Mathematics Education Research Group of Australasia (MERGA) about the previous year's VCE examinations and the results gained by CAS and non-CAS students (see Evans, Norton & Leigh-Lancaster, 2003, 2004, 2005; Norton, Leigh-Lancaster, Jones & Evans, 2006, 2007). In reference to the 2004 Mathematical Methods CAS Exam 2 specifically, it was reported that: "What appears to be the case from a general scrutiny of CAS examination results over three years is that CAS does "scaffold" and enable students to engage, and continue to engage, in extended response analysis questions, with comparatively good level of success" (Evans, Norton & Leigh-Lancaster, 2005, pp. 334–335).

Garner and Pierce (2005) described four categories of students using CAS at one of the initial CAS pilot schools. There appears to be a range of student responses to CAS, where at one end of the scale, there is the student who is enabled by CAS (see Garner, 2002; 2004) and the other end, the student who is a powerful and elegant user of the technology, efficiently choosing when and when not to use it (see Pierce & Stacey, 2001; 2004).

It has been reported that the use of CAS in internal and external assessment tasks for the VCE has changed how teachers and students viewed their mathematics:

> The unrestricted use of CAS has led to dramatic changes in pedagogy and assessment… CAS has proved to be a powerful learning tool that allows students to move between numeric, graphical and symbolic representations of a problem. It also allows students to observe patterns and explore concepts… Unrestricted access to CAS has challenged us, as educators, to start inventing new paradigms for the teaching and learning of senior mathematics (Garner, McNamara & Moya, 2003, p. 271).

Two Davids

Consider two students, each named David, who have studied senior mathematics subjects at the same school in recent years. David G is a talented student who could be described as a pure mathematician who selected to use his calculator quite sparingly. David G often kept his calculator closed on the desk in front of him and excelled in by-hand skills. He could, however, be called an elegant user of CAS who opened his calculator when he felt that the technology could be more efficient than his very expert by-hand algebra. David G successfully completed three Units 3 & 4 mathematics subjects as well as University Enhancement Maths during his final school years.

David M is currently also studying three Units 3 & 4 mathematics subjects as well as University Enhancement Maths. He is known as a "techno-wiz" and often shows other students ways to use the CAS. In the most difficult subject, called Specialist Mathematics, David M regularly sees the need for, and investigates ways of, using the CAS in the topics of Complex Numbers, Vectors and Calculus. He often starts the class with statements such as, "Last night I was finding out that…" His expertise is clearly accepted by the students and also by me, their teacher. It should be noted that David M is also an outstanding pure mathematician with concomitant by-hand skills. One example of David M's queries is that the CAS must convert Cartesian to Polar form and he was puzzled that when he tried to convert a complex number, the answer was not as he expected, as shown in Figure 1.

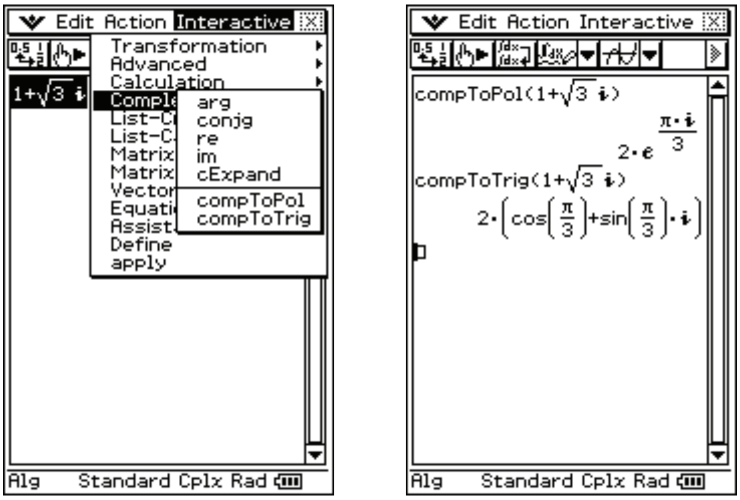

Figure 1. ClassPad 330 calculator showing attempts at Cartesian to Polar conversion.

Converting the complex number $1+\sqrt{3}i$, using the instruction *compToPol*, he found the answer as

$$2e^{\frac{\pi i}{3}}.$$

This was not what he expected and he asked for clarification. After discussion he found that *compToTrig* gave him the answer

$$2\left(\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)i\right)$$

which is clearly equal to, and a commonsense explanation of

$$2\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right).$$

Other student examples

Students talk to the teacher

While showing last year's Application Task to my Mathematical Methods students during Semester 1, 2009, I projected the task onto the screen via the data projector and flipped between the task and the *ClassPad* emulator on my computer. Teaching in this way encourages my students to follow what I am doing, as well as concurrently attempting the task. This approach encourages interaction between me (the teacher), my calculator, their peers and their calculators, their own calculator and their own by-hand skills. Such an environment provides a rich context in which to attempt the task. Comments heard from students were:

Student A: Oh, is that what happens.
Student B: I never knew that.
Student C: I didn't realise that last year.

Student D: Didn't know you could do that. I'd better write that down.
Student E: Yeah, yeah, know all that.
Student F: Oh wow, I could use that. [This student says this every time and then forgets.]
Student G: But there is another way to do that. I will show you.
Student H: I'll show him/her my screen. He/she must have it in degrees.
Student I: I've always used *Graph/Table* for my graphs. Do you think it is better to use *Main*?
Student J: What did you do on the weekend? [This said to a neighbour.]
Student K: Why should I have to rely on this futuristic piece of technology when I will never carry it with me for the rest of my life?

Students talk amongst themselves

The following quotes are from a discussion between students after they had handed in a technology active Year 12 Mathematical Methods test in Semester 1, 2009:

- "I tried to solve something in Complex but it was a big failure."
- "How do you solve that?"
- "Use the calculator! It is only worth two marks so why try to do it the long way?"
- "It solves them using Newton's method. It just tracks to find more solutions."
- "My goodness I don't believe it was so easy."
- "I don't think it is fair that we have had four years using the calculator and Reuben has just arrived."
- "Let's go back to the old days."
- "I used the calculator *so* much and that is why I was so quick. Every possible time I used *solve* for *x*."
- "In Stats I want more feedback — I want more tactile feedback from the keyboard."
- "Basically, when you are stuck, look on your calculator. No… before you are stuck, look on your calculator."

Responses

For every student comment there is a response that could be made. These interactions are an integral part of the multi-faceted CAS classroom. Experience in the VCAA Pilot Study suggests that CAS can increase the engagement of a larger cohort of students. CAS provides teachers and students with an additional tool, which, when used strategically may assist students to explore mathematics and tackle problems that they may previously have found daunting. My answer to Student K above was that the efficient use of CAS technology is where the student's algebra is enhanced by familiarity and the patterns that CAS affords. For the first time, Student K began to be aware that CAS could be a partner in learning rather than just a tool for solving. It is every teacher's hope that students are able to analyse their mathematics rather than merely learn a set of rules one year that they may not be able to apply the next year. If a student can move between multiple representations of data, then clearly a broader understanding has been attained. With the new CAS machines there are the number, graph and symbolic platforms that easily link, as well as statistics and dynamic geometry facilities:

> Classic writing on multiple representations for algebra has considered three representations (e.g., the numerical, symbolic and graphical representations of a quadratic function), but with dynamic geometry used to study real world problems we move seamlessly between five worlds: real world situation, dynamic geometry simulation, numerical representation, symbolic representation and graphical representation (Pierce & Stacey, 2008, pp. 301–302).

Conclusion

An article entitled "Switching Off" in the *Sunday Life* magazine of *The Age* newspaper, described the "electronic merry-go-round" ever-present in our lives. He was specifically writing about the mobile phone but the same can be said for the wider use of technology. The article ended with the observation:

> As the writer George Dyson warned in his book *Darwin Among The Machines*: "In the game of life and evolution there are three players at the table: human beings, nature and machines. I am firmly on the side of nature. But nature, I suspect, is on the side of the machines." (McCulloch, 2009, p.16)

I am always amused when pre-service teachers present summaries of articles from the educational community, and comment easily on the impact of technology all around us, and then those same teachers are most reluctant to use that technology in their own teaching practice. The question could be asked of these pre-service teachers whether they eschew all technology, or is it specific to the classroom.

Teachers are divided as to the pros and cons of technology, with the use of CAS the current debate. Goos and Bennison (2008) delineated teacher attitudes to technology in their classrooms. Items supported by teachers in the study were: "Technology erodes students' basic mathematical skills," "Technology does not add to students' understanding of mathematical concepts," and "It is time-consuming to teach students how to use technology" (p. 116). Goos and Bennison related teacher use of technology to their zone theoretical framework (p. 121). An interesting conclusion of this part of the study echoes many articles about the use of CAS and whether teachers are ready to incorporate them into their classrooms: "It is not clear whether frequent use of (graphics) calculators in the classroom led teachers to develop these beliefs, or teachers already convinced of the benefits of technology simply embraced … calculators when they became available" (Goos & Bennsion, 2008, p. 124).

Teachers in the early part of the VCAA Pilot Study reported that the CAS classroom "sinks or swims with the teacher" (see Garner & Leigh-Lancaster, 2003; Garner, 2004), and there was some discussion as to whether the easy use of CAS was dependent on a certain personality type of the teacher. The resentment commented on below can be seen to be influenced by the teacher's role modeling when using the technology.

> [S]ome (students) have openly resented being taught an alternative method of answering a question, fearing that the new knowledge might get confused with the rote patterns they had memorised and which had given them at least limited success in the past. … The mathematics of the future is about choice — and students require exposure to the various features of their CAS so they may develop flexibility in approaching a problem. (Wander, 2008, p. 378)

It has been said that there is a misunderstanding between the teachers who actually have to carry out new educational practices and the people who are pushing, and theorising about, the practices. This argument reflects a common theme at mathematics education conferences where researchers and practising teachers meet together. Having been concurrently in both fields as the practising teacher, and also the researcher, the dilemma truly exists. I would say that it is most worthwhile for teachers to branch out and learn about their teaching in the context of educational theories. This can only enhance their teaching in the classroom. I would also encourage researchers to include teachers as partners in research for the mutual benefit of both parties. In a presentation at ICME-11 in Monterrey, Mexico in July 2008, Garner reported:

> Since change is upon us, it is imperative that this change is captured to suit students' learning of mathematics and to see if we can use these new tools to increase breadth and depth of the mathematics we teach… This is an opportunity for the maths education community to use CAS in any way which may improve students' access to, and success with, mathematics (Garner, 2008).

Acknowledgment

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THE PREVALENCE OF YEAR 7 STUDENTS WHO HAVE NOT DEVELOPED A RELIANCE ON RETRIEVAL STRATEGIES FOR SIMPLE ADDITION

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The approach taken in this study produced an estimate of the prevalence of Year 7 students who have not developed a reliance on retrieval strategies for simple addition. Of the participants who scored below the benchmark on Western Australia's state-wide numeracy achievement test, nearly half displayed this type of number fact difficulty. Low performing Year 7 students would rarely receive instruction in simple addition but this research suggests that, for at least half of these students, this may be exactly what they need.

Introduction

Developing *a reliance on retrieval* for simple addition problems is a complex learning process requiring an understanding of number principles and correct practice with efficient counting strategies (Hopkins & Lawson, 2002). Simple addition performance dominated by (a) correct answers and (b) the use of *retrieval strategies* – encompassing *direct retrieval* (just knowing the answer) and *decomposition strategies* (deriving the answer by applying a known fact) – is referred to in this paper as being indicative of a reliance on retrieval strategies.

It is clear from the research literature and from classroom observations that some students do not follow a pattern of typical development: they do not develop a reliance of retrieval strategies for simple addition but rely on counting on from the minimum addend (referred to as using the *min counting* strategy) at an age well beyond their peers (Geary, 2004; Robinson, Menchetti & Torgesen, 2002). What is not clear from the research literature is the prevalence of this problem.

The aims of this study were to (a) develop criteria for characterising typical and atypical simple addition performance among Year 7 students to identify those who have not developed a reliance on retrieval strategies (b) establish the validity of the criteria, (c) estimate the prevalence of Year 7 students with this type of number fact difficulty in the state of Western Australia.

Background

As well as the frequent use of the min counting strategy, the simple addition performance of students who have not developed a reliance on retrieval strategies is likely to be characterised by (a) frequent inaccurate performance either when counting or retrieving (Geary & Brown, 1991; Geary, Hamson & Hoard, 2000) *or* (b) accurate but slow counting performance (Hopkins & Lawson, 2006).

Developing criteria for identifying students who have not developed a reliance on retrieval strategies for simple addition is complicated by unclear findings as to when a reliance on retrieval can be expected for typically achieving students and by a difficulty specifying what, exactly, constitutes "a reliance." The research suggests that by the time students are around 12 years old, if the min counting strategy dominates the strategy mix then performance reflects more than a delay in development. It is further complicated by findings suggesting that some students (labelled "perfectionists" by Siegler, 1988) retrieve answers but do not state the answer because of a high confidence threshold; they choose to count on but they do not *have* to count on. For these reasons, criteria were not developed *a priori*; instead, data capturing simple addition performance were scrutinised to find suitable criteria and cut-off scores for classifying simple addition performance.

The objective was to classify simple addition performance into four categories: *typical*, *perfectionist*, *inaccurate* and *slow counting*. While not all low achieving mathematics students would be expected to show an over reliance on counting for simple addition (given that numerous factors may account for low achievement in mathematics), the literature indicates that students who have not developed a reliance on retrieval strategies are highly likely to be low achieving in mathematics. From the literature it can be hypothesised that students displaying inaccurate and slow counting performance will show poor achievement on a numeracy assessment. If this is found to be so, then the validity of the criteria for identifying these groups is supported and these students will be referred to as having *not* developed a reliance on retrieval strategies.

Method

In total, data capturing simple addition performance were collected for 200 Year 7 students attending 13 Government primary schools in the Perth metropolitan area, in the state of Western Australia. The purposeful sampling procedure involved randomly selecting primary schools from four metropolitan districts that were chosen to reflect the range of schools found in the metropolitan area based on socio economic status. The 13 principals received an information letter about the study in the mail, and subsequently returned signed consent forms to participate. Information sheets were then sent to parents of Year 7 students via the schools. Consent forms with parent and student signatures were obtained from all participants prior to testing.

Numeracy achievement scores for each participant were based on results from the Western Australian Literacy and Numeracy Assessment (WALNA). A specifically designed computer program was used to present a set of simple addition problems and record the strategies used by each participant to complete the problems. Accuracy and the time taken to perform each problem were also recorded.

Students worked one-on-one with a research assistant as they completed a set of 36 simple addition problems. A research assistant initiated the presentation of each problem by pressing the space bar; unbeknown to the student, this also activated a timer. The students were asked to solve the problem and respond verbally with the answer. As soon as an answer was given, the research assistant pressed the space bar a second time to stop the timer and activate a display requiring students to input their answer. The students were then prompted to indicate what strategy they had used by selecting one of four options: “I counted,” “I just knew it,” “I did something else,” (explained to students as meaning a decomposition strategy) or “I don’t know.” This procedure was repeated until a set of problems was completed. The problems were presented in random order and included all single-digit addition problems with addends greater than 1 (2 + 2 to 9 + 9), written in the form “$m + n =$“ where $m \leq n$.

The accuracy of self-reports to identify strategies used for mental addition has been confirmed by Siegler (1987). The approach used in the present study of identifying strategy use based on self-report and observation has been widely used in studies involving simple addition (e.g., Canobi, Reeve & Pattison, 1998; Geary et al., 2000; Hopkins & Egeberg, in press).

Results

The study cohort comprised 200 Year 7 students, 116 of which were female (58%), around 12 years of age (M = 12.38 years, SD = 0.43 years). Numeracy achievement scores for the study cohort (M = 473.20, SD = 76.646) did not differ significantly from the population’s mean score (M = 472) [$t(196) = 0.219$, $p = 0.827$]. The percentage of the population below the numeracy benchmark score was 19.1% and the percentage of the study cohort who achieved a numeracy score below the benchmark was 16.5%. These indicators suggest that the numeracy achievement scores for the study cohort were slightly higher than for the population but that the study cohort can be considered representative of the population.

Establishing criteria

The number of problems in the 36-problem set that were performed correctly using retrieval strategies by each participant ranged from 2 problems to 36 problems (M = 25.23, SD = 7.66). A histogram showing the number of times a retrieval strategy was correctly recorded for each student was inspected to see if it was possible to identify an appropriate cut-off to establish the first criterion. There was no obvious delineation. A cut-off of 17 correct problems directly retrieved or derived represented the point on the distribution corresponding to one standard deviation below the mean for the study cohort (and one half of problem set) and was chosen for the first criterion: students used a retrieval strategy correctly on 17 or less problems in the set.

The number of problems directly retrieved correctly ranged from 2 to 36 problems (M = 21.93, SD = 7.73) and the number of problems *derived* (that is, for which a decomposition strategy was used) ranged from 0 to 21 problems (M = 3.31, SD = 4.02). A prominent characteristic of differences in simple addition skill was whether or not participants used decomposition at all. From these data, the second criterion was chosen; that no problem in the set was correctly derived.

The frequency of incorrect trials for each participant indicated that the majority of students performed all problems correctly (42.5%) or made one error (29%). The number of problems in the set that were performed incorrectly ranged from zero to thirteen problems ($M = 1.33$, $SD = 1.913$). Again there was no obvious delineation in the distribution to suggest a cut-off value. A third criterion was chosen: that the number of incorrect trials was four or greater. This value represented a cut-off corresponding to one and a half standard deviations above the mean for the study cohort (a more generous cut-off than that chosen for the first criterion, to take into consideration errors that may be due to pressure imposed by collecting data).

Based on these three criteria, students' simple addition performance was categorised into three atypical groups. Student performance was categorised as indicating *slow counting* if the first and second criteria were met but not the third criterion: that is, students correctly used retrieval strategies on half or less than half of the problem set (showing a weak reliance on retrieval) and displayed no correct use of a decomposition strategy on any problem, but generally responded with a correct answer. Students were categorised as displaying *perfectionist* performance if they met the first criterion (indicating they had a weak reliance on retrieval) but not the second criterion (indicating at least some correct use of decomposition) and not the third criterion (they generally responded with a correct answer). Students who met the third criterion (that is, they recorded four or more incorrect trials) were categorised as displaying *inaccurate* performance. Table 1 summarises the categorisations into the three atypical groups. Otherwise students were characterised as showing *typical* performance.

Table 1. Definition of the three atypical groups.

	Students used a retrieval strategy correctly on 17 or less problems in the set	No problem in the set was correctly derived (i.e., no decomposition)	The number of incorrect trials was four or greater
Slow Counting	✓	✓	✗
Perfectionist	✓	✗	✗
Inaccurate	-	-	✓

Students referred to as perfectionists are argued to be using counting because they have a high confidence threshold for retrieval. They are more confident using counting to arrive at the right answer but they *can* retrieve answers, unlike students with a number fact difficulty who have not built a peaked distribution of correct answers in memory allowing retrieval to occur (Siegler & Shipley, 1995). Students in the perfectionist and slow counting groups both displayed a weak reliance on retrieval strategies (or an over-reliance on min counting) but were distinguished by the criterion of whether or not they used a decomposition strategy. Students in the slow counting group did not derive a correct answer on any trial and students in the perfectionist group correctly derived an answer on at least one trial. Given that the use of a decomposition strategy requires the retrieval of an answer to a simple addition problem, this criterion was used to distinguish between these two groups.

Criteria validity

The distribution of numeracy achievement scores for each of the four groups of students categorised by simple addition performance is displayed in Figure 1.

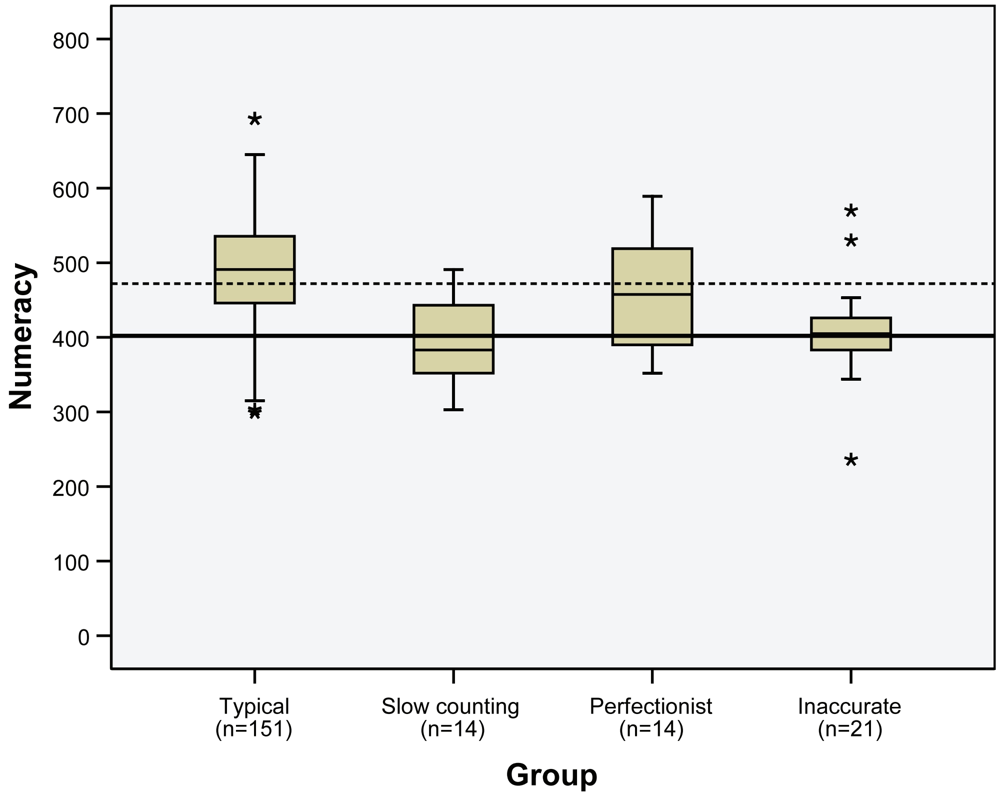

Figure 1. Boxplots displaying numeracy achievement scores for students categorised according to simple addition performance. Asterisks represent outliers. The weighted black like represents the benchmark numeracy score for the population (WAMSES=401) and the dotted line represents the mean numeracy score for the population (WAMSES=472).

An ANOVA revealed a significant difference in mean numeracy achievement scores across student groups [$F(3,192) = 15.359$, $p < 0.000$; $\eta^2 = 0.194$]. The measure of effect size indicated that approximately 19% of variance in numeracy achievement scores could be explained by differences between how students performed simple addition. Planned contrasts were made between the typical group and atypical groups.

Students displaying typical performance scored significantly higher on the numeracy assessment (M = 490.83, SD = 69.26, n = 151) than students characterised by slow counting [M = 395.85, SD = 63.43, n = 14, $t(159) = 4.770$, $p < 0.001$)] or students displaying inaccurate performance [(M = 405.81, SD = 66.43, n = 21; $t(167) = 5.290$, $p < 0.001$]. These findings support the validity of the criteria used to identify these two atypical groups as inaccurate and slow counting were associated with low mathematics achievement, as hypothesised.

Students displaying typical performance scored higher on the numeracy assessment (M = 490.83, SD = 69.26, n = 151) than students characterised by perfectionist performance (M = 459.71, SD = 79.50, n = 14) but the difference in mean scores was found to be non-significant [$t(160) = 1.586$, $p = 0.115$)]. This finding suggests that perfectionist performance is unlikely to be associated with low achievement and therefore does not appear to constitute a number fact difficulty.

The mean numeracy achievement score for the slow counting group was compared with the score for the perfectionist group and was found to be significantly lower [$t(25) = -2.296$, $p = 0.030$]. This finding provides support for the validity of the criterion used to distinguish between perfectionists and slow counters, as a difference in numeracy achievement was found between these two groups. Evidence supporting the validity of the criteria for identifying these two groups was further sought by examining differences in reaction times (RTs) to min counting trials.

A telling characteristic of students with a number fact difficulty is that they rely on min-counting for problems with minimum addends of 2, 3 and 4, and produce reaction times on these trials that are often outside the temporal limit of working memory (Hopkins & Lawson, 2006). Although students in the perfectionist group are also likely to also use a min-counting strategy for these problems, there is no reason why they would display a similar pattern of RTs. An inspection of RTs revealed these problems were generally counted on within a 3s span of working memory by the perfectionist group but not by the slow counting group.

The errors made by each of the 21 students in inaccurate group were examined for error patterns. The number of errors for students in this group ranged from 4 to 13 problems ($M = 5.9$). The most frequent problem that produced an error was 5 + 8 (8 of the 21 students performed this incorrectly) and the second most frequent problem was 5 + 7 (7 of the 21 students performed this incorrectly). Errors were coded into three categories: over by one, under by one, and other (indicating that the error was out by two or more). The frequency of the error types according to strategy are displayed in Table 2.

Table 2. Strategy frequency and error type for incorrect trials made by students in the number fact difficulties (inaccuracy) group.

	Over by 1	Under by 1	Other	TOTAL
Counting	28	21	32	81 (65.8%)
Retrieval	7	13	9	29 (23.6%)
Decomposition	1	1	7	9 (7.3%)
Don’t know	0	2	2	4 (3.2%)

Most commonly, a min-counting strategy was used on incorrect trials, indicating that these students were not just simply guessing the answer. They were using a min-counting strategy, which was just as likely to produce an answer over by one, as an answer under by one, or an answer out by more than one. By inspection of incorrect answers given, the there is some evidence to suggest that a common procedural bug with the min-counting strategy involved counting on the same addend twice: out of 81 trials where counting produced an error, on 27 trials (33.33%) the answer given was twice that of one of the addends.

Prevalence

Among a representative sample of 200 Year 7 students, 35 students (17.5%) displayed a simple addition number fact difficulty characterised by slow counting or inaccuracy. Of the 99 participants who scored below the average numeracy achievement score for the population, 31 (31.3%) displayed this number fact difficulty; of the 33 participants who scored below the benchmark, 16 (48.5%) displayed this difficulty. In comparison, among the 97 participants who scored at or above the population mean, only three (3.1%) displayed this difficulty[9]. It is important to note that of the 35 students identified as displaying a simple addition number fact difficulty, the performance of 21 students (60%) was characterised by inaccuracy and the performance of 14 students (40%) was characterised by accurate but slow counting. While both inaccurate and slow counting impede the development of a reliance on retrieval strategies, it is likely that inaccurate performance will be easier to address than slow counting.

Discussion

It is easy to imagine a strong, iterative relationship between procedural knowledge associated with simple addition and conceptual knowledge associated with number. It is likely that students identified by their simple addition performance as being inaccurate or slow counting have not gained valuable experience with numbers to develop or reinforce basic number concepts associated with part-whole relations. A number concept that has received less attention in the research, but which appears to be strongly related to simple addition and critical for mathematical development, is that of place value. Students in the present study were making counting errors on problems that added to ten, e.g., 4 + 6 = 8 and arriving at 10 for problems that did not, e.g., 4 + 5 = 10. Such inaccurate performance undermines the development of even the most foundational understanding of place value concepts.

The approach taken in this study allowed researchers, for the first time, to estimate the prevalence of low achieving Year 7 mathematics students who have not developed a reliance on retrieval strategies for simple addition. The prevalence is particularly striking as simple addition skill is rarely a focus of instruction beyond Years 2 or 3. Students who do not attain the numeracy benchmark in Year 7 in Western Australian Schools would rarely receive instruction with such a basic skill as simple addition, but this research suggests that for at least half of these students, this may be exactly what they need.

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[1] Four participants did not sit the numeracy component of the WALNA

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MEASURING TEACHING EFFECTIVENESS IN LOWER SECONDARY MATHEMATICS CLASSES

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Developing appropriate measures of teaching effectiveness and teaching quality are imperative for future research to be able to validate (or refute) findings and theories pertaining to excellence in teaching secondary mathematics. One aim of our research project was to examine the appropriateness of using value added estimates of student achievement gains as a measure of teacher effectiveness in Year 8 mathematics classes. The findings revealed important issues for consideration by anyone involved in, or influenced by, the measurement of effectiveness.

Introduction

In documents outlining standards for excellence in mathematics teaching (AAMT, 2006; NCTM, 2000), the idea of teaching *quality* is often equated with the idea of teaching *effectiveness*. If teaching quality is considered in terms of what the teacher does and teacher effectiveness is considered in terms of student outcomes, then these two constructs appear quite distinct. A critical question requiring empirical investigation then becomes apparent: does teaching quality (as it is defined or promoted to be) lead to (predict or explain) teacher effectiveness?

The Trends in International Mathematics and Science Study (TIMSS) data and 1999 TIMSS Video Study (Hiebert et al., 2003) drew attention to a need to improve the quality of teaching in lower secondary mathematics classes (Desimone, Smith, Baker & Ueno, 2005; Hollingsworth, Lokan & McCrae, 2003; MacNab, 2000). In addition, teachers, schools, and education departments are becoming more and more accountable for improving student outcomes. While many may look to the research for answers, a close examination of the literature reveals few published studies have examined whether or not the teaching practices commonly associated with teaching quality or excellence actually account for teacher effectiveness as measured by student achievement gains (Hill, Rowan, Lowenberg & Ball, 2005; Schoen, Cebulla, Finn & Fi, 2003). Limiting factors have been the development of appropriate measures of teaching quality based on teaching practice, and appropriate measures of teacher effectiveness based on achievement gains made by students.

Our research project (Louden, Rohl & Hopkins, 2008) explored the relationship between children's growth in literacy and numeracy, and teachers' classroom teaching practices. The numeracy section comprised two parts: in Part One we examined the appropriateness of using value added estimates of student achievement gains as a measure of teacher effectiveness and in Part Two we outlined the development of a new observation schedule, referred to as the Teaching of Mathematics Observation Schedule (ToMOS), that could be used *in situ* to help identify and measure specific teaching actions that are commonly associated with quality teaching. A summary of findings from Part One is presented in this paper.

Measuring teacher effectiveness

The objective of Part 1 of this study was to undertake a value-added analysis of student achievement gains in mathematics to obtain a measure of teacher effectiveness for participating Year 8 teachers.

Contemporary value-added models are often used to partition variance in student achievement gains into different levels (e.g., the student level, teacher or classroom level, and the school level). There have been many attempts to quantify the effect teachers make based on the amount of variance to be explained at the teacher or classroom level and limitations of this approach have been noted. For example Rowan, Correnti & Miller (2002) produced markedly different estimates depending on the model of analysis used. The secondary school setting for this research project provided an interesting scenario to explore further the appropriateness of using a multilevel analysis of student achievement gains to indicate teacher effectiveness. Participating teachers worked in a secondary school environment where some were required to teach more than one Year 8 mathematics class. If estimates of student achievement gains at the class level are indicative of teacher effectiveness then the analysis should produce similar estimates for the same teacher across different classes. This prediction was tested.

Method

A convenient random sample of government secondary schools across Western Australia was contacted and selected based on a positive response from the head of the mathematics departments to an information flyer detailing the study. The 24 participating schools included 17 metropolitan schools and 7 country schools. The heads of mathematics departments from each school provided survey information for their school. In total 95 teachers agreed to participate and provided written consent indicating this. Participating teachers were asked to provide survey data on their qualifications and experience, and data on their students' backgrounds. The number of teacher participants from the same school ranged from one teacher to eleven teachers. The number of Year 8 mathematics classes taught by the same teacher ranged from one class to six classes. Overall, 132 classes of students participated in the research.

A numeracy achievement score for each student was collected towards the end of Year 8 and matched with their achievement score obtained for the numeracy part of the Year 7 Western Australian Literacy and Numeracy Assessment (WALNA). The Year 8 numeracy assessment designed specifically for this study consisted of items drawn from a bank of WALNA items yet to be used at the state-wide level and was comparable with the Year 7 WALNA in terms of length and composition.

Results

A two-level variance components model was fitted to the Year 8 numeracy achievement scores. The unconditional model (Model A) indicated that the proportion of variance in numeracy achievement was partitioned into 68.1% of variance at the student level and 31.9% at the class/teacher level. The conditional model (Model B) incorporated Year 7 numeracy scores as a predictor of Year 8 numeracy scores and accounted for 57.7% of variance in numeracy scores. Model C incorporated both Year 7 numeracy scores and ATSI status as predictors of Year 8 numeracy scores and accounted for 58.2% of variance attributed to possible student effects on achievement gain. Model C was used to estimate class residual scores. Adjusted mean-point estimates of residuals for each class were then calculated. These class residuals are displayed in Figure 1.

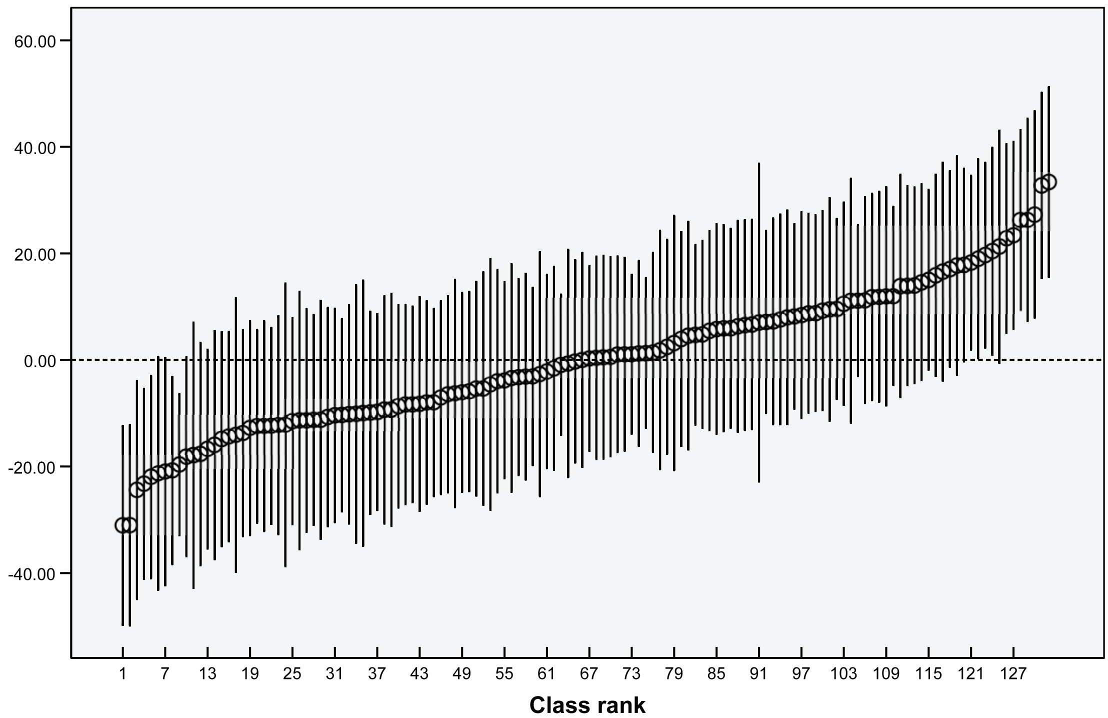

Figure 1. Each circle represents the adjusted residual calculated for each classroom. The whiskers indicate 95% confidence intervals. The y-axis represents residual scores measured in WAMSES.

A conservative approach to try and identify highly effective teachers was taken. Eleven classes where the achievement gain adjusted residual and the lower boundary of the 95% uncertainty interval was above that expected were highlighted. Taking a similar approach to identify less than effective teachers, an examination of classes where the achievement gain adjusted residual and the upper boundary of the uncertainty interval was below that expected, revealed seven classes. This left 114 classes, which were identified as achieving as expected growth. Descriptive statistics for classes identified as showing exceptional achievement are presented in Tables 1 and 2.

Table 1. Classes achieving higher than expected learning gains.

Class Rank	Class Residual	SEI	School Area	Year 8 Streamed	Year 7 Class mean*	Class size (students)
132	33.37	105.30	Metro	Yes	503.91	25
131	32.74	100.97	Metro	No	543.92	28
130	27.31	98.59	Metro	No	517.63	21
129	26.30	99.23	Metro	No	492.58	21
128	26.29	109.48	Metro	Yes	506.11	28
127	23.37	105.3	Metro	Yes	473.50	26
126	22.79	109.48	Metro	Yes	564.79	26
124	20.39	107.20	Metro	No	565.78	21
123	19.66	105.30	Metro	Yes	499.07	28
122	18.98	100.97	Metro	No	441.20	24
121	18.27	95.43	Metro	-	541.27	30

*Year 7 Class grand mean = 457.26.

Table 2. Classes achieving lower than expected learning gains.

Class Rank	Class Residual	SEI	School Area	Year 8 Streamed	Year 7 Class mean*	Class size (students)
1	-31.04	98.59	Metro	No	451.95	22
2	-31.02	96.88	Country	No	400.29	27
3	-24.42	94.11	Metro	-	411.73	15
4	-23.27	95.39	Country	No	438.30	25
5	-21.98	98.59	Metro	No	428.42	21
8	-20.77	95.59	Country	No	448.83	24
9	-19.67	104.98	Metro	Yes	452.73	55

*Year 7 Class grand mean = 457.26.

It is interesting to note that *none* of the eleven classes achieving higher than expected learning gains were initially low achieving classes: Year 7 class means were around or above (mostly well above) the grand mean for the study cohort. This could indicate that (a) classes of initially low achieving students are more likely to be allocated teachers who are least capable of producing exceptional achievement gains, or (b) it is harder to produce exceptional achievement gains in classes comprising low achieving students. The second explanation is supported by the finding that class residuals were significantly correlated with Year 7 class means ($r = 0.483$, $p < 0.000$). This means that around 23% of variance in achievement gains can be explained by prior performance. The second explanation is also supported by the marked difference found in residuals for classes that were taught by the same teacher. This is best illustrated using the extreme cases where classes were considered to make exceptional learning gains.

Five of the eleven teachers who taught classes identified as achieving higher than expected learning gains taught at least one other class participating in the study. In Table 3, the residuals calculated for these classes are displayed, along with information pertaining to each teacher. Two of the seven teachers who taught classes that achieved lower than expected learning gains also taught more than one Year 8 class and these residuals are displayed in Table 4.

Table 3. Residuals for classes taught by the same teacher.

Teacher No. of years teaching mathematics (Qualification in maths ed.)	Class Rank	Residual	Year 7 Class Mean*	Class size (students)
Teacher A	130	27.31	517.63	21
30 years (degree + DipEd)	71	1.05	444.77	25
Teacher B	128	26.29	506.11	28
3.5 (Masters)	98	8.77	477.60	24
Teacher C	127	23.37	473.50	26
24 (Masters)	46	-7.08	432.26	23
Teacher D	124	20.39	565.78	21
20 (degree + DipEd)	86	5.94	518.11	20
Teacher E	121	18.27	541.27	30
17 (degree + DipEd)	16	-14.37	431.00	22

*Year 7 Class grand mean = 457.26

Table 4. Residuals for classes taught by the same teacher.

Teacher No. of years teaching mathematics (Qualification in maths ed.)	Class Rank	Class Residual	Year 7 Class mean*	Class size (students)
Teacher G	3	-24.42	411.73	15
7 (no maths qualification)	25	-11.52	432.83	19
	76	1.41	432.57	20
Teacher F	8	-20.77	448.83	24
1 (degree + DipEd)	84	5.47	448.95	24

*Year 7 Class grand mean = 457.26

These findings highlight the limitations associated with using a multilevel analysis and adjusted class residual (representing student achievement gains) as a measure of teacher effectiveness.

Discussion

Developing appropriate measures of effectiveness and quality are imperative for future research to validate (or refute) findings and theories pertaining to excellence in teaching secondary mathematics. The results of the value-added analysis revealed that class residuals were not appropriate indicators of teacher effectiveness. The findings revealed considerable variance in estimates of student achievement gains made by different classes taught by the *same* teacher. At best, class residuals may be interpreted to indicate levels of effective *teaching* rather than levels of effective *teachers*. The distinction is an important one for both researchers and teachers to consider.

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TIMSS 2007: PERFORMANCE IN MATHEMATICS OF EIGHTH-GRADERS FROM ASIA-PACIFIC COUNTRIES

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TIMSS (Trends in International Mathematics and Science Study) 2007 is the fourth in a series of international mathematics and science assessments conducted every four years. TIMSS is designed to provide trends in fourth and eighth grade mathematics and science achievement in an international context. This paper reviews the mathematics achievement of eighth-graders from nine Asia-Pacific countries (Australia, Chinese Taipei, Hong Kong, Indonesia, Japan, Korea, Malaysia, Singapore and Thailand) that participated in TIMSS 2007. The achievement data show that Chinese Taipei, Korea, Singapore, Hong Kong and Japan ranked as the top five countries respectively. The paper also reviews some classroom characteristics and instruction practices of these Asia Pacific countries to explore factors that may explain student achievement, in particular of Australian students.

Introduction

TIMSS (Trends in International Mathematics and Science Study) 2007 is the fourth in a series of international mathematics and science assessments conducted every four years. TIMSS is designed to provide trends in fourth- and eighth-grade mathematics and science achievement in an international context. In TIMSS 2007, 49 countries participated at the eighth grade level. The Asia-Pacific countries that participated at the eighth grade were Australia, Chinese Taipei, Hong Kong, Indonesia, Japan, Korea, Malaysia, Singapore and Thailand. Data was collected from participating students, their teachers and school leaders with the help of assessment tasks and background questionnaires. The TIMSS 2007 International Mathematics Report (Mullis, Martin & Foy, 2008) is a comprehensive report of all the data collected and analysed for mathematics assessment of grades four and eight students. This paper draws on the data from the report and discusses the achievement of grade eight students from Asia Pacific countries. It also examines the respective data on classroom characteristics and instructional practices to explore factors that may explain student achievement, in particular of Australian students.

Student participants and tests

Representative samples of eighth graders participated in the study. They were in their eighth year of formal schooling with average ages ranging from 13.9 to 14.5 years. The TIMSS 2007 tests (Ruddock, O'Sullivan, Arora & Erberber, 2008) comprised of both mathematics and science items. Fourteen different booklets containing a selection of the 215 mathematics and 214 science items were administered to the students. Each student completed the test in one booklet. Testing time was 90 minutes. The 215 mathematics items (117 multiple choice and 98 constructed response type) were classified by content domain and cognitive domain. The four content domains were Number, Algebra, Geometry, and Data and Chance, while the three cognitive domains were knowing, applying and reasoning (Mullis, Martin, Ruddock, O'Sullivan, Arora & Erberber, 2005).

Mathematics achievement

Table 1 shows the ranking and average scale scores of the nine Asia Pacific countries that participated in TIMSS 2007. Five of these nine countries were in the top five ranks. Chinese Taipei was in the first position followed by Korea, Singapore, Hong Kong and Japan. Australia was 14th, Malaysia was 20th, Thailand was 29th and Indonesia was 36th in position. The average scale scores of the five Asia-Pacific countries that were ranked in the top five positions were significantly higher than the international average. There was no significant difference between the average scale scores of Chinese Taipei, Korea and Singapore.

The human development index is indicative of how developed a country is. The nine Asia-Pacific countries with decreasing human development index were Australia, Japan, Hong Kong, Chinese Taipei, Singapore, Korea, Malaysia, Thailand and Indonesia. From Table 1, it is apparent that the human development index may be a predictor of the average scale scores of the grade eight participants if we consider Australia as an anomaly.

Table 1. Rank, average scale scores and human development index of Asia-Pacific countries.

Country	**Rank**	**Average Scale Score**	**Human Development Index**
Chinese Taipei*	1	598 (4.5)	0.932
Korea*	2	597 (2.7)	0.921
Singapore*	3	593 (3.8)	0.922
Hong Kong	4	572 (5.8)	0.937
Japan	5	570 (2.4)	0.953
International Average	-	500	-
Australia	14	496	0.962
Malaysia	20	474 (5.0)	0.811
Thailand	29	441 (5.0)	0.781
Indonesia	36	397 (3.8)	0.728

Standard errors are shown with ( ).

* No significant difference between average achievement

International benchmarks of mathematics achievement

The international benchmarks presented as part of the TIMSS 2007 data (Mullis, Martin & Foy, 2008) help to provide participating countries with a distribution of the performance of their eighth-graders in an international setting. For a country the proportions of students reaching these benchmarks perhaps describe certain strengths and weaknesses of mathematics education programs of the country. The benchmarks delineate performance at four points of the performance scale. Characteristics of students at each of these four points are elaborated in the next section.

Table 2 shows the percentage of students from the nine Asia-Pacific countries reaching TIMSS 2007 international benchmarks of mathematics achievement. It is worthy to note that 45% of students from Chinese Taipei were at the Advanced benchmark and in all the five Asia-Pacific countries that were ranked as the top five, more than 60% of their students were at the High benchmark level and almost 90% of their students were at the Intermediate benchmark level. In contrast, particularly for Australia, a country that is comparable in human development index with the five Asia Pacific countries that have ranked top, only 6% of their students were at the Advanced Benchmark level and 24% at the High benchmark level. For Malaysia, 50% of the students were at the Intermediate benchmark level and for both Thailand and Indonesia 34 % and 19% respectively were at the Intermediate benchmark level.

Table 2. Percentages of students reaching TIMSS 2007 international benchmarks of mathematics achievement.

Country	**Advanced benchmark (625)**	**High benchmark (550)**	**Intermediate benchmark (475)**	**Low benchmark (400)**
Chinese Taipei	45 (1.9)	71 (1.5)	86 (1.2)	95 (0.6)
Korea	40 (1.2)	71 (1.1)	90 (0.7)	98 (0.3)
Singapore	40 (1.9)	70 (2.0)	88 (1.4)	97 (0.6)
Hong Kong, SAR	31 (2.1)	64 (2.6)	85 (2.1)	94 (1.1)
Japan	26 (1.3)	61 (1.2)	87 (0.9)	97 (0.3)
Australia	6 (1.3)	24 (1.8)	61 (1.9)	89 (1.0)
Malaysia	2 (0.5)	18 (2.1)	50 (2.7)	82 (1.9)
Thailand	3 (0.8)	12 (1.7)	34 (2.2)	66 (2.0)
Indonesia	0 (0.2)	4 (0.6)	19 (1.4)	48 (1.9)
International Median	2	15	46	75

Standard errors are shown with ( )

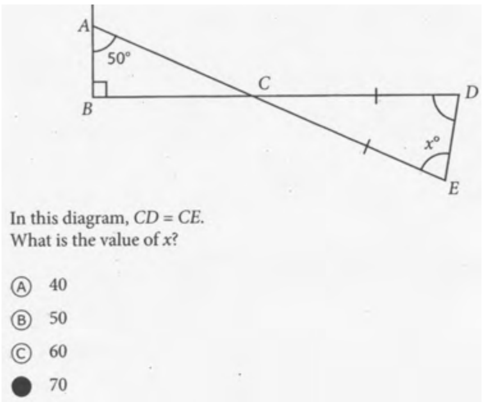
What can students at each of these benchmarks do?

Table 3 shows what students at each of the four international benchmarks were able to do.

Table 3. Descriptions of international benchmarks.

International benchmark	Characteristics of students
Advanced - 625	Students can organise and draw conclusions from information, make generalisations, and solve non-routine problems.
High - 550	Students can apply their understanding and knowledge in a variety of relatively complex situations.
Intermediate - 475	Students can apply basic mathematical knowledge in straightforward situations.
Low - 400	Students have some knowledge of whole numbers and decimals, operations, and basic graphs.

Figure 1 shows an item of the advanced benchmark that students reaching the benchmark are likely to answer correctly.

Content Domain: Geometry **Description:** Uses properties of isosceles and right angled triangles to find the measure of an angle	**Country**	**Percent Correct**
	Singapore	75 (1.7)
	Chinese Taipei	73 (2.2)
	Korea	73 (1.8)
	Japan	71 (1.9)
	Hong Kong SAR	69 (2.8)
	Thailand	36 (2.1)
	Malaysia	36 (2.7)
	Australia	32 (2.8)
	Indonesia	19 (2.0)
	International Avg	32 (0.3)

Standard errors are shown with ( ).

Figure 1. An advanced international benchmark item.

Classroom characteristics and instructional practices

This section reviews the analysis of some of the data collected by the background, teacher and student questionnaires for TIMSS 2007 and reported in Mullis, Martin and Foy (2008). Specifically, the data concern class size, teachers' emphasis on mathematics homework, the use of homework, types of mathematics homework, and the emphasis on sources to monitor students' progress in mathematics. These data are explored for possible explanations of student achievement, particularly for Australian students.

Does class size affect achievement?

There is an abundance of research literature that both refutes and also confirms that class size does correlate with student performance. Failing and high achieving students have been found to benefit most from smaller classes where instruction is tailored to meet their specific needs (Glass & Smith, 1979). Otherwise class size does not appear to affect achievement. Table 4 shows that Australia had the highest percentage of students with small class size of 1–24 students and that the average achievement of these students was lower than their Australian counterparts in classes with larger class size of 25–40 students. In the five top performing countries (i.e., Chinese Taipei, Hong Kong SAR, Japan, Korea, and Singapore), at most 10% of the students were in classes with 1–24 students. In Chinese Taipei, Hong Kong SAR, Japan and Korea, average achievement was higher in classes with larger sizes. In Singapore, however where a class size of 1–24 students was a rarity, it appears that class size did not affect the average achievement of the students.

Table 4. Class size and average achievement.

Country	**1 – 24 Students**		**25 – 40 Students**		**41 or More Students**	
	% of students	Average achievement	% of students	Average achievement	% of students	Average achievement
Australia	30 (2.8)	471 (6.3)	70 (2.9)	511 (5.3)	0 (0.1)	-
Chinese Taipei	4 (1.8)	549 (29.9)	85 (3.3)	593 (4.6)	11 (2.7)	660 (11.0)
Hong Kong SAR	10 (1.9)	513 (23.5)	44 (4.3)	555(10.1)	46 (4.1)	604 (7.2)
Indonesia	6 (1.7)	374 (13.7)	61 (4.2)	400 (5.1)	33 (4.1)	396 (8.6)
Japan	10 (2.1)	555 (5.9)	85 (2.7)	567 (2.9)	5 (1.6)	645 (24.7)
Korea	4 (1.4)	558 (15.6)	78 (2.6)	596 (3.1)	18 (2.3)	607 (7.2)
Malaysia	1 (0.8)	-	80 (3.2)	470 (5.8)	19 (3.1)	486 (10.9)
Singapore	2 (0.6)	-	76 (2.5)	593 (5.2)	22 (2.5)	592 (7.2)
Thailand	11 (2.4)	406 (11.2)	47 (3.7)	416 (5.7)	42 (3.1)	479 (9.3)

- Insufficient data to report achievement

Does homework enhance the learning of mathematics?

The learning of mathematics often extends beyond school hours and into the homes of students where they complete the assigned work. Table 5 shows the index of teachers' emphasis on mathematics homework (EMH). The index is based on teachers' responses to two questions about how often they usually assign mathematics homework and how many minutes of mathematics homework they usually assign. A High EMH level indicates the assignment of more than 30 minutes of homework for about half of the lessons or more. A Low EMH level indicates no assignment or an assignment of less than 30 minutes of homework for about half of the lessons or less. A Medium EMH level includes all other possible combinations of responses.

Table 5. Index of teachers' emphasis on mathematics homework (EMH).

Country	**High EMH**		**Medium EMH**		**Low EMH**	
	% of students	Average Achievement	% of students	Average Achievement	% of students	Average Achievement
Australia	5 (2.0)	497 (30.8)	46 (4.0)	520 (5.4)	49 (4.0)	477 (5.9)
Chinese Taipei	38 (4.2)	613 (8.0)	37 (4.6)	608 (5.0)	25 (3.5)	562 (7.4)
Hong Kong	31 (4.5)	586 (10.9)	52 (4.6)	582 (9.0)	17 (3.5)	532 (16.1)
Indonesia	41 (4.9)	403 (9.2)	50 (4.9)	409 (7.9)	9 (2.5)	386 (13.7)
Japan	8 (2.0)	564 (7.7)	33 (3.8)	575 (4.7)	59 (3.8)	568 (3.9)
Korea*	17 (2.8)	609 (7.7)	28 (2.8)	591 (5.8)	56 (3.3)	597 (4.0)
Malaysia	34 (4.0)	478 (8.6)	54 (4.2)	475 (6.7)	11 (2.3)	458 (15.7)
Singapore	43 (2.8)	612 (5.7)	39 (2.7)	595 (6.5)	18 (2.3)	542 (12.8)
Thailand	43 (4.3)	448 (7.8)	48 (4.2)	436 (9.0)	9 (2.2)	438 (14.1)

* Has a policy to assign mathematics homework.

From Table 5, it is apparent that for Chinese Taipei, Hong Kong, Malaysia and Singapore the level of emphasis directly related with the average achievement of the students, i.e., students whose teachers placed high emphasis on homework had high average achievement scores. For Australia, Indonesia and Japan it appears that the optimal was medium emphasis. Korea was the only country with a policy on homework. Both Korea and Thailand recorded an unusual pattern of average achievement and homework emphasis, i.e., high average achievement for high emphasis and lowest average achievement for medium emphasis. The data in Table 5 do not appear to show any simple relationship between emphasis of homework and achievement.

Needless to say, emphasis on homework alone cannot be indicative of its impact on student achievement. The purpose and nature of homework are equally important in ascertaining its impact on student learning. Teachers assign students homework and may or may not follow-up on its completion. Table 6 shows that 90% of the students in Indonesia and Thailand, 85% in Singapore, 82% in Hong Kong, 81% in Malaysia, 80% in Korea, 66% in Chinese Taipei, 65% in Japan and 63% in Australia reported that their teachers monitored whether or not the homework was completed.

However, a marginally smaller percentage of students in all the countries reported that their teachers corrected their assignments and gave feedback with the exceptions of Korea where only 12% did so and Japan, in which only 25% did so. In Chinese Taipei, more than half of the students reported that homework was used as a basis for class discussion. In all the other countries, less than two-fifths of the students reported the same. Also, in Chinese Taipei almost three-fifths of the students reported that their homework assignments contributed towards their performance grades; in Indonesia almost half of the students reported this practice while in all the other countries less than a third did the same. The data in Table 6 show no apparent relationship between aspects of homework and achievements of the students, in particular that of Australian students.

Table 6. Use of homework by teachers.

	Percentage of students whose teachers always or almost always			
Country	Monitor whether or not the homework was completed	Correct assignments and give feedback	Use homework as a basis for class discussion	Use the homework to contribute towards students' grades/marks
Australia	63 (3.3)	59 (3.9)	15 (3.3)	21 (3.2)
Chinese Taipei	66 (4.3)	50 (4.4)	53 (4.3)	59 (4.2)
Hong Kong	82 (3.5)	77 (3.2)	24 (4.0)	29 (4.0)
Indonesia	90 (2.4)	84 (2.9)	23 (3.4)	47 (3.4)
Japan	65 (3.6)	25 (2.9)	5 (1.7)	17 (2.6)
Korea	80 (2.1)	12 (2.0)	5 (1.6)	28 (3.1)
Malaysia	81 (3.2)	68 (3.2)	38 (4.0)	13 (2.9)
Singapore	85 (1.9)	80 (2.2)	28 (2.3)	20 (1.9)
Thailand	90 (2.5)	75 (3.8)	30 (3.8)	24 (3.8)

Standard errors are shown with ( ).

The data categorised homework as doing problems, doing sets of questions or gathering data and reporting. In Australia 66%, Chinese Taipei 79%, Hong Kong 64%, Indonesia 84%, Japan 51%, Korea 59%, Malaysia 67%, Singapore 75% and Thailand 73% of the students always or almost always were doing problems or question sets as homework assignments. On the contrary 1% or less of the students in Australia, Chinese Taipei, Hong Kong, Japan, Korea and Singapore; 17% in Indonesia, 7% in Malaysia, and 9% in Thailand reported always or almost always gathering data and reporting as homework assignments. It appears that homework assignments were similar in nature for students in Australia and the five top-ranked countries.

Sources to monitor students' progress in mathematics

In this section, we look at the source that the majority of the teachers in each of the countries placed emphasis on to monitor the progress of their students in mathematics. Teachers of 78% of the students in Australia, 81% in Hong Kong, 67% in Indonesia, 71% in Japan, 77% in Singapore, and 67% in Thailand placed major emphasis on classroom tests to gauge their students' progress in mathematics. Teachers of 54% of the students in Chinese Taipei, 64% in Korea, and 58% in Malaysia placed some emphasis on their professional judgement to gauge their students' progress in mathematics. It appears that both top performing, moderate and low performing countries appear to use similar sources to monitor students' progress in mathematics.

Concluding remarks

The data presented in this paper are limited and not comprehensive. Therefore with caution, some questions about the correlates of achievement and the learning of mathematics are raised. As this presentation is meant for participants attending the AAMT conference, I wish specifically to address some of the findings related to eighth-graders from Australia. From the limited discussion in this paper, it appears that although the Human Development Index appears to be a predictor of achievement in some countries, this was not so for Australia. The international benchmarks of mathematics achievement also show that only about a quarter of the Australian students were at the high benchmark compared to more than 60% of their counterparts in Chinese Taipei, Korea, Singapore, Hong Kong and Japan respectively. From Figure 1, it is apparent that eighth-grade Australian students lacked in deductive reasoning using simple properties of triangles. This leads one to speculate that perhaps the teaching of mathematics in many Australian classrooms lacked emphasis on application of knowledge and higher order thinking.

Bearing in mind that other factors related to schooling such as size of population around schools may be directly related to class size, it appears that in Australia, students in relatively larger classes (25–40 students) performed better on the TIMSS tests. The data in Table 4 shows that for many Asia-Pacific countries, very large class sizes (41 or more) yielded highest average achievement. In Australia, about 50% of the students were in classes where there was low emphasis on homework; further, the average achievement of these students was lower than that of their Australian counterparts in classes where there was medium emphasis on homework. This may suggest that placing more emphasis on homework may improve the achievement of Australian eighth-graders. Australia was also the Asia-Pacific country for which the lowest percentage of students reported that their homework was monitored for completeness.

The data in Table 6 did not show any simple relationship between achievement and aspects of homework, such as homework assignments being graded and feedback given to students, use of homework for class discussion and contribution towards student grade in mathematics. Like the other Asia-Pacific countries, Australian teachers almost always engaged students in doing problems or question sets for homework.

Teachers of almost 80% of the students in Australia reported that they used class tests to monitor the progress of their students. This was similar to the other top-performing, moderate- and low-performing Asia-Pacific countries. Perhaps what teachers do with the outcomes may hold some cues.

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NUMBERS IN THE MEDIA

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Traditionally mathematical and statistical concepts and skills have been taught either "context free" or using textbook examples. Research has indicated the value and importance of authentic tasks for student motivation and engagement. It has also highlighted the importance of everyone becoming numerate and statistically literate. This means being able to understand, interpret and critically assess numbers in the "real world." This paper argues that the media as a resource provides teachers with current and quantitative material that can be used to help students concurrently develop their understanding of mathematical and statistical concepts and to apply them in "real world contexts." Furthermore, media derived resources are useful for students to use to develop through their own investigations their understanding of mathematical and statistical concepts. Two contexts have been chosen to illustrate this; one uses numbers in text and the other uses tables. These are contexts within which students can also develop their critical thinking skills in terms of social issues. The paper suggests frameworks for students to use when analysing and interpreting the data in the tables.

Introduction

The research literature over the last two decades has outlined the importance of developing informed citizens who are able to evaluate quantitative material presented in a range of ways in their lives (Steen, 1997; Gal, 2003). We are living in a "supercomplex" world (Barnett, 2000) in which information is transmitted through a range of media including television, newspapers, radio, podcasts, *YouTube* and other areas of the Internet. All forms of media in addition to reporting valid quantitative information in the form of numbers in text, tables, graphs or charts, present information that is inadequate or contains incomplete data, or logical deficiencies. In such reporting, effects may be linked or attributed to causes without adequate justification. There are also cases reported in the media where the original collection of data was neither valid nor reliable and the conclusions drawn are consequentially questionable.

People need to be aware that material is usually deliberately chosen to persuade people to adopt a particular point of view, as Best (2001, p. 18) points out:

> Certainly we need to understand that people debating social problems choose statistics selectively and present them to support their point of view. Gun-control advocates will be

more likely to report the number of children killed by guns, while opponents of gun-control will prefer to count citizens who use guns to defend themselves from attack.

Given media is non-neutral in its reporting of statistical information this provides an opportunity and a responsibility for educators to foster in students an understanding of the pliable nature of statistical evidence. Therefore, it is argued in the following that there are significant benefits for students and teachers from using material from the media, in print or electronic form, in the classroom. Firstly, the use of such material provides opportunities for authentic tasks to be identified that can motivate and engage students as advocated by Zevenbergen (1997) and Watson (2004). Secondly, using information from the media can help teachers address questions such as, "Why are we doing this?" often asked by students, by highlighting the need for an understanding of mathematical and statistical concepts to make sense of material encountered in everyday life. Thirdly, this material, selected by the teacher to be used in conjunction with pedagogically sound learning activities (Watson, 2006) can be used to help students develop their understanding of mathematical and statistical concepts such as averages, sampling, variation, risk and inflation in current contexts. As such, it can be used to provide opportunities for students to decide what they need to know in order to understand a given situation. In this way the teachers and students are using the media as a tool for learning mathematical and statistical concepts in context.

Fourthly, through this engagement students can develop the ability to look critically at data located by themselves in the media. By noting how the data has been collected, collated and presented students can then evaluate the validity and implications of that reported material. In this way students are applying their knowledge and developing the critical reading skills necessary to be informed citizens (Steen, 1997).

Due to constraints on space, two contexts presented in the media and mathematical and statistical concepts related to them are explored. The first context is "diet, lifestyle and health risks" in which quantitative information is reported as numbers in text, and the second "opinion polls" which relies on a tabular representation,

These contexts have been chosen because they require an understanding of both statistical and mathematical concepts including: probability; risk; sampling; variation within and between samples; bias and interpretation of tables as well as number concepts such as fractions, decimals, percentages and proportions.

Diet, lifestyle and health risks

In the "diet, lifestyle and health risk" context the media highlight relationships between the development of conditions such as cancer, diabetes 2 or depression, and factors such as obesity, smoking, alcohol, fast food, or lack of exercise. For example, there are claims of links between diet and bowel cancer; mobile phone use and tumours; alcohol and breast cancer and child diet and depression (e.g., Blastand & Dilnot, 2007; Hope, 2007). There are some mathematical and statistical concepts that people need to understand to be able to confidently interpret the results of these reports including the concepts of randomness and size of samples related to health studies, as well as relative and absolute risk, percentages and natural frequencies.

In November 2007, public concern and anxiety was caused by media comments based on a report from the World Cancer Research Fund on the links between lifestyle, diet and cancers. On 1 November 2007, *The Sun* newspaper in the UK published the headline “Careless pork costs lives” stating that there is an estimated 20% excess risk of bowel cancer per 50 g of processed meat eaten per day, and recommending that people avoid processed meats such as bacon, salami and sausages. A later edition of *The Sun* that same day carried the headline “Save our bacon” (Morton, 2007). This information was also reported by other newspapers. For example the *MailOnline* (Hope, 2007) asked on the same day “Is anything safe to eat?” These articles are currently available on-line (Morton, 2007; Hope, 2007), however, no indication of the population size from which the sample was drawn was reported.

In the article reported in *The Sun* the risk of bowel cancer is given as a *relative risk*. Using relative risk, the risk of contracting bowel cancer for people who eat at least 50 g of processed pork per day is compared with the risk of contracting bowel cancer for people who eat less than 50 g of processed pork per day. In order to make sense of this it is necessary to know the risk of bowel cancer for someone who does not eat 50 g of processed pork daily, that is the *absolute* risk for those people *or* the *baseline* risk. The chance of someone, who does not eat the specified amount of pork, contracting cancer over an average lifetime is the “baseline” with which one can work. This is about 5% (Pearson & Spiegelhalter, 2009); that is, about 5 people in 100. The daily consumption of at least 50 g of processed pork changes the risk. It is 20% higher than for those who do not eat the specified amount of processed pork. Since 20% (or one-fifth) of 5% is 1% then it is necessary to add on the 1% to the 5% to get 6% as the absolute value for those who eat at least 50 g of processed pork daily.

There are other ways of expressing risk that do not involve percentages explicitly. These can take a positive or negative point of view. Taking the positive point of view, 95 out of 100 people who eat less than 50 g processed pork (such as a bacon sandwich) daily are likely to be free of bowel cancer. This is reduced to 94 out of 100 for those who consume 50 g or more of processed pork daily. Taking the negative view, 5 out of 100 people who *do not* consume 50 g or more of processed pork daily are likely to experience bowel cancer. This is increased to 6 out of 100 for those who *do* consume 50 g or more of processed pork daily.

In summary, to comprehend “risk” statements in the media, students need to be familiar with the ideas of relative risk, absolute risk and baseline risk, and with negative and positive ways of expressing risk. They could consider advantages and disadvantages of the different ways, and need to be able to perform calculations linking the various risks.

As a further example, students could consider the relationship between alcohol and breast cancer, as announced in the BBC National TV news bulletins of 12 November 2002. The bulletin noted that the world’s largest study of women’s smoking and drinking behaviour reveals that for every extra alcoholic drink a woman consumes on a daily basis, her risk of breast cancer increases by 6 per cent. Firstly students would need to be told the baseline risk – the absolute risk of contracting the disease among women who do not drink alcohol. In discussing this claim Blastand & Dilnot (2007, p.85) inform us that “about 9% of women (which statisticians describe in natural frequencies as 9 out of 100 women) will have been diagnosed with breast cancer by the time they are 80”. To express the risk as an absolute risk for those women who have one alcoholic drink every day then the students would first calculate 6% of 9 women to get 0.54

women. For those women who have a drink every day, there are predicted to 9.54 (9+0.54) women out of every 100 who will get breast cancer, that is 954 out of 10 000 without a decimal point or rounded to about 19 out of 200 which is better because it can be comprehended more easily.

Expressing this in the positive mode with natural frequencies: for women who do not have daily alcoholic drinks, 91 (100-9) out of 100 will be free of breast cancer, which is reduced to 90.64 out of 100 (or 9,064 out of 10 000) with one drink every day. In the negative form: for women who have no drinks daily, 9 women out 100 will get breast cancer which is increased to 9.54 out of 100 (954 out of 10 000) with a daily drink.

In addition or as alternatives to the processed pork and breast cancer examples, students could be given other scenarios reported in the media, with the baseline risk identified for them, and they could then be asked to write the risk in a number of ways. Then they could do their own research on the Internet or in newspapers for other examples of health risks that are reported statistically, many of which are readily found in Australian print or on on-line media.

Opinion polls

A common way of gauging public opinion on social issues is through opinion polls which cover a range of social, political, economic and environmental issues; recently polls on issues such as daylight saving, nuclear power, political preferences, carbon emissions policies and climate change have been conducted. The findings are often displayed in print newspapers or their electronic versions, or on the websites of the organisation conducting the polls, often in tabular form. When these tables are incorporated into an article they are often used to endorse the author's point of view. Rarely are both sides of an argument presented equally.

There are some important statistical concepts that people need to understand to be able to confidently interpret the results of opinion polls. They need to understand concepts of size and randomness, bias, and variation within and between samples (Watson, Kelly, Callingham & Shaughnessy, 2003; Watson, 2004) and that the questions asked can influence the responses of the interviewees.

In Western Australia, a referendum about daylight saving will occur on 16 May 2009. The question to be asked is: "Are you in favour of daylight saving being introduced in Western Australia by standard time in the State being advanced one hour from the last Sunday in October 2009 until the last Sunday in March 2010 and in similar fashion for each following year?"

Some results from a survey conducted by Patterson Market Research for *Westpoll* for *The West Australian* newspaper were published in an article on 11 April 2009 on the front page of *The West Australian* (Phillips, 2009) with the heading "On a Knife Edge." The article included Table 1 and reference to other findings.

Table 1[10]*. Poll on daylight saving*

Westpoll Do you support daylight saving?

	March '07	March '09	April '09
Support	34	42	47
Oppose	62	57	51
Don't know	4	1	2

Westpoll conducted April 6-8 through phone interviews with 400 voters across WA by Patterson Market Research.

In the article, Phillips (2009, p.1) commented on the overall percentage change in support and opposition and included percentages concerning specific categories of people by age "Younger West Australians registered the greatest support for daylight saving, with 58 percent of voters under 35 polled backing its introduction, compared with 42 per cent over 35". In addition, Phillips reported that "ALP voters were more likely to vote yes, with 54 per cent indicating their support, behind Greens voters (52 per cent) and Liberal-Nationals voters (43 per cent)"

This article was followed on 21 April 2009 with a photo of three young men at the beach on the front page, urging 370 000 young people to vote in favour of daylight saving. In the associated article (Strutt, 2009, p. 4), said: "Almost 370,000 voters aged 18 to 34 who have never voted on daylight saving and who polling shows are overwhelmingly in favour of putting the clocks back an hour will have their say on the issue at next month's referendum."

The reader should note the use of the word "overwhelmingly" and question whether this is a correct use of the term. The comparison is made between 58% and 42% of the 400 sampled. Older students who know something about hypothesis testing could consider this question: is the proportion of 232 out of 400 (58%) significantly different from 168 put of 400 (42%)? An hypothesis test for the comparison of two proportions indicates that it is significantly different.

There are four main steps forming a framework for analysing this and other opinion poll data proposed here. These are to consider in turn:

1. What was the question asked?
2. What was the method of sampling and data collection?
3. What was the sample size?
4. What is the interpretation?

These steps are applied below to the data concerning daylight saving in Table 1:

Step 1: What was the question asked?

In this case the question asked is shown in the table as, "Do you support daylight saving?" The question has the positive attributes that it appears to be clear, unambiguous and non-leading. (However, when checked against the Patterson Market Research webpage the actual question to be used is the actual longer referendum question cited above).

[10] Table 1 is reproduced by courtesy of *The West Australian*.

Step 2: What was the method of sampling and data collection?

This table does not indicate whether the sample was randomly selected from the population who will be voting. Neither is there any indication of this in the text of the article. (Information concerning the sample size and randomness can be found on-line at Patterson Market Research (2009).) For any valid statistical conclusions to be drawn the sample must be random (Watson, 2004). Simple random sampling ensures that all people have an equal chance of being selected and that the responses by the sample are likely to be representative of the population.

If students have not previously grasped the concept of randomness there are a number of activities that can be undertaken in the classroom. Students can use dice, random numbers from tables, calculators or computers to generate random data. They can conduct computer simulations to explore randomness. There are examples for classroom activities in the teaching materials published by the Curriculum Council of Western Australia (2009).

Step 3: What was the sample size?

The sample size is given as 400 for the poll undertaken on April 6-8. No information is given for the March 2007 and March 2009 polls. This seems like a reasonable sized sample but discussion in the classroom could focus around the question of size of the sample and the disadvantages and advantages of small and large samples (Chance, delMas & Garfield, 2004).

In addition, the sample size of 400 could provide the motivation and opportunity to proceed to classroom activities that investigate the effect of sample size on mean and standard deviation. To do this, samples could be taken from a known population and students can see that the samples need to be "big enough" to ensure that the results for the mean and standard deviation are stable. Starting with a known population of about 200, students can take random samples of size 5 and compare, 10 and compare, 30 and compare until the estimated population mean is reasonably stable. This also leads into consideration of confidence intervals for older students. The Curriculum Council (2009) provides materials for just such activities.

Step 4: What is the interpretation?

Firstly, the students could interpret the table of data looking at the changes overall in the support of daylight saving. At a glance there appears to be stronger support for daylight saving in April 2009 than before. Questions should be asked about the sample sizes from the previous surveys and how the samples were selected. It is useful for students to ask their own questions in steps 1, 2 and 3 before they look at how their conclusions compare with those of newspaper authors. After the students have considered the table, they could be provided with the two articles by Phillips (2009) and Strutt (2009) from *The West Australian* to see how these two authors have interpreted that, and additional data.

In this interpretation task, a *Five Step Framework* for the interpretation of tables can be used to help students to develop appropriate strategies for interpreting tables (Kemp, 2005; Kemp & Lake, 2001). The framework rubric is provided in Figure 1.

Step 1: Getting started

Look at the title, axes, headings, legend, footnotes and source to find out the context and expected reliability of the data.

Step 2: WHAT do the numbers mean?

Make sure you know what all the numbers (percentages, '000s etc.) represent. Look for the largest and smallest values in one or more categories or years to get an idea of the range of the data.

Step 3: HOW do they change or differ?

Look at the differences in the values of the data in a single data set, a row, column or part of a graph. Repeat this for other data sets. This may involve changes over time, or comparisons within categories, such as male and female, at any given time.

Step 4: WHERE are the differences?

What are the relationships in the table or graph? Use your findings from Step 3 to help you make comparisons between columns or rows in a table or parts of a graph to look for similarities and differences.

Step 5: WHY do they change?

Look for possible reasons for the relationships in the data you have found by considering societal, environmental and economic factors. Think about sudden or unexpected changes in terms of state, national and international policies or major events.

Figure 1. Five Step Framework.

When using the framework initially the teacher devises questions suitable for the particular table being examined. As students become experienced with it they are able to devise their own questions. In Figure 2 the framework is applied to Table 1.

Step 1: Getting started
Q: From the title, what is the general topic being examined?
Q: From the labels on the left column, how are the groups being compared?
Q: From the labels on the top row, how are the groups being compared?
Q: Is there any evidence that the information is reliable?
Q: Are the collection dates evenly spread?
Q: What information is given about the sample size for each polling date?
Step 2: WHAT do the numbers mean?
Q: What is the meaning of the 47 in the third column?
Q: Which year has the highest percentage of people in favour?
Q: Which category has the lowest percentage of people in favour?
Step 3: HOW do they change?
Q: How does the percentage of people in support change over the three collection times?
Q: How does the percentage of people who oppose change over the three collection times?
Q: How does the percentage of people who don't know change over the three collection times?
Step 4: WHERE are the differences?
Q: Compare the differences for 'in favour' and 'against;' for each time frame, look at the relationship between the percentages.
Q: Compare the 'don't know' responses with the support percentages.
Step 5: WHY do they change?
Q: Did any of the values in the table surprise you?
Q: Suggest possible reasons for the differences in the percentages for the responses considering that two of the dates are three years apart and the third is only one month later than the second.

Figure 2. Application of the Framework to Table 1 on daylight saving.

In this case the interpretation of the data in the table is not complicated because the table does not report responses in terms of gender, age or political persuasion.

There are also more complex tables of results from opinion polls published in the media; one example is given in Table 2. This poll concerning nuclear power had two parts in which different, but related, questions were asked. Results were reported in *The Australian* 6 March 2007.

Table 2. Nuclear power

Are you personally in favour or against the development of a nuclear power industry in Australia, as one of a range of energy solutions to help reduce greenhouse gas emissions?

	TOTAL	SEX		AGE			POLITICAL SUPPORT	
		MALE	FEMALEE	18-34	35-49	50+	COALITION	LABOR
	%	%	%	%	%	%	%	%
STRONGLY IN FAVOUR	20	26	14	17	18	24	33	13
PARTLY IN FAVOUR	25	27	24	32	25	20	28	22
TOTAL IN FAVOUR	45	53	38	49	43	44	61	35
PARTLY AGAINST	13	13	13	13	15	12	11	14
STRONGLY AGAINST	27	25	29	26	30	25	11	38
TOTAL AGAINST	40	38	42	39	45	37	22	52
UNCOMMITTED	15	9	20	12	12	19	17	13

This survey was conducted on 2–4 March on the telephone by trained interviewers in all states of Australia and in both city and country areas among 1207 people aged 18 years and over. Telephone numbers and the person within the household were selected at random. The data has been weighted to reflect the population distribution. The maximum margin of sampling error on the total sample is plus or minus 3 percentage points. Copyright at all times remains with NEWSPOLL. More information is available at www.newspoll.com.au.

The second question was: "And if it was decided that Australia develops a nuclear power industry, would you personally be in favour or against a nuclear power station being built in your local area?" The table in the newspaper indicated that the percentages "in favour" and "against" changed quite considerably. The total in favour shifted from 45% to 25% and the total against from 40% to 66%, indicating large changes in responses to somewhat similar questions. In the following the steps are applied to the data reported on the responses to the first question.

Step 1: What was the question asked?

Question one: Are you personally in favour or against the development of a nuclear power industry in Australia, as one of a range of energy solutions to help reduce greenhouse gas emissions?

This somewhat leading question encases the nuclear power question within the context of the reduction of greenhouse gas emissions. There has been a lot of publicity about the need to reduce greenhouse gas emissions as well as the advantages and disadvantages of nuclear power from both positive and negative perspectives.

Question two: And if it was decided that Australia develops a nuclear power industry, would you personally be in favour or against a nuclear power station being built in your local area?

This question would no doubt alert fears of the possibly unknown effects of a nuclear power industry so close to their homes. It is certainly possible that people would have not had access to sufficient information to make an informed decision.

Step 2: What was the method of sampling and data collection?

This poll was conducted by Newspoll. The footnote gives information about how the data was collected using a random sample via the selection of telephone numbers for city and country people.

Step 3: What was the sample size?

The footnote tells us that the sample size was 1207 people over 18 and justifies the sample size with reference to the maximum margin of sampling error of plus or minus 3 percentage points.

Step 4:What is the interpretation?

Figure 3 shows how the Framework can be used for this more complex table.

Step 1: Getting started
Q: From the title, what is the general topic being examined?
Q: From the labels on the left column, how are the groups being compared?
Q: From the labels on the top row, how are the groups being compared?
Q: From the footers, what evidence is there that the information is reliable?
Q: What are the meanings of the following terms: greenhouse gas emissions, nuclear power, uncommitted?
Step 2: WHAT do the numbers mean?
Q: What is the meaning of the 45 in the first column?
Q: Which category has the highest percentage of people in favour (total) of the development of a nuclear power industry?
Q: Which category has the lowest percentage of people in favour of (total) the development of a nuclear power industry?
Step 3: HOW do they change?
Q: How does the percentage of males in favour compare with the percentage of females in favour of the development of a nuclear power industry?
Q: How do the percentages of people in favour of the development of a nuclear power industry and supporting the Coalition or Labor compare?
Q: How does the percentage of people in favour (total) of the development of a nuclear power industry change with age?
Step 4: WHERE are the differences?
Q: Compare the people with Coalition and Labor support for ‘total in favour’ and ‘total against.’
Q: Compare the age differences for ‘in favour’ and ‘against.’
Q: Compare the uncommitted responses across the categories.
Step 5: WHY do they change?
Q: Did any of the values in the table surprise you?
Q: How well do these findings fit with related theories/stereotypes?
Q: Suggest possible reasons for the differences in the percentages for the responses.

Figure 3. Application of the Five Step Framework to Table 2

The analysis of poll data has focused on tables because that is the format usually used and generally people find the analysis of tables more complex than interpreting graphs. This framework is also useful for interpreting graphs (Lake & Kemp, 2001). Students could seek out reports from polls or surveys that are presented in graphical form.

Conclusion

This paper used examples drawn from the media in two different contexts. These examples show the reader how they can be used as a motivational and starting point for students to become aware of the concepts and skills necessary to analyse the information and data provided. Suggestions have been made for classroom activities to help develop the necessary concepts and skills to enable students to complete the analysis of the example. Subsequent examples from the media are included to indicate how students can apply their skills. Once students have the necessary concepts and skills they can explore the media and from their own research find examples that interest them. This work can be extended to looking at interpreting graphs in a similar way (Lake & Kemp, 2001).

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POPULAR MATHEMATICS

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Popular mathematics seems to be not very popular in either schools or society at large. In this paper, we briefly explore the meaning of popular mathematics, and consider some of the ways in which it might be of value to mathematics education in schools, particularly secondary schools. Various kinds of popular mathematics are identified and briefly illustrated, and the prospects for supporting and complementing the school mathematics curriculum discussed.

Introduction

In this paper, a brief introduction to the idea of popular mathematics is offered, recognising that encounters with mathematics ought not be restricted — although they frequently are — to the formal school curriculum or even to schools themselves. It is suggested that mathematics continues to have an image problem in Australia, if not elsewhere, and that the public understanding of mathematics is not always helpful for the learning of school mathematics. The paper does not claim to treat this broad topic in detail, but rather to suggest some of the ways in which popular mathematics and mathematics education in schools might be connected, and offering some advice on productive ways of doing this.

What is popular mathematics?

There does not seem to be general agreement (or even debate) on the meaning of the term "popular mathematics." Indeed, a *Google* search produces mostly examples of popular mathematics *things* (such as popular mathematics *books*, where the meaning is to refer potential customers to books that are purchased or borrowed from a library by many people). Turney (2008) refers to "popular science" in relation to "an attempt, in some medium, to make scientific ideas accessible to non-professional audiences" (p. 5). Following Turney, this paper is concerned with published attempts to engage non-specialists with aspects of mathematics.

There are many kinds of attempts to do this, including:

- books written for the general public;

- film and television features;
- the Internet;
- newspapers and magazines;
- incidental mathematics, such as that in museums.

We will exemplify and discuss the merits for school purposes of each of these.

Budd and Lim (2004) have summarised some useful discussion on the broader issues associated with "public understanding of mathematics" in the report of a discussion group at the International Congress on Mathematics Education in Copenhagen. While the ideas presented are sound, and worthy of action at a range of levels, the present paper is focussed on the more immediately manageable activities in a school setting.

Popular mathematics books

As for science, printed books are the most widespread source of popular mathematics materials and hence potentially the most accessible. This species of books comprised very few members until comparatively recently. Early examples, deservedly much-acclaimed, included Courant and Robbins' *What is Mathematics?,* the work of superb writers such as Warwick Sawyer (e.g., *Mathematician's Delight*), Martin Gardner (with various off-prints from *Scientific American* magazine) and the four-volume compendium, *The World of Mathematics*, edited by James R. Newman. The last of these was issued to many schools in the 1960s when the federal government first began providing direct support for schools, albeit in oblique ways, through libraries and science laboratories.

In recent years (the past thirty years or so), a large number of books have been published that might be regarded as popular in the sense that they are written to engage in some way an audience of non-specialists (at least, non-specialists in a particular field of mathematics). Many of these are potentially of considerable benefit to secondary school mathematics education, although it seems that only a few of them ever find their way into secondary schools. Amongst the sub-categories of such books might be:

- general introductions to mathematics or particular mathematical topics, e.g., Sardar, Ravetz & Van Loon (2005);
- glossy coffee-table books, lavishly produced in beautiful colours, e.g., Bentley (2008);
- books highlighting the role of mathematics in everyday life, e.g., Eastway & Wyndham (1999);
- books directed at younger readers, e.g., Stewart (2006);
- books offering historical perspectives on either ideas or the people associated with them, e.g., Mankiewicz (2000);
- books focussed on puzzles or problems, e.g., Enzensberger (2000);
- mathematics books, aimed at mathematically literate readers, but not specialists in the relevant field, e.g., Odifreddi (2004).

This list includes a good example[11] in each category, although in each case there are many members in good standing available in today's catalogues. Figure 1 shows a sample of recent popular mathematics books.

[11] Lest readers think otherwise, it is important to note that these few examples in no sense exhaust the very many good examples that might have been included here. The constraints of this paper, with the need to include a reference list well in excess of the allowable paper length, have meant that only examples can be given.

Figure 1. Some recent popular mathematics books.

No claim is made here that a better taxonomy of categories cannot be devised. Nor is it suggested that the categories are mutually exclusive. (In fact, they are not.) With the possible exception of the last category above, however, books of these kinds have the potential to provide students with significant opportunities to engage with mathematical ideas and to improve their sense of the nature of mathematics — provided of course that the books find their way into the hands of younger readers. They also have the potential to significantly affect students' career aspirations at the very time when they are making educational decisions of lasting significance to their lives. Devlin (1995) provided an interesting set of opinions from notable successful popularisers of mathematics (including Ian Stewart, John Allen Paulos and William Dunham as well as himself). It is clear that the academic world does not usually regard the writing of such books to be entirely respectable, although the authors themselves regard it as an important part of their work. As Ian Stewart, a very successful popular mathematics author, noted: "[M]any still do not appreciate that unless somebody tells the public what mathematicians are doing, support for the subject (and I mean appreciation rather than money) will dry up" (Devlin, 1995).

Whilst it appears obvious that books can have no effect unless someone tries to read them, it is worth noting that many students seem to have relatively few opportunities to do so in most situations, for a range of reasons, to do with prevailing circumstances of school libraries, school mathematics curricula and commercial bookshops.

Good data seem not to be available, but informal evidence suggests that many school libraries contain few, if any, popular mathematics books of the kinds described above. Reasons for this are unclear, and doubtless relate to practices at individual schools, but they probably include the limited knowledge by librarians of what is available, a general reluctance of mathematics teachers to suggest acquisitions for the school library and, possibly, limited personal knowledge of what is available by mathematics teachers themselves. While these are speculations, they are supported by considerable personal conversations with librarians and teachers, as well as students. Readers are invited to examine the collection of mathematics materials in the school library they know best, in order to test these hypotheses against some local data.

School mathematics curricula in Australia do not generally seem to be constructed on an assumption that students might be expected to read about mathematics anywhere much except their textbook, often tightly written to match the course at which it is aimed. (Indeed, close inspection of many textbooks gives an impression that students are not expected to read much at all.) Thus, it is rare that students are invited to seek information elsewhere, such as a library, an encyclopaedia, or in the media and equally rare that they are encouraged to look elsewhere for extra material to follow up on interesting ideas that have arisen.

Finally, many bookshops do not even recognise popular writing in mathematics as a field of publishing at all. While there are (a few) exceptions, many bookshops accessible to the general public do not have an identifiable named section that clusters together popular mathematics books. A fairly common practice is to have a section concerned with "Science" or "Popular Science," which might include a few stray examples of popular mathematics, such as those shown in the photograph in Figure 2, which shows books on measurement, chaos and the invention of the number zero, among others. While these are unquestionably good examples of popular mathematics books, only an avid explorer would be able to find them in a typical non-specialist bookshop.

Figure 2. Popular mathematics as a branch of science.

Apart from the linguistic problem (that is that, although it is of very considerable value to scientific work and study, mathematics is most certainly not a branch of science), this practice has a number of unwanted side-effects: anyone looking to purchase a popular mathematics book will have some trouble finding one; the fact that there is no category of mathematics books in many bookshops reinforces the existing impression (even amongst shop assistants!) that no such books exist; people are most unlikely to purchase a popular mathematics book as a gift for a child (or an adult), although the purchase of books is a popular means of gift-giving in our society. The commercial world relies on turnover for its practices, of course. The self-fulfilling prophecy that there is little demand for mathematics books is effectively reinforced by practices that make sure that none are available for perusal[12].

The situation seems to be a little better in many local government libraries, at least in Western Australia[13]. Perhaps because there is a conscious effort to provide for a wide readership and perhaps also because there are well-known Dewey classification numbers (starting with 510) with a consequent expectation that there are some members in good standing on the shelves, students might be better advised to go to their local library instead of their school library in search of popular mathematics books. It might also be that purchasing practices for a state library system routinely add recent popular materials to the catalogues as a public service, unlike schools, which may be systematically more reliant on recommendations from school staff.

Film and television

We are surrounded by the modern media of film and television. Large-screen televisions are already popular in affluent Australia, and there are very few Australian households without television at all. Our youth are immersed in a television culture of enormous and pervasive influence. To date, however, these media have had relatively little effect on the popularisation of mathematics: indeed, mathematics is scarcely visible in these environments.

Films and television programmes have so rarely involved mathematics that it is almost possible to list the examples of the last decade:

- Films
  - *Good Will Hunting*
  - *A Beautiful Mind*
  - *Proof*
- Television series
  - *Life by the Numbers*
  - *Numb3rs*
  - *The Story of Mathematics* (BBC)

While things may not be *much* better in other learning areas, they are in fact better. Our collective societal passion for fiction over non-fiction has meant that English and drama are very well served by these media, where science and mathematics are not, for the most part. Looking beyond fiction, however, it seems that most non-fiction in the media of film and television does not have a mathematical focus. There are many programs on science, history, travel and other cultures, and a wealth of sporting programs, but mathematics is conspicuous by its absence. Even on subscription television, the situation is not much better. There are channels dedicated to science (such as *Discovery* channels), geography and nature (*National Geographic*), history (*History Channel*), travel channels, health channels and others, but only rare mentions of mathematics and the people behind it.

As Budd and Lim (2004) suggest, one of the difficulties of finding mathematics in the media, or at least positive images of mathematics in the media, is the widespread view that mathematics is made up mostly of numbers and symbols. Another difficulty is

[12] As an example, I recently scoured a very large shopping centre in Perth's northern suburbs and found a total of only two mathematics books in bookshops, leaving aside mathematics school textbooks.
[13] I am unable to comment on the situation elsewhere in Australia in this respect.

that the place of mathematics in the modern world is mostly hidden, as observed by Keith Devlin (1999), in the (perhaps unfortunately titled) PBS television series, also published as a book, *Life by the Numbers*. Indeed, a major motivation for this particular series was to highlight the ways in which mathematics "makes the invisible visible".

While there are few major media productions of these kinds, they do not necessarily even get an airing in Australia. Of the three television series referred to above, only *Numb3rs* was shown on free-to-air television in Australia. *Life by the Numbers* was not shown at all, and *The Story of Mathematics* has (to date, at least) only been shown on subscription television.

A consequence of these observations is that students seem unlikely to encounter much evidence of mathematics in films or television, and the associated learning, both affective and cognitive, unless steps are taken by the school and by mathematics teachers to seek out suitable materials. Unlike the situation elsewhere in the school curriculum, mathematics in film and television will only be seen by students if we make an effort to find and use good examples of them.

The Internet

The Internet offers fresh opportunities for mathematics to be made available to a non-specialist audience, including the audiences of students and their parents, as well as the more mathematically aware audience of mathematics teachers themselves. Indeed, Kissane (2009) has suggested that "reading interesting materials" is one of the categories of Internet use likely to be productive for students, and offers a number of examples of suitable publicly available materials of this kind.

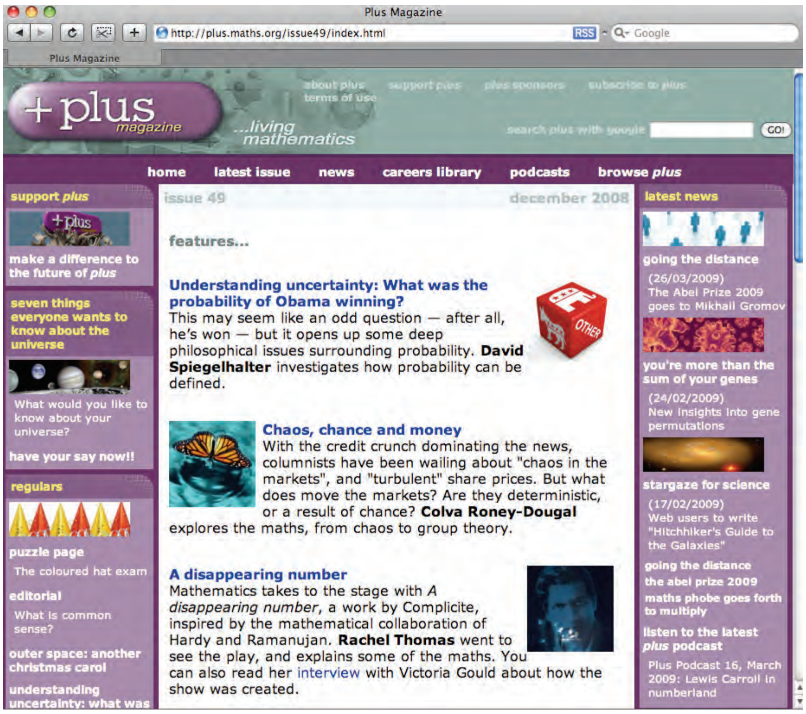

Figure 3. A recent issue (49) of Plus magazine (Millennium Mathematics Project, 2008).

Perhaps the best of these to date is *Plus*, an Internet magazine produced five times a year by the excellent Millennium Mathematics Project team at Cambridge University in the UK. A recent example is shown in Figure 3. This magazine contains a variety of short articles in each edition, together with searchable archives, up-to-date mathematics news and lately even downloadable podcasts and posters. The articles typically include both expository articles regarding the applications of mathematics, the nature of mathematics, mathematics in wider society and interviews with people for whom mathematics is an important part of their career. Although relatively sophisticated, the articles are generally accessible to senior secondary school students and to a broadly educated public. The site continues to be available without charge because of sponsorships from various organisations, including government agencies, which presumably recognise the importance of improving the image and public understanding of mathematics.

As well as magazines, other kinds of written materials intended for a popular audience are available on the Internet, several of which are referred to by Kissane (2009). These include mathematical columns of various kinds (such as might be found in magazines, but written expressly for the Internet) and mathematics posters.

While much good material is available on the Internet, and hence technically in the "popular" domain, students are unlikely to encounter these materials and to gain from using them, unless they are explicitly brought to their attention in some ways. These include linking via school web sites, specific recommendations for accessing sites, project work that directs them to websites or the use of elements from the websites in teaching.

Newspapers and magazines

There seem to be no magazines for a popular audience whose main focus is mathematics, although some science magazines have occasional mathematical components in them. (An outstanding example was the Mathematical Games column in *Scientific American*, edited for many years by Martin Gardner; however, this column no longer exists). A consequence of this omission is that students will rarely encounter mathematics (directly, at least) when browsing in a newsagency or in other places where magazines are found, such as doctors' waiting rooms, fish-and-chip shops, and so on.

Some magazines and now essentially all daily newspapers do contain mathematical elements in the form of puzzles, however. One example is the weekly *BrainTrainers* column in the *Weekend Australian* newspaper, frequently containing small mathematical puzzles or puzzles which require mathematical thinking. In the wider community, the remarkable rise to fame of *Sudoku* is the most visible recent example of popular mathematics: in a few short years, it has become a conspicuous part of most daily newspapers in Australia, as well as those of most other countries, attesting to the drawing power of elementary mathematical logic. This puzzle in particular offers opportunities for the mathematics classroom, based on the deductive reasoning required to complete the grids involved, and some have offered ways of doing this with lower secondary students.

Incidental mathematics

While mathematics or its effects are indeed pervasive, they are frequently not noticed, as Keith Devlin has observed, and take some effort to extract. The modern world contains extensive evidence of everyday mathematics, which alert observers will notice in supermarkets, buildings, timetables, architecture, sport, households and many leisure activities such as games. Unfortunately, students are not naturally “alert” in this sense, and energy must be spent helping them to see the mathematics in activities and situations that are not presented in an overtly mathematical way.

Some public institutions, such as museums, offer glimpses into the world of mathematics, although these are rarely accessible easily to most school classes. Throughout Australia, there are a small number of “science” museums, which typically contain some mathematical elements, but these are accessible mostly to city children and their families. More widely, evidence of mathematics is sometimes seen by tourists, perhaps the most spectacular examples of which are the Alhambra in Southern Spain, the pyramids of Giza in Egypt and the archaeological ruins of the ancient European worlds around the Mediterranean Sea. Examples of incidental mathematics can be seen in many mosques, in which wonderful patterned tiles are to be seen, such as those in Figure 4, from the Islamic Arts Museum in Kuala Lumpur, Malaysia.

Figure 4. Ceramic tiles of Islamic origin.

Of course, incidental mathematics of these kinds is directly accessible to only a privileged few Australian students, so that realistic alternatives to international travel are needed. The Internet offers some opportunities to explore incidental mathematics of this kind, once students know what search terms to use. For example, using search terms such as “Islamic,” “art” and “mathematics” together leads to many websites allow students to be virtual tourists or even to engage in producing mathematical materials themselves.

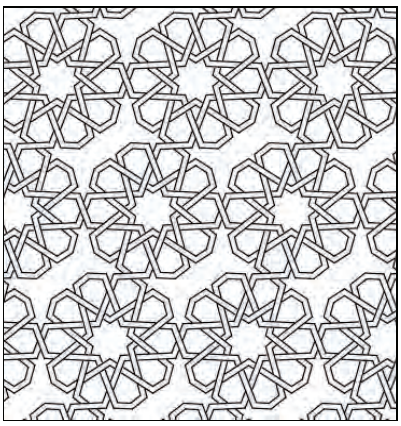

Figure 5. Islamic star pattern, generated with Taplats (Kaplan, 2003).

In the latter category, the *Taplats* software (Kaplan 2003) allows students to generate some kinds of Islamic patterns for themselves, such as that shown in Figure 5. As well as helping students to see that others have found mathematical patterns to be pleasing to the eye, such activity might encourage students to appreciate that mathematical activity has a creative side, and is not concerned entirely with the "correct" use of symbols and numbers.

Potential benefits for mathematics education

The biggest potential benefit of some attention to popular mathematics would seem to be the possibility that students might be helped to see mathematics as an intrinsic part of their culture, rather than something that is confined to the school mathematics classroom. In an age in which students are attracted in many directions, many of which are away from mathematics, some attention to increasing the popularity of mathematics might be worth the effort involved. Some of that attention might well be directed not only at students, but also to their parents and to the wider community.

It is not clear whether mathematics is not visible in the popular world because it is not popular, or whether mathematics is not popular because it is not visible in the popular world; indeed, this seems to be a version of the chicken and the egg argument. However it seems plausible that some attention to popularising mathematics, both within school and outside it, may be worth the trouble involved, especially if it helps to generate more awareness of the nature, significance and intrinsic interest of mathematics. This paper has suggested some possible ways of starting out to do this on the small scale of a school, recognising that larger scale efforts, such as those summarised by Budd and Lim (2004) will also be an important part of the exercise of making mathematics more popular and thus mathematics education better.

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WHAT DOES THE INTERNET OFFER FOR MATHEMATICS STUDENTS?

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The Internet offers a range of new opportunities for school students to learn mathematics and about mathematics. This paper analyses these opportunities, based on publicly accessible web sites that do not require direct payment by users. A typology of five different sorts of opportunities is presented, justified and exemplified: (i) interactive opportunities; (ii) reading interesting materials; (iii) reference materials; (iv) communication; and (v) problem solving. The ways in which the Internet is accessed in practice (both at home and at school) are recognised to be of significance to the prospects for successful educational use of these resources, as are the pedagogies invoked by teachers.

Introduction

Internet access is now very widespread in Australian homes, possibly even more so than in many schools. According to the most recent Australian census (Australian Bureau of Statistics, 2008), home Internet access has increased sharply recently to 67% of Australian households (rising fourfold from a mere 16% in 1998), with most of these now using broadband. While access is uneven (with significantly greater access in homes in major cities than for remote Australia), it is clear that it is rapidly increasing, and can be expected to increase further with recent Australian Government policies to provide widespread and fast broadband access.

Although the developmental work is unfinished, it is already clear that the proposed National Mathematics Curriculum will be significantly influenced by the learning opportunities created by the Internet: Paragraph 50 of the November 2008 *Framing Paper* (National Curriculum Board, 2008) makes it clear that it is no longer acceptable for digital technologies to be regarded as optional extras, and proposes that they be embedded into the curriculum. Similarly, the recent *Position Statement on Technology* of the National Council of Teachers of Mathematics (2008) in the USA reaffirmed previous versions in noting that the availability of technology is of key importance to modern mathematics education, affecting both what is taught and learned as well as how it is taught and learned.

These observations make it clear that the Internet is now an important element of the technology that is available for education, possibly offering new opportunities for students to learn mathematics and to learn about mathematics, as well as new opportunities for teachers of mathematics to structure student learning. The focus of this paper is on the kinds of learning opportunities for mathematics in particular that are freely available to relatively unsophisticated users, and in particular students and their teachers who are not familiar with developing Internet materials or websites. A major purpose of the paper is to organise, describe and exemplify[14] some of the opportunities now available, as well as to briefly suggest what promise these seem to hold for mathematics education.

Typology of potential uses

In this section, five different kinds of Internet uses for mathematics are identified and exemplified:

1. interactive opportunity;
2. reading interesting materials;
3. reference materials;
4. communication; and
5. problem solving.

The emphasis is on how students (as distinct from their teachers[15]) might use websites to learn mathematics or about mathematics, although some of these can certainly be used effectively with groups of students by a teacher.

There is not space here to provide a large number of examples for any of these five categories. Instead, the reader is referred to Kissane (2009) where there are links to many different examples, together with brief comments on their particular significance[16]. The examples have been carefully chosen to highlight good uses of the Internet (in the author's opinion) and to reduce the need for those interested in such materials (whether students, teachers, parents or others) to rely on browsing with search engines to find valuable resources.

Interactive opportunity

In recent years, the Internet has been used to allow students to interact directly with mathematical objects, in a variety of ways. In many cases, the experiences provided are very difficult to provide in other ways and engage the students directly in mathematical activity and a corresponding need to think about what they see on the screen. Most of these require that the browser has particular capabilities, such as being Java-enabled or having a plug-in for Flash software. These days, many sophisticated computers have such capabilities without the user even knowing that they do, so that significant expertise is not needed to take advantage of them.

Virtual manipulatives provide an opportunity for students to interact directly with (virtual) mathematical objects. Manipulations with physical objects have long been

[14] The paper contains some references to websites as examples. As websites change addresses and even disappear altogether, the reader is cautioned that some links in this paper may no longer work.
[15] Teacher uses may include other kinds of materials, such as online and electronic journals and resources for lessons, as well as use of materials that are intended for direct use by students.
[16] The website also refers to a sixth category, Webquests, which is not discussed here, partly because it is most likely to arise from mathematics teacher planning, rather than be directly accessed by students and partly for reasons of space.

considered productive for learning aspects of mathematics, leading to widespread use of materials for younger students such as multibase arithmetic blocks, pattern blocks and Cuisenaire rods. Virtual versions of these have the same mathematical properties and have the extra advantage of being in unlimited supply, unlike "real" manipulatives. The evidence for the use of these is promising (Galindo, 2005), both for young children and for older students (Moyer and Bolyard, 2002).

An example in this category from the excellent National Library of Virtual Manipulatives (2009) is shown in Figure 1. This Java applet allows older students to explore the properties of a rotation of 50^{o} about a point. It would normally be quite difficult to provide students with opportunities to explore the nature of such transformations in an effective way; in this case, the virtual manipulative provides a good alternative. Virtual manipulatives are not restricted to unsophisticated mathematical environments; the Demonstrations project at the *Wolfram MathWorld* site (Weisstein, 2009) has thousands of examples at higher levels.

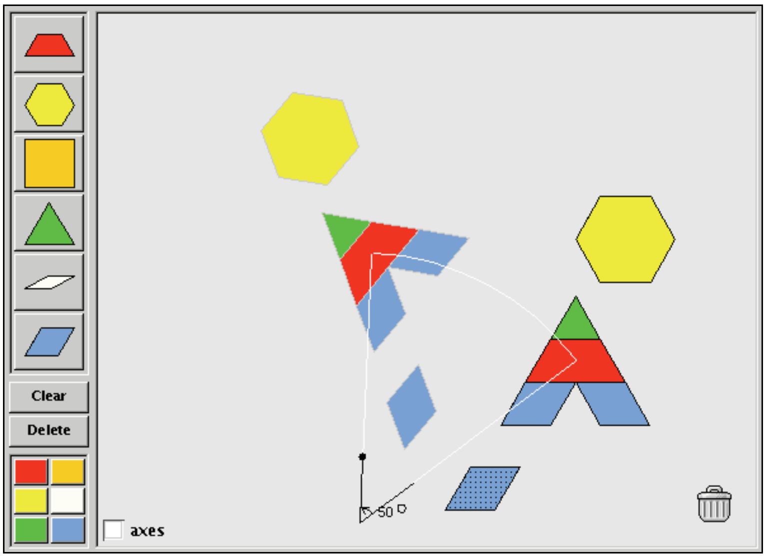

Figure 1. Transformations — Rotations from the National Library of Virtual Manipulatives (2009).

Some websites offer significant advice to teachers as well as students. Advice for students is usually necessary in order to direct student interaction, although advice for teachers is also important, to help clarify the intentions of website designers. In some cases, such as the NCTM's *Illuminations* website (2009), detailed lesson plans are provided for teachers and detailed instructions are provided for students. In other cases, such as the National Library for Virtual Manipulatives (2009), detailed advice[17] is also offered for parents, on the apparent assumption that the Internet is accessed at home as well as at school. Some interactive material provides sophisticated tools for mathematics, rather than individual lessons or activities. A good example of this is described by Hart, Hirsch and Keller (2007), supporting an innovative new curriculum exploiting such opportunities.

Interactive objects can be used with a whole class, as well as individually, and many are well-suited to the use of interactive whiteboards. Indeed, some are designed for this purpose, such as many of the Interactivities on the *NRICH* site in the UK (Millennium

[17] Of course, the provision of advice is no guarantee that it is good advice, or that it will be read, or followed.

Mathematics Project, 2009a). Some interactive objects take the form of games, for which it is expected that more than one person be involved in the interactions. A comprehensive set of interactive materials has been compiled by Rex Boggs (2009).

Reading interesting materials

There are many interesting materials related to mathematics on the Internet, in sharp contrast to many school libraries. Many libraries restrict themselves to mathematics textbooks, which are often not very interesting to students, especially if they already have a textbook of their own. Of course, the nature of school libraries varies immensely across Australia, for a range of practical reasons, not the least of which is the cost of books and the availability of suitable materials for students of different ages. Another factor in the availability of reading materials is the ease with which they can be found: few mathematical magazines are available widely and even many bookshops hold limited reading materials of interest to students, such as works in popular mathematics.

As well as good written material, some Internet readings may have an interactive element[18], good illustrations, hyperlinks and so on. Some materials intended for the general public are suitable for students, especially older students, and there are also good materials written expressly for students. As well as being of direct relevance as resources for school projects, high quality readings may kindle interests in mathematics that would otherwise not be sparked by more conventional school experiences. When mathematical expertise is in short supply, as it seems to be in many schools, materials that generate interest in mathematics amongst students may be of critical importance to the future.

Figure 2. Plus magazine, Issue 50 (Millennium Mathematics Project, 2009c)

Better readings on the Internet for school students are likely to be fairly short and liberally sprinkled with images or even interactive elements. Some take the form of

[18] This makes it clear that the typology does not define mutually exclusive categories.

regular magazines, such as the excellent companion websites *Plus* and *NRICH*, based at Cambridge University in the UK. Figure 2 shows a recent issue of *Plus* magazine, which provides regular and stimulating mathematical reading material for sophisticated secondary school students, as well as for mathematics teachers.

There is a range of materials available in all categories, to suit the needs of a range of students (and others). Not all "readings" are in the form of articles. For example, Figure 3 shows the American Mathematical Society of America's *Mathematical Moments* website, which provides posters of various kinds intended for download and display for various audiences, and for some of these podcasts as well; the posters provide many examples of the relevance of contemporary mathematics to societies. Other websites in this category include visual materials, such as film and television, although these of course increase both access requirements and costs.

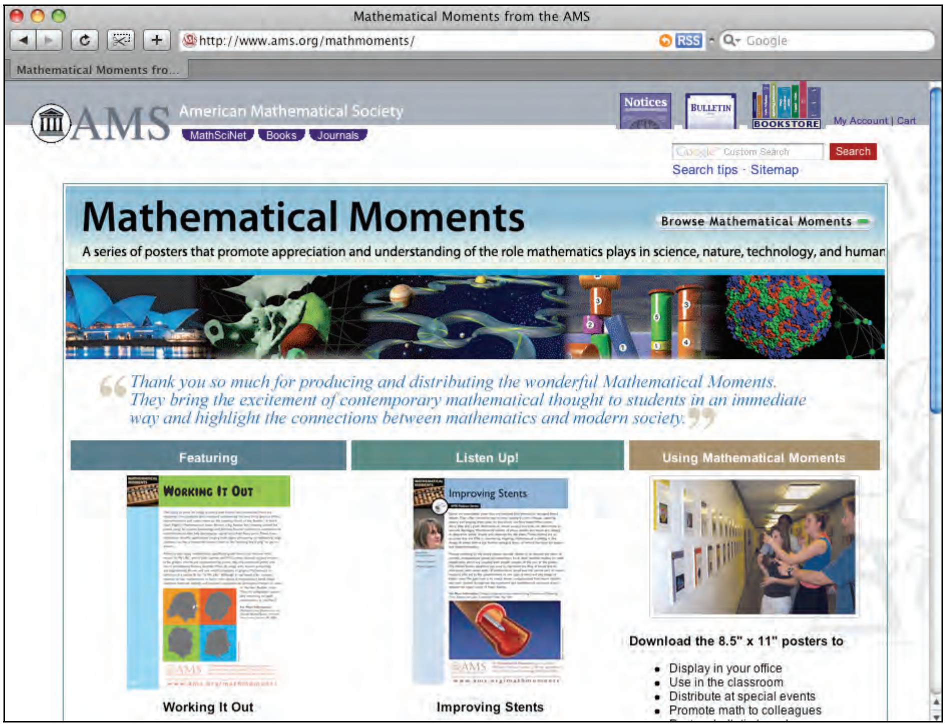

Figure 3. Mathematical Moments (American Mathematical Society, 2009)

Reference materials

The Internet is sometimes interpreted as a massive encyclopaedia, and can be used as a means of looking up various kinds of mathematical information for various purposes. These might be used by students directly from home, especially as few homes will have a mathematical reference source such as a mathematics dictionary or encyclopaedia. They might also be used in school, by both individual students and teachers or by a whole class, seeking clarification or information of a reference kind.

While printed mathematics dictionaries are available for both young readers and sophisticated professional mathematicians, it is unusual for students to have routine access to these. Indeed, it is still surprisingly unusual for school textbooks to routinely include a glossary — or sometimes even an index. For this reason, Internet mathematics dictionaries might well be more accessible to students than other kinds of mathematics dictionaries, and may be more helpful than standard dictionaries, because of the possibility of cross-linking of entries and even dynamic interactive definitions. Internet mathematics dictionaries are available for a range of year levels, from the least to the most sophisticated. The example shown in Figure 4 is intended for young children.

Encyclopaedias provide more detailed and extensive information than dictionaries, which (at least in paper forms) focus on the meanings of particular terms. Encyclopaedias provide more than merely meanings, but offer support to readers to locate ideas in contexts, including historical, practical, theoretical and social contexts. On the Internet, there is a gradual blurring of the distinctions between dictionaries and encyclopaedias, with increasingly each of these having some of the characteristics of the other. For this reason, students using quality mathematical reference materials on the Internet will generally be offered more information and support for learning mathematics than they expected, or would be likely to obtain, by consulting paper-based sources. Some encyclopaedias, such as *Wolfram MathWorld* (Weisstein, 2009) deliver an extraordinary depth of information that no school or home could ever hope to provide.

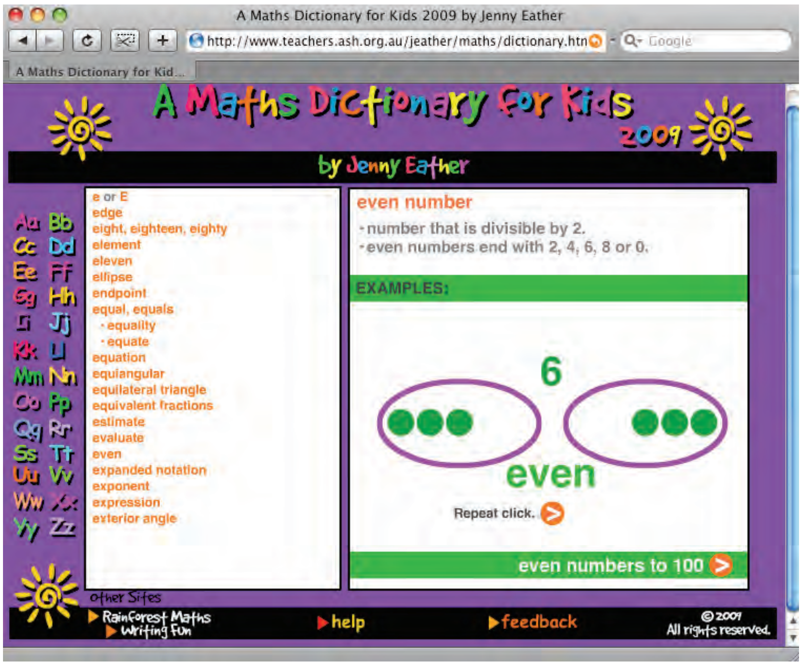

Figure 4. Mathematics dictionary for younger students (Eather, 2009).

While frequently neglected in school curricula, the history of mathematics is accessible to students on the Internet. The website maintained by O'Connor and Robertson (2009) is an outstanding example, providing opportunities for students to trace the development of mathematical ideas into the twenty-first century, and to see the roles played by mathematicians and others over the centuries. Such a perspective is rarely provided by school mathematics curricula, which also suffer from the problem of appearing to be unaware of mathematics of modern times. While the website is not expressly designed for secondary school student use, it still gives access to a sense of perspective as well as a realisation that mathematics is both an ancient and modern discipline, of immense significance across all ages and cultures.

Communication

The Internet offers opportunities for students to communicate with other students or teachers, regardless of geographical location. For students, some opportunities to be part of a wider mathematics community are provided. An example of this aspect of Internet use is the *Ask Dr. Math* site (The Math Forum@Drexel, 2009) illustrated in Figure 5. Students have been asking mathematical questions of the fictitious Dr. Math at the Math Forum site in the US for more than a decade now, and both questions and (often multiple) answers have been archived to avoid repetition.

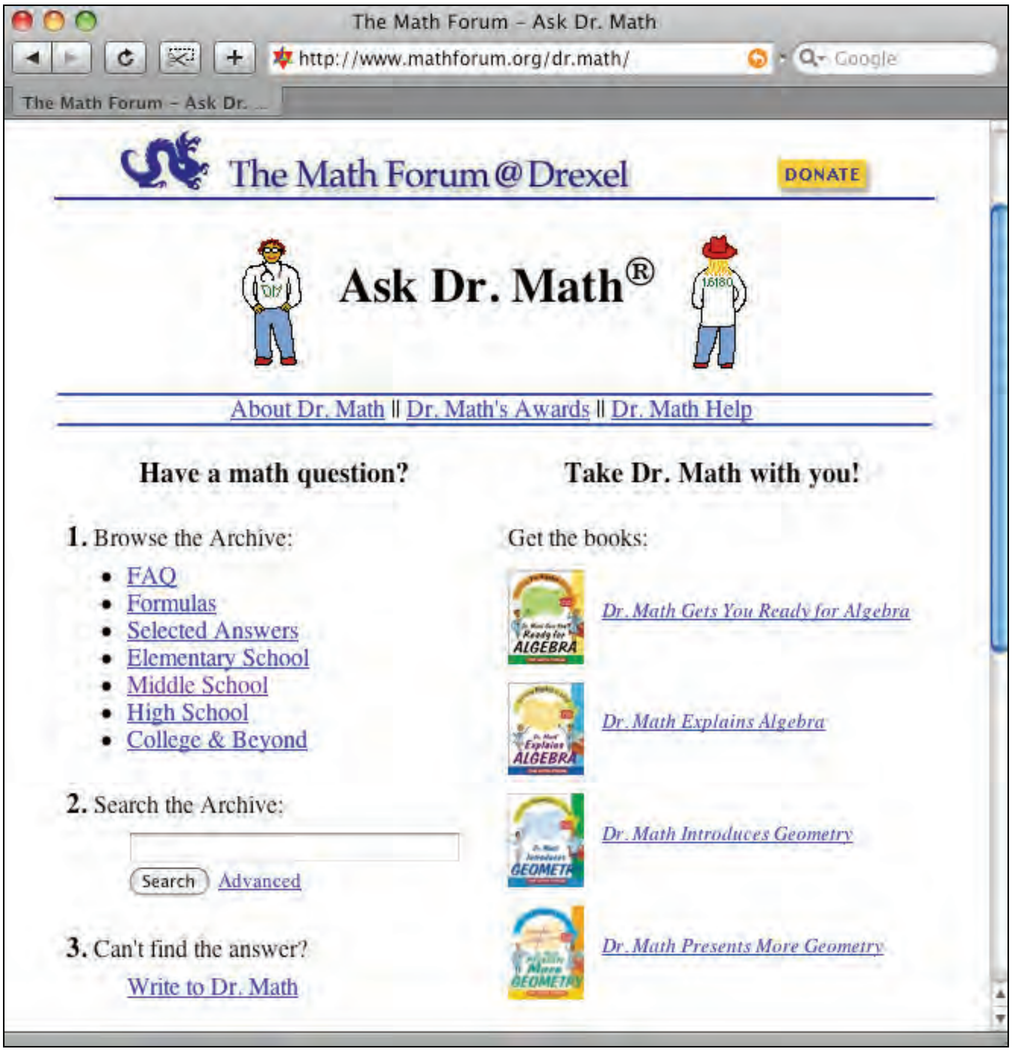

Figure 5. Ask Dr. Math home page (Math Forum @ Drexel, 2009)].

In a similar way, the *Ask NRICH* facility provides opportunities for more direct communication between students and others, as part of the extensive *NRICH* site (Millennium Mathematics Project, 2009b). Much of this communication is directed at students getting help with thinking about mathematical tasks and ideas, frequently by undergraduate students. In addition to the communication opportunities explicitly targeted via *Ask NRICH*, regular responses from students all over the world to mathematical questions and problems help to reinforce the fact that mathematics is of international interest and is universally engaging.

Problem solving

While a great deal of problem solving ought to happen in regular classrooms, the Internet can theoretically offer some extra benefits to students, including a regular supply of suitably targeted problems, opportunities to share solutions and even opportunities to get professional feedback on their work.

Figure 6. NRICH problems for March 2009 (Millennium Mathematics Project, 2009c).

There seems little argument for a set of problems on the Internet that could just as easily be written in a textbook, with nothing extra added. Figure 6 shows an example of how the Internet might offer increased opportunities for problem solving, with the very large *NRICH* site. The typical month shown includes a range of mathematical problems for school children at a range of levels of sophistication. Some of these problems include interactive elements, so they comprise more than merely descriptions of problems (as might be found in textual resources). As well as problems, the site provides hints, a printable version, notes for students or teachers and opportunities to contribute solutions online. Importantly, the site also publishes solutions from students themselves to previous problems, highlighting the range of ways in which problems can be solved as well as the range of students around the world interested in such activity.

Internet pedagogy

It is one thing to have access to good resources for learning, but another for these to be used effectively. A focus of this paper is on the sorts of resources that are independently accessible by students of mathematics, who may access the Internet at school, but who also may do so at home, without direction from the school. In practice, however, it still seems likely that most Internet use will be suggested by and supported by the work of mathematics teachers, so that some attention to this is warranted.

Access to the Internet in schools continues to be quite varied, with a range of possible mechanisms. Some schools rely on computer laboratories, with individual machines connected to the Internet via a network. In some cases, individual classrooms have access, at least for whole class use via a data projector, possibly also with an Interactive Whiteboard. Some classrooms contain more than one Internet-connected computer, allowing use by small groups of students. In some classrooms (very few in Australia, although this is increasing, as a result of government initiatives) Internet access is routinely available to all students all the time (through the use of laptops for individuals and wireless broadband networks). These variations are important influences on the ways in which the Internet can be used for learning mathematics.

The uses available to teachers are of course constrained by the available facilities; Alejandre (2005) has provided an extensive discussion of the various ways in which teachers might use their available classroom resources effectively. Galindo (2005) has argued that effective use of the Internet requires careful educational planning by teachers to ensure that students obtain the benefits expected. Some websites have been designed with educational settings in mind, and so offer advice to teachers (and in some cases, parents) on effective use.

It seems likely that Internet resources of the kinds described here will be most effective if they are integrated into the classroom and the curriculum in some systematic way; this will require teachers to be aware of the possibilities and to refer their students to them appropriately. One way of doing this is for mathematics teachers to systematically compile links to web sites for access by students within a particular class or school. In some cases, the resources are clearly supplemental to the curriculum, while in others they may prove to be central. While some resources can be well used as a whole class activity, others are better used with groups of students, while still others are likely to be effective with students working individually, either at school or outside school. Indeed, some Internet resources may fit best into the curriculum as homework activities, where students and their parents might together engage in mathematical activity suggested or supported by a particular URL.

Where facilities allow, many web sites lend themselves to use of an interactive whiteboard in a classroom. Miller, Glover and Averis (2008) at the University of Keele have researched the use of interactive whiteboards and offer much practical advice for teaching. An appealing summary of this work is the phrase, “At the board, on the desk, in the head”, elaborated upon in the excellent *Mathemapedia* entry (National Council for Excellence in the Teaching of Mathematics, 2009). Miller (2009) also offers (by subscription) links to hundreds of resources for secondary school mathematics. In Australia, a substantial collection of interactive resources has been nicely compiled by Boggs (2009).

When reliable classroom Internet access is problematic, as it is in many schools, it is worth noting that some materials may be downloaded from the Internet for educational use or purchased for classroom use, so that “live” access in a classroom or home is not necessary. Two good examples of this are the packages made available on *NRICH* (Millennium Mathematics Project, 2009b) and the CD-ROMs that can be purchased from the National Library of Virtual Manipulatives (2009), details of which are easily found on the respective websites.

The use of the Internet for Webquests is briefly explored and exemplified by Kissane (2009). These offer a systematic way for students to draw on a range of Internet resources. McCoy (2005) offers a good treatment of the possibilities and has constructed websites with several good examples.

Conclusion

This paper has explored a typology of ways in which the Internet might be used by relatively unsophisticated users to support and augment the teaching and learning of mathematics. The five categories of uses are distinctly different and offer different potentials, which might be constrained to some extent by the different circumstances through which Internet access is available to learners. The typology is exemplified by Kissane (2009), with links to many examples, together with brief supporting advice. There is a much to offer students, as well as their teachers, already on the Internet, without subscription charges, and every prospect that the available offerings will be increased in the next few years, as developers continue to make their work available to a global audience.

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GAMES: A CATALYST FOR LEARNING OR BUSY WORK?

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There is an assumption that the use of mathematics games is common in primary classrooms. This raises the question as to whether the use of these games improves outcomes for children. Teachers were invited to participate in a study to examine the use of games in school. We analyse some of the data from the literature and will share the criteria used by teachers when choosing games for their classrooms. A variety of games will be presented and participants will be invited to examine these games and develop their own set of criteria, which may be compared to those in this study.

Introduction

There appears to be a general consensus among teachers, particularly at primary level, that playing games assists in the teaching and learning of mathematics. When asked to pinpoint what aspect of games helps children learn mathematics, reference is often made to "fun" or motivation. Some games are utilised on a regular basis on the assumption that regular playing improves skills. Given that the time to teach mathematics is limited, this issue requires further investigation.

Consider the following statements made by two teachers who attended a professional development activity as part of a project on the use of mathematics games in schools:

Teacher 1: I try to teach something via a game, or more to incorporate a game into part of our learning program. Then it's much better or more effective.

Teacher 2: We've decided to incorporate Thursday games; so every Thursday, instead of having our mat session, we have Thursday games.

The first teacher was referring to the use of games that were targeted to a particular mathematical concept to be taught or practised. The second teacher seemed more concerned with children having "fun." It is the second type of use of mathematics games that concerned the authors. However, another question arises, even for teachers who use games for a very specific purpose, regarding whether games are effective and, if so, what makes them effective?

A team from Edith Cowan University (ECU), in collaboration with Industry Partners The Association of Independent Schools of WA (AISWA) and R.I.C. Publications, were awarded a grant to investigate the use of mathematics games in primary schools in

Western Australia. This investigation involved offering extensive professional development to a group of teachers from a variety of independent schools, with follow up in-school support. As part of this process, the teachers were expected to conduct some aspect of action research on the topic, and report back their findings to the group. This paper draws on those experiences.

Design of the study

As this study was exploratory in nature, it was only conducted with a small sample of 16 teachers. Various data were collected throughout the project, culminating in semi-structured interviews. Interview data were collated and analysed to identify common themes. This paper focuses on some of the interview data in order to identify and highlight more effective ways of using games.

Literature review

A review of the literature indicated a paucity of research on the use of games in the teaching of mathematics. Much of the literature dated from the 1980s. Two recent studies by Bragg (2006) and Asplin (2003) also indicated a lack of research in this area. Bragg completed a major study on the use of games in the teaching of mathematics. The study raised many questions about the efficacy of using games in the teaching of mathematics.

What is a game?

Defining a game has proved to be problematic, although several definitions were consulted (Bell & Cornelius, 1988, Oldfield, 1991a, Gough, 1999). Rather than provide a succinct definition, these authors tended to provide criteria for defining a game. Differences of opinion exist as to what constitutes a game. Gough (1999) stated that games should involve more than one player, while Bell and Cornelius (1998) included single player games in their discussion. As the researchers wanted to emphasise the structure and the mathematics inherent in a game, the following criteria from Oldfield (1991a, p. 41) were used as a starting point.

- The activity is governed by a set of rules and has a clear underlying structure to it
- The activity normally has a distinct finishing point
- The activity has specific mathematical cognitive objectives

One of the aims of the research, being reported here, was to flesh out these criteria and add to them, based on working with the teachers. During the initial session with the teachers it became apparent that, while teachers had a vague set of criteria for selecting games for use in mathematics lessons, few had given it much thought. For some teachers, the fact that an activity was badged as a game or came in a book with the word "games" in the title was enough to make it a game suitable for use in class.

Types of games

After defining what constituted a mathematical game, Oldfield (1991a) attempted to categorise games according to type. Included among his types were concept games, practice games and strategy games. In all, Oldfield listed 12 different types of games.

The researchers felt that there was some overlap between the various types listed and because this research project was rather modest, decided to focus on the three types:

- concept,
- practice and
- strategy games.

Foremost in teachers' minds were practice games.

Achievement

The work of Bright, Harvey and Wheeler (1985) is quoted often in the literature to support the use of games in the teaching and learning of mathematics. They concluded that, while games can be effective, simply using a game did not guarantee learning would take place. Onslow (1990), who studied the use of games as a means of overcoming misconceptions, noted, "One cannot assume that conceptual obstacles will be overcome solely by playing a game" (p. 591). He acknowledged that while games improve motivation, this was not enough. He went on to state, "Children do not make implicit links between doing and understanding without guidance" (p. 591).

Discussion

The role of discussion is emphasised in the literature (Booker, 2000; Oldfield, 1991b, Onslow, 1990), which is one reason why Gough (1999) believed that there must be at least two players involved in playing a game so that there is more opportunity for discussion. Oldfield (1991b) expanded on what discussion implies, indicating it was much more than simply asking questions. Booker (2000) explained that a classroom environment where talk is encouraged and children are expected to justify and describe their thinking is important if children are to gain the most from playing a game. The various interactions throughout the game provide an opportunity for discussion and sharing of thinking. This places the onus on the teacher to try to monitor the children's language, ensure that appropriate discussion is taking place and that the underlying thinking is correct.

Motivation

Bragg (2005, p. 254) concluded that the games used in her study "appeared to have had a positive effect on mathematical learning and student on-task behaviours." However, she noted a discrepancy in the attitudinal data collected via attitude scales and data collected from interviews. The former suggested that attitudes had become more negative while interview data implied that attitudes had become more positive. It is possible that the games had been overplayed and the children became bored or the challenge was lost, or the games proved to be too much of a challenge. Clearly choosing appropriate games to match the needs of learners is important.

Findings

Four main themes emerged from the interview data. The first of these concerned the place of games in classroom activities. Most teachers agreed that games were being used by classroom teachers for lesson starters, concept development and extension. Several of the teachers admitted that, prior to taking part in the research, their use of games had been limited to busy work for early finishers; games were also used with more able students, who were given various games as extension work.

The second theme concerned the types of games that teachers preferred to use. Most teachers felt the games they used needed to have simple rules that the children could pick up with little difficulty; the most popular of these were games of skill, with interaction between students being another requirement. Several teachers expressed a preference for a variety of games that could cater for small or large groups. Games that were quick to organise and pack away were appreciated.

The third theme of the descriptions of the teachers' favourite games, and what they believed were the children's favourites. As one of the games introduced to the research group, the card game *Numero* in particular was nominated most frequently by teachers.

The way in which participation in the project changed the teachers' approaches to using games in their classrooms was the fourth theme. Many participants reported that the research component of the project supported their use of games and gave them the confidence and inspiration to explore the use of games further.

The types of games teachers prefer to use

The interview question on this topic elicited a wider range of responses than we anticipated. Many of the comments were more concerned with criteria[19] for what makes a suitable game in general rather than identifying a type or genre of game. Table 1 below shows the comments and their categories.

Table 1. Teachers' preferred types of games.

Description	*Number of responses*	*Genre*	*Criteria*
Interactive	2	✓	
Challenging	1		✓
Simple, easy to explain	3		✓
Played in a small group	3		✓
Place Value	1		✓
Mental practice	2	✓	✓
Relies on skill rather than luck	2		✓
Not too noisy	1		✓
Quick and easy to set up	1		✓
Not too many pieces	1		✓
Clear links to concepts	1		✓
Children use manipulatives alongside game	2		✓
Where children move around physically	2		✓
Strategy games	2	✓	
Everyone participates at same time (not taking turns) i.e., whole class	2		✓
Games that the teacher can observe	1		✓
Practise a skill	2	✓	
A variety – chance for children to experience them all	2		✓
Card games	1	✓	
Board games	2	✓	
Visual, spatial awareness games	1	✓	
Games that are relevant to children's stage of learning	1		✓
Number games	1	✓	
Games that are quick to play	1		✓
Games to make rather than buy	1		✓

[19] We will consider these criteria in more detail when examining a sample of games in the conference workshop.

The review of the literature on games brought to light the various categories into which games fall. Among the types mentioned were: concept games, strategy games, practice games, interactive games, misere games, games of skill, games of luck, card games, board games and electronic games.

Teacher change as a result of participating in the research

A recurring comment in response to a question regarding the effects of the research itself on the participants in the project was that the teachers had become more inclined to use games in their classrooms. They could see the benefits of games, and felt that they were better able to discriminate good games from poor ones. They appreciated that the project gave them the luxury of time to look more closely at games, and several said they had become inspired to use games more than they had previously. Their experience with the project gave them more confidence in using games, with the added benefit of research to back this feeling. One teacher commented, "Before, it all seemed a bit too complicated and I was doubtful of the educational benefit. But now I've seen my children using *Numero* and the *Trading Game*, I see there is actual improvement and they seem really enthusiastic". Concern about the time taken to learn and play games was mentioned by another teacher, who went on to say, "You always get this perception that time is short and you've got to cover the course and now I know I can cover the course while doing this."

Other comments centred on the enthusiasm of the children when playing games, that they "were not just having fun, but learning from it." The teachers believed that the use of games kept children more focussed. One teacher described teaching the game of *Numero* to parents so that they could play with their children. He believed that, "Games have helped in that way in terms of getting parents involved with their children and now they are taking games home and playing with their children. So it has got them involved in doing maths activities with the kids that it wouldn't have done before."

An improvement in their children's skills was mentioned by a number of teachers. One mentioned that her children's skills in fractions and division had improved as a result of playing *Numero*. Others talked about an improvement in their students' basic number facts.

Two teachers noted an improvement in their children's use of language. "They are talking about what they are doing and I think they get a better understanding through playing a game." Also they referred to the advantages of peer tutoring and self-correcting, and how games can become a supporting factor.

A further reported change was that the participants came to be regarded in their schools as "experts" on mathematics games, and frequently were asked to give advice to peers on various games.

One final comment came from a recently graduated teacher:

> You can be doing times table or mental calculations and it is the same thing but it can be done in a more fun environment rather than reading tables out or getting them to do a hand written tables activity or something like that. Keeping the students focused and on task or playing a game for no purpose at all. They keep asking to play thumbs up, heads down and you think really there is no educational value in a game like that. Some of the maths games are really exciting and they are still learning and they don't necessarily know that.

Examining some maths games: Advice for action

Earlier, reference was made to the criteria that teachers listed when discussing the types of games they preferred to use (see Table 1). Consider the following games: *Contig* (Lewis, 2005, p. 8), *Mastermind*, *Nim*, *Mary's Game* (Swan, 1997, p.20), *Battleships* and *Equal/Unequal Parts Fraction Game* (Booker, 2000, p. 82). Also the reader is invited to consider one of their own preferred maths games, from the perspective of the criteria in Table 1. Do the games meet these criteria? What further criteria could be added? In the light of the criteria, would they be considered as games? Would they be useful in your class? How would you classify them?

Other questions arise when composing a list of criteria. Are there criteria that fit one genre of game that would not be as important for a different genre? For example, do we look for something different in a practice game than in a concept game? Do criteria for computer or electronic games differ because a student is playing against the computer rather than against a class member; or against a student in a different school, state or even country?

The researchers would suggest that the ability to adapt a game to suit the needs of the variety of any learners in the room is one that should be added in the light of findings by Bragg (2005). The opportunity for discussion is another that could be added to the list. Consider what questions might be used to encourage discussion or what further task, such as redesigning the board, could be used to stimulate discussion.

Conclusion

The authors challenge the notion that simply playing games is good pedagogy. There is a lack of evidence to support this idea. In fact, in some cases, playing poor games or mismatching of games to student needs may be akin to colouring in a cartoon character on a worksheet – simply busy work. However, well-chosen games that meet specific criteria and are matched to the needs of the students can become the catalyst for discussion and debate about mathematics.

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A STUDY OF TEACHERS' LEARNING AND TEACHING OF COMPUTATIONAL ESTIMATION: GETTING STARTED

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The value of computational estimation has recently been recognised in the US but it has not been given much prominence in Australia. This paper describes the first stage of a research study that analyses a professional learning program designed to develop teachers' learning and teaching of computational estimation in primary classrooms in Australia. The professional learning program entails the researcher working collaboratively with seven teachers to enhance their teaching of computational estimation. Disseminating to the seven teachers the research literature about computational estimation strategies is a central component of the program. This paper reports on some of the initial findings of the study.

Background

For the last three decades there have been persistent assertions that there is value in teaching primary school students how to undertake computational estimation (McIntosh, Reys, Reys, Bana & Farrell, 1997; Reys & Bestgen, 1981; Reys & Reys, 2004; Sowder, 1992). Computational estimation is a process in which some or all of the numbers in an arithmetic problem are approximated to simplify the computation. There are situations in our lives where an estimate is just as useful as an exact calculation. For example, when deciding how much money to take on holiday for daily purchases, an estimate is more sensible than an exact calculation. There are two main areas in mathematics education where this type of estimation can be useful – as a useful device for checking exact calculations (Reys & Bestgen, 1981) and as a useful contributor to the development of number sense (McIntosh et al., 1997; Neill, 2006; Sowder, 1992). Despite the value of computational estimation, it has yet to take its rightful place in the primary classroom. There are a several reasons for this. For example, there has been little settled research available to inform the teaching community because researchers in this field have taken various epistemological positions, and there has been a lack of classroom observation of teaching practice. The latter, arguably, is something of "a chicken and egg" problem between lack of attention in the mathematics curriculum and lack of credible research on its teaching (Trafton, 1994, p. 77).

Introduction to the research project

The research reported here is the first stage of a PhD study being undertaken by the author at Edith Cowan University. The study examines a professional learning program designed to develop teachers' pedagogical content knowledge of computational estimation, and seeks an understanding of what this program means for the participants. The pedagogical content knowledge required for teaching mathematics was recently defined by Sowder (2007) as:

- an overarching knowledge and belief about the purposes for teaching (mathematics);
- knowledge of student understandings, conceptions, and potential misunderstandings (in mathematics);
- knowledge of mathematics curriculum and curricular materials; and
- knowledge of the instructional strategies and representations for teaching particular topics. (p. 164)

The year-long professional learning program entails teachers reflecting on their own practice and working collaboratively with the researcher and other members of the program. Ideally, as the program progresses, the teachers will become more confident in their number sense ability, gain a comprehensive understanding of the computational estimation strategies, and develop effective teaching approaches for those strategies. However, the researcher is mindful "that many a well-laid plan" is not realised and seeks a detailed account of what happens as the program unfolds. There are three phases to the study. This paper is reporting on the initial phase.

Methodology

A multiple case study approach was chosen for this research as it aims to describe and analyse a small number of teachers and their students, rather than generalise from any findings. Underpinning the methodological approach is social constructivist theory (Crotty, 1998; Patton, 2002, p. 96) as the research seeks to gain an understanding of the participants' behaviour.

The participants for the study are seven primary school teachers and their Year 6 pupils, all recruited from low-fee independent schools in metropolitan Perth. The principals of the schools were invited to be involved and asked to recommend teachers who had been teaching for at least three years and were confident and competent in their mathematics teaching. The teachers are taking part in three days of professional learning, as part of the year-long research project. The data collection includes interviews with teachers and students, observations of the professional learning days and the classrooms, work samples of estimation tasks, and reflective diaries of the researcher and teachers. The data collection will be coded and analysed using the computer software *NVIVO 8*. The research project has just finished its first phase and the initial findings are reported here.

Which computational estimation strategies will be introduced?

After reviewing the research literature, the researcher identified the most useful and appropriate estimation strategies for Year 6 pupils. The following six strategies were selected:

Table 1. Suitable estimation strategies for the primary school.

Strategy	**Identified by**
Intuition	Lovitt and Clarke(1992)
Rounding	Dowker (1992) Trafton (1994) Reys (1984)
Compatible numbers	Dehaene (1997) Reys (1984) Leutizinger, Rathmell and Urbatsch (1986)
Front end loading	Dowker (1992) Trafton (1986) Allinger and Payne (1986)
Benchmarking	McIntosh (2006) Carter (1986)
Range	DeNardi(2002)

These strategies can be used with all operations, although some lend themselves naturally to certain operations more than others. For instance, when students are dividing, the compatible numbers strategy is especially valuable because students can approximate using a factor of another number. For example $26 \div 4 \approx 24 \div 4$.

These more straightforward strategies are all reformulating ones, that is, they are strategies that make the calculation simpler by leaving the operation intact (Reys, Bestgen, Rybolt & Wyatt, 1982). Other strategies, which involve changing the mathematical structure of the problem or compensating for the approximation, are more complex and it is proposed that these are best taught at secondary school. An elaboration of the six strategies follows.

Intuition

This strategy involves students using their past mathematical experiences, such as manipulating quantities of concrete materials, in order to estimate but without articulating the steps they took. This estimation strategy is valuable as it encourages inexperienced students to “have a go” and can reinforce the belief that approximate answers can be valuable. Students may use this strategy in one or all of the numbers in the problem. It is very common in younger and inexperienced estimators. For example the student may say: “The average of 8, 12, 6, and 15 is about 10 but I can’t say why.”

Rounding

This group of strategies is one where students make the computation easier by changing the numbers to a power of ten. The decision as to which power of ten will be dictated by the arithmetic problem. Depending on the context, students may also choose to round one or all of the numbers in the problem and may round up or down. The following example is where a student rounds both numbers down.

$$34 - 22 \approx 30 - 20$$

Compatible numbers

Students will use this strategy when they identify that it is easier to calculate with some numbers than with others. Sometimes both numbers will be approximated and at other times only one number will be changed. In arithmetic certain numbers are compatible with each other as the example below highlights:

$$0{\cdot}23 \times 40 \approx \frac{1}{4} \text{ of } 40$$

Front end loading

This can be one of the first strategies that students encounter and the great benefit of it is its simplicity. Students focus on the left-most digits of the calculation. One reason that this strategy may not be used by students is that this underestimate may not be very accurate, especially where the magnitude of the number is large.

$$\begin{array}{r} 435 \\ +\ 328 \end{array} \approx \begin{array}{r} 400 \\ +\ 300 \end{array}$$

Benchmarking

Students will build up mental reference points in mathematics that can be used as an estimation tool. As with the other strategies mentioned, it may be appropriate to use this strategy on one or all the numbers in the problem. When asked to calculate problems involving fractions students can estimate using well-established fraction concepts, for example:

$$\frac{7}{8} + \frac{3}{8} \approx 1 + \frac{1}{2}$$

Range

This strategy is where students calculate the range into which the answer to a calculation will fall. In a calculation one or all of the numbers may be approximated to calculate the lower and upper range. An example of this strategy would be:

3.7×6

The answer will fall between the lower limit of $3 \times 6 = 18$ and the upper limit of $4 \times 6 = 24$.

So the answer will fall between 18 and 24.

Computational estimation is in the curriculum documents — why is it not taught?

Anecdotal evidence created through visiting schools suggests that estimation strategies are not being taught in the primary school. Early findings from this research study provide confirming evidence. All the teachers interviewed before the professional learning began indicated that computational estimation was an important component of mathematics. Most described in their interviews that they often suggested to students that they estimate, although they did not instruct them on how they should do this.

Teacher 1, who had been teaching for many years, explained that: “As sixes we do estimation as part of the lesson in any case, like if you are doing multiplication; ‘What do you think it is going to be?’”

Teacher 3 said she got her students to estimate in her mathematics lessons: “When we have done our algorithms, I will say, ‘Is that a sensible answer?’ If we are taking one number away from another number and we have a higher number than we started with, we say, ‘Is that sensible?’”

Teacher 2 had noted that students did not naturally estimate in the way that adults might:

> You’re multiplying that number to that number and because they are columns they come up with a silly answer they don’t estimate. They don’t think about what they are doing. They don’t think about how wrong it possibly could be prior to doing the algorithm.

Teacher 4 also noted that students did not intuitively undertake estimation. When asked what he thought of when he heard the word estimation, teacher 4 said that he thought estimation was:

> …necessary and not done enough. Not done enough by the children even though you tell them. They still don’t understand why you want them to estimate before they have done the sum. More often than not they do the estimation afterwards.

The majority of the teachers said they wanted their students to estimate but, apart from activities where estimation was taught in an algorithmic fashion using the rounding strategy, the teachers were not teaching their students how to estimate. The following responses were given to the question: What problems do you have in teaching estimation? These responses give some insights as to why computational estimation has not been taught:

> We become so focused on learning and understanding the process of the algorithm that the focus is on that, and we often don’t allow them to take the time to estimate (Teacher 3).

> Estimation is one of those things. It’s like health or RE and something has to go. It’s the thing that gets dropped. We will pick it up somewhere; we will pick it up next year; we will pick it up the year after and that it’s unfortunate that everybody says that. It keeps getting dropped and at the end it hasn’t been done (Teacher 5).

> It probably does get pushed out, doesn’t get highlighted (Teacher 4).

It is also apparent that most of the teachers interviewed were not aware of estimation strategies, other than rounding. When asked what strategies they were aware of, their answers included:

> Yeah, rounding to the nearest ten, nearest hundred rounding up, rounding down but that’s it, no other strategies (Teacher 4).

> I try and teach as many strategies, like when we do mental maths on the board. Round up, Round down (Teacher 6).

> Researcher: Have you named any strategies with your pupils?
> Teacher 1: Like guess and check?
> Researcher: A bit like that.
> Teacher 1: Not particularly, no.

Whilst it is appreciated that the purposeful sample of seven teachers is too small from which to generalise, the teachers involved were considered by their principals to be confident and competent in mathematics. Their pedagogical content knowledge about computational estimation is therefore likely to be fairly representative of the state.

The way forward

It is clear from the early data gathered in this research that teachers believe that computational estimation is important but at present do not have the pedagogical content knowledge necessary to teach these strategies effectively. The professional learning program that forms part of this project is aimed at developing the teachers' pedagogical content knowledge of computational estimation using a collaborative approach. Teachers have been provided with an opportunity to reflect on their practice and identify areas in which they would like to focus for the professional learning. There will be opportunities to continue this process of reflection as the school year progresses. Suggested teaching approaches of computational estimation are being evaluated and trialled by teachers in their classrooms. The outcomes of these activities will be reviewed in the professional learning sessions. This cyclical design experiment approach (Shoenfeld, 2007) will allow for an analysis of how effective are various teaching approaches in enhancing students' learning of the above estimation strategies. At the end of the year, the researcher will conduct a final analysis of this professional learning professional program and evaluate how the pedagogical content knowledge of computational estimation of the seven teachers' developed and how it influenced their teaching.

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PLACE VALUE:
WORKING WITH CHILDREN AT RISK IN YEAR 3

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This paper discusses intervention strategies implemented over a six-month period with two groups of students. These students were identified by their teachers as having difficulties in mathematics after the first term of Year 3. Focussed place value activities involving trading games were presented to the students together with learning activities in the topics of fractions, patterns, addition facts and doubling numbers. Students were encouraged to communicate their thinking as they worked with concrete materials, occasionally record with pictorial and symbolic representations and reflect on their learning through writing. These activities proved successful for the ten students who regularly attended.

Introduction

Almost twenty years ago, after working in four early childhood classrooms for most of the year, the author stated, "I am convinced that early trading activities… are essential and valuable for understanding number systems and operations" (Bruschi, 1990). Benefits gained for that cohort of Year 1 and Year 2 students included: abundant use of mathematical language; mathematical thinking and strategising; the ability to record and justify their place value understandings; and enhanced social interactions in small groups. Over time, these benefits have been reinforced through experiences of working in classrooms with young children, in university courses with pre-service teachers and during professional development with practicing teachers.

Background to non-proportional place value activities

Although materials such as MAB base 10 and pop sticks are the most commonly used materials for numeration (Price, 1998) a range of materials and activities are required to develop deeper understanding of the number system (Association of Independent Schools of South Australia, 2004). There is some question as to the value and effect of MAB base 10 because the children do not construct the relationships inherent in the blocks. Furthermore working with too many objects in a group before bundling or exchanging is a difficulty for young children. Anderson and Collyer (1988) noted:

> Much of what is suggested by writers in this area is related to work in base 10. The use of base 10 requires that children have already developed a considerable understanding of number… there is a need for children to be given the opportunity to develop [place value]

> concepts at a much earlier age and to use numbers with which they would be much more comfortable (p. 88).

Around 3–5 years of age, students are often able to see at a glance how many there are in a small group without counting (Department of Education and Training of Western Australia, 2004). This skill known as *subitising,* is useful for early place value activities in different "lands" because children can work comfortably with numbers of up to five or six objects without relying heavily on counting skills. Hence, working initially with smaller groups (three, four, five or six objects) can help students gain confidence through visualising.

Furthermore, Reys, Lindquist, Lambdin and Smith (2007) stated:

> [M]any children who count correctly have absolutely no concept of place value. In most cases the confusion or misunderstanding can be traced back to lack of counting and trading experiences with appropriate materials and the subsequent recording of results. Early and frequent hands on activities… are essential prerequisites to meaningful understanding of place value (p. 181).

The development of children's mathematical ideas and relationships whilst using concrete materials is essentially a social process where language helps to build understanding (Rogers, 1997). Interacting in small groups encourages students to extend their current skills and thinking through experimentation, problem solving and discussion. "Children can go well beyond their own capabilities in collective activity or under the guidance of adults… as children's Zone of Proximal Development (ZPD) can be opened up during their play" (Rogers, 1997, p. 36). The ZPD is a valuable concept for educators as it allows teachers to recognise current and potential growth. Furthermore, when a more experienced adult or peer works alongside the student in a "community of practice" the potential for learning is greatly increased. Goos (2008) stated:

> Contemporary socio-cultural theory proposes that learning involves increasing participation in socially organised practices, and the notion of a situated learning in a community of practice composed of experts and novices (Lave & Wenger, 1991; Wenger, 1998) has been fruitfully applied to educational settings (p. 235).

The intervention activities undertaken with children in this project involved four or five students and an adult educator actively engaged in mathematical activity, opportunity for students to build on prior knowledge and experience, discussion of learning as it occurred and after, and opportunity for reflection.

Participants and environment

The school in which this project was conducted is located in the outer suburbs of Adelaide, South Australia and has a population of 360 students. The teaching and learning environment in Year 3 for 2008 consisted of a large carpeted area divided into three sections: whole group activities and discussion space; two smaller floor spaces for launching of lessons and small group discussion; and two seated sections with tables and chairs for individual, partner or small group work.

Teachers selected students for the project from the Year 3 class of 50 students (aged 7–8). The students were identified as having low mathematical achievement, possible avoidance behaviour and/or low self-confidence in mathematics. Initially eleven students were allocated for "maths activities." Two large group lessons and 12 small group lessons of between 30 and 40 minutes were conducted over a six-month period. There were some students absent on individual days and one child did not continue after four sessions for health reasons.

After the two initial sessions that had been held in the library, the students were separate into two groups based on whether they belonged to Teacher A or Teacher B. Twelve sessions were held in a temporarily unoccupied classroom with children seated around two connected tables. Both groups met for approximately 40 minutes weekly or fortnightly from June to November. Group (1) of five students (three girls and two boys) from year three *Teacher A* met with the author before recess play and Group (2) of five students (girls) from *Teacher B* met after recess play but inclusive of eating recess. The "communities of practice" therefore consisted of one educator and a small group of students engaging in learning activities in a separate space from the classroom.

Curriculum relevance

It was essential to determine what the students already knew about early number and place value. Furthermore, because the greater Year Three class were working on fractions at the commencement of the project, those outcomes were also considered using the curriculum adopted by the school (DETE, 2001).

Using observation and informal questioning of students, children were placed mostly at standard 1, outcome 1.6 for Number, with respect to the South Australian curriculum. More specifically, all students could sort, order and compare collections of objects, and count to determine the size of a collection (under 20 counters). All students were aware of ten as a unit in counting and could name the value of larger collections of materials. Some students could not show or read the correct numeral for a two-digit number nor indicate what the place value tens or hundreds meant in a numeral. All students could name fractions as portions of a whole but some did not recognise the importance of equal size pieces. Students were able to compare the size of parts of a whole given spatial models but not order fractions symbolically.

The aim was for all students to achieve standard 1 (as described above) by the end of the project and achieve components of outcomes 2.6 and 2.7 in Number specific to place value. That is, students would be able to use manipulative materials to extend the numeration system to tens, hundreds and thousands; deconstruct numbers into smaller parts and reconstruct them in different ways; and use mathematical language to describe and represent numbers.

Activities used and adapted

Meaningful understanding of place value is complex and requires connections between face and place value; the zero as a placeholder; the base ten system; and the additive property (Reys et al., 2007). Furthermore, Thomas and Mulligan (1999) noted, the multiplicative nature of base 10 is a critical problem. The project reported in this paper focussed on well structured, repetitive place value activities which included sorting, comparing and collecting counters or blocks (trading up for addition), giving back (trading down for subtraction) and winning a prize (simple multiplication). Activities and materials used were progressed or adjusted according to the needs of students individually and as a group.

Determining prior knowledge of students

For the prior knowledge activities, the materials chosen were small bears in red, green, blue and yellow, round coloured counters, sorting mats, base ten mats, dice, *Numero* playing cards, circle fraction cake and pattern blocks. Year 3 numeracy benchmarks (Curriculum Corporation, 2000) were used as a guide, that is: read and write whole numbers up to 999; compare and order numbers up to 99; demonstrate their knowledge of hundreds, tens and units. The task used the following order of activity:

- Three cards were shuffled and placed on each student's place value mat to determine if the student could say the number, this was repeated with two cards if the student was unable to do so.
- Some coloured bears (quantity under 20) were given out, and each child was asked to estimate and count the objects. Students were asked to group the bears.
- A small handful of counters was placed on a trading mat (with coloured spots as headings) and students were asked to trade them. If a student had greater than ten of a given colour, the basic rule of non-proportional trading was introduced for base ten. For example, "If we have ten yellow that equals one blue." Any "illegal" collection needs to be traded (for example, 12 blues and 4 yellows became one green two blues and four yellows or 124).
- Students were asked to show the value of their collection on a base ten recording mat (hundreds, tens, ones) in pictures (circles) or symbols.
- Students were questioned, for example, "What does the '5' in 56 mean? Is it the same as the '5' in 65? Why? Why not? What does the zero in 620 mean? How is it different to 602? Who has the largest collection, child *x* or *y*? Explain."

The students were enthusiastic and positive throughout the session but for one student, who, towards the end, stated that she really did not want to be in the group because "I know everything about numbers!" A four-digit number was written and Madie (pseudonym) was asked to read the number. Because she was unsuccessful the task was repeated using four playing cards, randomly chosen by Madie and placed on the mat. Madie was unable to state the number correctly or explain the value of the digits. Later, *Teacher A* discussed Madie's attitude and the reasons why her behaviour might have been unpleasant, even aggressive.

Early trading activities

Initially students were introduced to the rules of trading or exchanging in *Threeland*. The following procedure was undertaken in a small group: each child was given a trading mat; a collection of counters was placed in the centre of the table; a die with only one and two on it was introduced; and depending on the skill and confidence of the group, one child was nominated to be the banker. Year Three students could trade correctly without a banker. The game was played as follows: each child rolled the die; collected yellows; traded if possible (three yellows equal one blue, three blues equal one green, and three greens equal one red)**.** When a red was gained the game stopped.

As children became expert in the game (usually after one or two plays), the land was changed to *Fourland* (the die has 1, 2 and 3 on it), *Fiveland* (the die has 1, 2, 3 and 4 on it) and *Sixland* (the die has 1, 2, 3, 4 and 5 on it). It was easy to determine when students needed a change because they asked questions, for example, "What happens in *Sevenland*? *Hundredland*?" After trading in various lands, conventional dice, with values one to six, were introduced. This allowed students to strategise, for example a six in *Threeland* is equal to two blues not just six yellows.

Focussing on Base 10

When students were ready, they were introduced to the conventional numeration system, *Tenland*. There were several ways of doing this: either using the coloured trading mat and two special dice which have small numbers (0, 1, 2, 3, 4 and 5) and large numbers (5, 6, 7, 8, 9 and 0) or the base 10 mat and two normal dice or a dodecahedral die. A conventional die was used because adding two dice required students to find the total. This gave valuable practice in solving addition facts up to 12. Furthermore, the complication of "doubles" naturally arose: "Do we decide as a group to have another turn or do we double the value of the throw?"

After time, these activities led to considerable problem solving according to students' ability. For example, some students started using a counting strategy similar to giving change in shops. If they had eight yellows or units on their mat and they threw a sum of four on the dice they would count on two more: "nine, ten," trade in their mind and give a blue (ten) to themself leaving just two on their mat. This thinking developed over time, was individualistic and could not be rushed. Students trying to help their peers did so with mixed success.

More advanced activities

After some time of successfully trading up (approximately five to seven sessions) the idea of trading down (or subtraction) was introduced. Students started the game with one hundred (green) on the mat and were required to give away yellows each roll of the die. The first student to successfully give away the exact amount on the mat was the winner.

The next complication introduced to the game was "prize time." During a normal trading up game, prize time would be randomly declared. Students were required to roll the die and double, triple or quadruple the value of their throw (similar to "utilities" in *Monopoly*). This adaptation involved early number facts for multiplication. The students became quite excited, as it was possible to become wealthy.

Finally, the major adaptation that occurred in the last four sessions with the students in this project was a change of materials. The students were not developing more advance strategies for counting, trading and for giving automatic responses to addition facts for ten. For example, if a six and four were thrown, the response of ten was not immediate. To overcome this problem, Unifix in towers of ten were introduced as the core material instead of coloured counters. This allowed students to see the relationship of ten more clearly when trading and become more familiar with addition facts for ten.

Recording of results in the game

At various stages during the above activities students were requested to record their results. They were often asked to choose a playing card that represented the quantity for each place value, then they were requested to draw up a place value mat, record with pictures and/or symbols and say something about what they had learned, either verbally or in writing. Recording in words, pictures and symbols added additional depth to observations because students were required to explain or justify their thinking. Questioning of students' understanding included being asked to state the place value of each digit and the overall results. For example, questions included “How many more ones would you need to get ‘*x*’ tens? How many more tens to get ‘*y*’ hundreds? How do you know your result is bigger or smaller than ‘z’ child’s result?” The students’ answers were often scribed. Anecdotes were written to the teachers in students’ mathematics books.

In the last few weeks of the intervention project students were asked to throw a dodecahedral die (numbered 1 to 12) for each place value: ones, tens and hundreds; exchange if necessary; and give a final number. This helped to determine how close the students were in meeting the intended outcomes.

An example of one student’s recording, Nathan (pseudonym), is below.

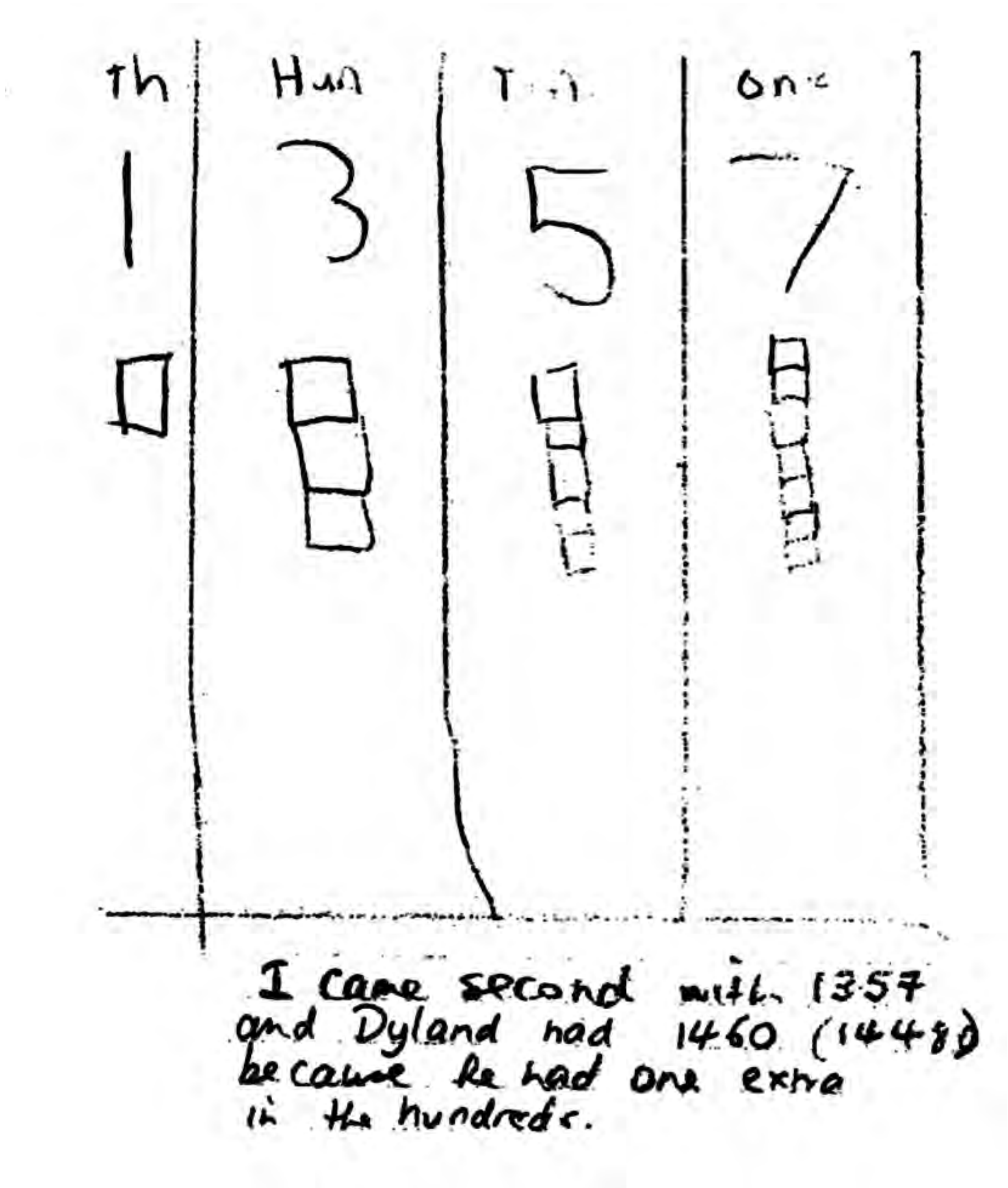

Figure 1. Nathan’s recording of the place value dice game.

Outcomes and findings

By the end of the intervention period (June to November) all students were able to read, write and discuss numbers to four digits confidently. They were able to compare the size of two and three digit numbers. Students understood the role of zero as a placeholder, the additive nature of base ten and they were able to explain their processes, their discoveries and express their feelings about the activities and mathematics in general.

In terms of the SACSA Framework stated above, students had achieved outcomes 1.6 and 2.6. Furthermore, according to *First Steps Mathematics: Number*, children in this project had achieved level 3 outcomes for Key Understandings 4, 5 and 6. That is: students could say the number names into the teens (level 1), could use the decades up to and over 100 and count forward and backward for numbers to 100 (level 2), and readily use the names of the first several places from the right (level 3) (DETWA, 2004, p. 41). Students could also additively partition for example, 2706 into two thousand + seven hundred + six (DETWA, 2004, p. 53) in a standard way. A smaller number of students within the group could partition whole numbers in non-standard ways (DETWA, 2004, p. 61).

The following quotes in response to questions "What have you learned today? What can you tell me about this result?" offered insights into the success of the intervention.

Jessica: My result today was 105. I came second in the game. It was fun because I learned new things about adding numbers (10 September 2008).

Daniel: I had the same as Jessica. That is five units, no tens and one hundred. I liked the game because we all played well and co-operated (10 September 2008).

Madie: I have 157, that is seven units, five tens and one hundred.
Anna: How many more would you need to get to one thousand?
Madie: Well, ten greens equals one red, so I would need three more units, four more tens and [pause with help from another student] eight hundreds (17 September 2008).

Claudia: I have 122, one hundred and twenty two. My best throw was a double six because that gave me an extra turn and it was worth 12 (17 September 2008).

Anneliese: One hundred and five, I've got a zero in my tens. The game helps me with my maths and adding up numbers and learning how to say numbers like one O five. It has the zero there so now I know (17 September 2008).

Unexpected outcomes

As a result of place value activities students were also able to: count on from a two digit number to get to one hundred (or a thousand), like the grocer's method of subtraction; use the commutative law for addition and multiplication in action; double numbers up to 24; and represent four digit numbers with blocks. Students also gained knowledge and experience with chance and particularly enjoyed the dodecahedral dice.

Conclusion

This project confirmed that early trading activities in small groups are essential and valuable for helping students develop deeper understandings of the numeration system and number operations (Bruschi, 1990). Furthermore, when place value base 10 activities are combined with strategic lessons in concurrent learning, students who had previously been seen to struggle in Year Three mathematics were able to gain greater success overall. Additional benefits included an increased confidence as learners of mathematics, an ability to communicate their ideas and to reason mathematically with their peers in small groups. The numeration foundations established with students who were participants in this project will further enhance their later understanding and skills with mathematical concepts such as estimation, money, the four operations, decimals and metric measurement.

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TWO APPROACHES TO ONE LESSON: THE ROLE OF SYMBOLIC ALGEBRA IN A CAS-ENABLED CLASSROOM

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Computer algebra systems (CAS) have become increasingly sophisticated and are now used by teachers and students to present, picture, analyse and solve mathematical problems. The role of computer algebra in such processes has been central to many debates on the relative merits and pitfalls of the technology. This paper examines the varying approaches to computer algebra taken by two teachers using similar versions of the same pre-planned lesson. In the two classrooms where technology could support a range of possible approaches, the lesson design, teacher expectations and preferences, and students' responses all influenced the enacted lessons.

Introduction

The growing use of computer algebra systems within mathematics education has led to an abundance of representations and methods for analysing and solving real-world problems. With the option of using CAS technology to discuss a mathematics problem with a class, the teacher is faced with decisions regarding the role that symbolic algebra might play in teaching the lesson and in students' learning of the underlying techniques and concepts.

This paper focuses on the differing approaches to symbolic algebra taken by two teachers, as they guided students through a lesson featuring CAS-enabled solutions to a problem involving quadratic functions. The lesson was designed to give teachers a model of a technology-rich mathematics lesson, which would enable the various representations of the problem (numerical, geometric, graphical and algebraic) to be analysed and linked. For lesson documents, see RITEMATHS (2009).

In the sections that follow we review literature discussing the issue of students doing algebra with CAS or with pen and paper, and describe the differing approaches of the two teachers.

Background literature

The use of technology, especially CAS, offers what Artigue (2001) describes as an "explosion of methods." A given problem may now be explored or solved using a range of different strategies. This means that the teacher must decide whether, for a given lesson, certain strategies should be given preference. Such decisions will be influenced by many factors including the teacher's perceptions of the value of using technology such as CAS for students' learning. For example, in a survey conducted by Pierce and Ball (in press), mathematics teachers were asked to respond to the statement "Students don't understand maths unless they first do it by hand" and found 4% strongly disagreed, 48 % disagreed, 24 % were neutral, 18% agreed and 5% strongly agreed (n=92, percentages rounded). There is evidence (Tynan & Asp, 1998; Kieran & Damboise, 2008) that working with CAS allows students to generate many answers quickly, and examine them to see patterns. Then, as a result, students' pen-and-paper technical work benefits and their confidence in their algebra increases along with their interest and motivation. However some teachers (Pierce, Ball & Stacey, in press) perceive both technical difficulties, and the "black box" nature of CAS, as possible significant obstacles to students' learning.

In the following sections we give brief details of the present study and describe key features of two different presentations of essentially the same lesson. These presentations differed in the extent to which symbolic algebra was used to explore a given problem and the way CAS was used to support this work.

This study

In 2008 a lesson entitled "Marina's Fish Shop", featuring CAS-enabled solutions to a problem involving quadratic functions in Year 10, was written for and trialled by nine teachers at two secondary schools in Victoria. These schools were selected because they were introducing CAS technology (*TI-Nspire*) at the Year 10 level, and at each school there were teachers both with and without previous experience of teaching mathematics with CAS. In this paper we concentrate on the differing approaches to symbolic algebra taken by Doug at School A and Eric at School B. School A was the site where Version 1 of the lesson was taught before it was refined through the lesson study process for presentation at School B as Version 2 (for details, see Pierce & Stacey, 2009). Recordings and transcripts of planning sessions, classroom presentations, post-lesson interviews and focus group sessions with observing teaching colleagues and researchers provided data enabling differing approaches to be compared.

The lesson (based on *Marina's Fish Shop*, a task statement for which is shown in Figure 1) was prepared by the research team in consultation with School A teachers. It required exploration and solution of area problems. Figure 1 shows the task set for Version 1, which was positioned at the end of School A students' work on quadratic functions. Version 2 was used later for revision of quadratics at School B, and included Q1(i) and Q2 as extensions. Doug (School A) and Eric (School B) were both experienced teachers who were confident technology users. In accord with timetables Doug taught a single 80-minute lesson while Eric taught the lesson across two 50-minute sessions.

Marina's Fish Shop is remodelling its premises, and Marina wants a new neon sign. She has designed a logo consisting of a silver square and a blue right-angled triangle making a "body and tail" shape. The sign is to be above the premises, and Marina wants it to take up the entire 10m length of the shopfront.

…[more story context]

Your task is to investigate the various signs which might fill in the 10 metre space, and to answer these questions:

Q1. Find an approximate and exact (surd) value for the body length (b) so that:

(i) the body and tail areas are equal
(ii) the total area to be lighted is a minimum.

Q2. Suggest a compromise sign design which will address both of Marina's concerns equally.

Figure 1. Marina's Fish Shop task.

School A (Doug)

In discussion with researchers prior to his presentation Doug expressed concern about the technical skills students needed to "capture" data values within *TI-Nspire*'s "Graphs & Geometry" application, as well as the algebraic skills assumed throughout the lesson. The researchers emphasised that either pen-and-paper or *TI-Nspire* could be used at any stage.

The preliminary tasks of Version 1 of Marina's Fish Shop involved exploration of configurations for the body-and-tail (square-and-triangle) possible within an overall sign width of 10 metres. This exploration was first done by manual ruler measurement of the dimensions of two "fish signs" on a worksheet, and then using measurements generated by manipulation of a flexible diagram on the calculator screen. Geometric ideas and labelling queries featured in the teacher-student discussion, and students' varying results were listed on the whiteboard. Students then estimated solutions to the problem based on the calculator-generated values. Algebraic content at this stage was limited to formulae for areas of squares and triangles. Doug chose to summarise the students' estimates and effectively present an interim numerical solution to each problem (equal areas and minimum areas), noting, "We've got a pretty good feel for what's going on. But we want a bit more accuracy." Students then "captured" measurement data from the flexible diagram, stored this in the "Lists & Spreadsheet" application and linked it to a scatter-plot. This produced an "ooh-ahh!" moment within the class as the quadratic pattern emerged. Here the solution to the problem was approximated again through examination of the implied parabola provided by the technology.

The next task introduced a formal algebraic expression of the problem in terms of the variable *b*, representing body length of the fish (the diagonal of the square). Another variable *e* was introduced as the side length of the square. Doug told students they would now do “a little bit of algebra.”

As the links between the geometric properties of the shapes were being formalised by an application of Pythagoras’ theorem, Doug remarked, “If we had more time we could use our calculators”.
He decided to present pen-and-paper methods, showing the transformation from $e^2 + e^2 = b^2$ to $e = \sqrt{2b}$ by hand, and then demonstrated that the area of the fish body was $\frac{1}{2}b^2$. The lesson ended, with the area of the tail (triangle) and total areas to be found at another time.

In the focus group following the lesson, the discussion turned to the importance of algebra. Doug remarked:

> It’s a real shame that we didn’t get to that [last part of the lesson]. A part of me wonders whether in restructuring [the lesson], the algebraic thing that we just got into, you could separate that out. It would have been just nice to go on with the regression and then be able to take the actual quadratic, create a new graph, let them do the “point-on” thing, which they are good at… find the minimum, compare it to the original estimates, and then go back and do the algebraic approach quite independently as a separate activity… When I did this first… and I was working through it, did the scatter-plot, and then I got into the algebra and I just felt my spirits sink… I just thought, “…there’s too much,” and I thought “… there’ll be kids who feel like this as well.” And whereas I think if I had gone on with the regression it would have had some kind of continuity.

School B (Eric)

Eric met with the researchers before his two sessions presenting Version 2. The issue of giving algebra a more central role in this revised lesson was discussed. The by-hand measurement of static diagrams, which Doug’s students laboured over, was replaced with general exploration and specific calculations intended to lead directly to algebraic generalisations. Eric indicated that his students should be capable (with teacher guidance) of coping with the technology and algebra required.

Eric took the lesson at a faster pace than observed at School A. He decided to model the body area calculation for one specific configuration using by-hand algebra, to which he could refer later when working on the generalised case. Eric substituted alternative variables (*b* for body length and *x* for side lengths), not just specific case values into Pythagoras’ Theorem, speaking to his class thus:

> Let’s work it through together. Verbally… Pythagoras’ Theorem, right? The hypotenuse squared will be b^2. Let’s call these other two sides *x* and *x*. So $2x^2= b^2$; you got that in your head? $2x^2= b^2$, right? And therefore $x^2= b^2/2$, right. So, in terms of *x*, just this time in terms of *x*, what’s going to be the area of that square? In terms of *x*? *x* times *x*, right? It’ll be x^2, isn’t it? So what did we have here for *x*, we had, what did I say it was? … No, no in terms of *b*. $x^2= b/2$, right. So if $b/2$ is x^2, and the area is equal to x^2, therefore the equation for the area of the square in terms of *b* is? $b^2/2$, half b^2.

In the post-lesson interview later that afternoon, Eric discussed the inclusion of algebraic formulation in the lesson and his students’ ability to generate this themselves.

He said he led this process because they were running out of time and it was perhaps 12 months since the students had specifically worked with Pythagoras' Theorem. He noted:

> It's something that hadn't been done recently with them, so finding the length of a square given the diameter, sort of took them a little while to click into the context… Particularly the algebra involved in that. So they come to that equation… the area of the square is a half *b* squared… and I did almost tell them how to do the area of the tail, because I could see I was running out of time… I pretty well told them it was 10 minus *b*. A couple of students I could see had picked it up, but not many of them were coming to it themselves and it then got to a stage where then, "You know, believe me, this is what it is." Without them sort of working it through. [Researchers: Area of the triangle, or the algebra?] The algebra… Finding the length of the side of the square given the diameter.

The following day, Eric continued the lesson by using *TI-Nspire* to produce a scatter-plot of the relationship between the body-length *b* and the total area of the fish. As the technology produced a regression equation in the variables *y* and *x*, Eric quickly led the students to link this with their earlier work with *b*:

> Write down the rule as shown in your calculator. I think you would have something like this: $y = 1.5x^2 + -20x + 100$. Anybody got any problems with the look of that? OK, you understand why you have the plus and the minus… Obviously in your calculator the plus keeps getting shown there, that doesn't really matter, you can understand that, can't you? … Is this the same rule that you used in step 15? … You can show it — do a little bit of algebra please quickly, you can do it either on your calculator… or do it manually.

At this point, Eric used the CAS to demonstrate an expansion of the expression

$$\frac{1}{2}b^2 + \left(10 - b^2\right).$$

A variable-definition protocol on *TI-Nspire* forced him to change to another variable, yet he continued to model CAS use over by-hand methods.

In the focus group Eric reflected on his approach to presenting the algebraic material in the lesson. Discussion turned to the students' adeptness in using different variable names for the same variable, in different contexts or applications within the calculator, and Eric said the use of different letters for variables had been discussed regularly during the year.

Discussion

When designing the lesson it was intended that technology be used to allow students to explore a problem using a range of representations and so gain a rich understanding of the mathematics, in particular the algebra. In the lessons described above Doug chose to privilege by-hand skills and not use the algebraic capabilities of CAS, while Eric used CAS in an integrated way throughout the lesson. The individual emphases displayed above illustrate the evolving nature of the lesson design process as well as the varying professional judgements made by these teachers. In preparing Version 1, the researchers had intended that the initial activities involving measurement and estimation only play a supporting role to algebra, which would be a dominant representation of the problem. This proved not to be the case, as Doug privileged by-hand work and waited for every student to complete such tasks. His focus was on specific individual examples and little algebra was introduced. At School B the hands-on measurement activity was not used and, from early in the lesson, Eric emphasised general patterns and used some algebra.

Doug chose not to use CAS when he reached the algebra late in the lesson because he felt they did not have time. It seemed that he expected CAS work to take longer than pen-and-paper or that, given the choice, pen-and-paper was more important. Eric, on the other hand, used CAS throughout a faster paced lesson. Eric's class appeared more familiar with both the foibles and possibilities inherent with CAS use than did Doug's class. Using CAS in an integrated way seemed to assist the pace of Eric's lesson, for example entering a familiar formula and substituting values and using CAS to "Solve."

The differences between teachers' use of CAS as seen in these two presentations may be influenced by many factors, including the students' perceived mathematical abilities and technical skills, and the time of year the lessons were presented. Other factors seemed to be the teachers' perceptions of the relative value of pen-and-paper work as opposed to CAS, and the fact that a main goal of the lesson itself — expressing the problem algebraically — was made more explicit by its authors in Version 2.

Conclusion

This lesson demonstrates the rich range of both technology and pen-and-paper options available to mathematics teachers. It suggests the need for teachers to decide whether to specify either a broad goal of "solve by any method" or to explicitly privilege one approach (here, by-hand or CAS-enabled algebra) for the intended purpose of the lesson. As suggested by the literature many factors affected the choice of pen-and-paper or CAS for algebra. CAS used in an integrated way, with technology skills treated as incidental, appeared to support early generalisations and assisted the pace of the lesson. In a CAS-enabled mathematics teaching environment several approaches to a lesson may be possible. The class teacher must now make even more decisions in order to choose the best path for her or his students.

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WHICH IS BIGGER: 250 TONNES OR 17%?
A TALE OF SALT

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The potential for numeracy across the curriculum is illustrated in this paper. Based on a newspaper article about salt in the Australian diet, two avenues of investigation are suggested. One is based on exploring the meaning and implications of the numbers in the article, using percent. The other involves data collection from a local supermarket and the software *TinkerPlots*.

Introduction

As the National Curriculum Board continues its deliberations on Australia's four initial curriculum areas — English, History, Science and Mathematics — it is to be hoped that the acknowledgement of the importance of "Numeracy across the Curriculum" as stated in the draft Mathematics document is translated across the other three subjects. In the initial drafts of the other three, only History acknowledged the importance of data to the study of the subject. As writers continue the task of creating the four curricula, it will be imperative for them to communicate and collaborate with each other on the important issue of numeracy. In contemplating ways that mathematics educators can assist in this process, one way is to suggest organisers that can be used to structure curriculum development and a second is to provide down-to-earth relevant examples of how the organisers could work.

This paper suggests two organisers and provides a contextual example to illustrate how they can be developed for the classroom. The first organiser focusses on current affairs and the media. The use of newspaper and other media sources has been advocated for many years for the development of quantitative literacy skills and critical thinking (e.g., Watson, 1992; 1996; 2004). The suggested learning sequence for quantitative literacy is based on a three-tiered hierarchy that encompasses (i) understanding the mathematical terminology used in the extract, (ii) understanding how that terminology fits with and helps explain the context within which it is set, and (iii) thinking critically to understand the implications of claims and to question those that do not have adequate justification. The potential for developing cross-curricular links based on current affairs is unlimited.

The second organiser is based on data. The information-laden 21st century presents opportunity at every turn to use data to study phenomena, answer questions, and question theories and claims. To be useful in the classroom, however, data require contexts and a suitable interface for analysis. One interface is the software *TinkerPlots* (Konold & Miller, 2005) and several examples of its application in middle school contexts illustrate its potential (Watson, 2008; Watson, Fitzallen, Creed & Wilson, 2008; Watson & Wright, 2008). The *TinkerPlots* software was developed from a perspective of students creating representations suitable to the story they wish to tell rather than being constrained by fixed formats such as histograms or pie charts. It is possible that the context relevant for considering data is the same as that found in a media report. That is the case with the topic considered in this paper, *salt*.

Developing quantitative literacy

In February 2009, *The Mercury* newspaper in Hobart published a report entitled "Salt-rich diet gets a lashing" (Rose, 2009)[20]. The story was syndicated through the AAP and many of the other newspapers across Australia carried similar stories. It was based mainly on a press release from the Australian Food and Grocery Council (AFGC). A search of the web will show that the article reflects almost exactly the information from the press release. What are the opportunities offered by this article and topic for enhancing numeracy across the curriculum?

The salt content of food is a health issue for many people today. As stated in the article, "People with a high-salt diet place themselves at risk of hypertension (high blood pressure), increasing the risk of heart disease in later life and, according to a recent Australian study, Alzheimer's." The cross-curriculum link to Health and Well-being is clear but also there is a link to Science due to the chemical makeup of salt. There are also issues of critical literacy in relation to Media Studies and the question of why the AFGC would be putting out a press release on the topic. Given the range of numbers and measures presented in the article, there is great opportunity to develop and/or use numeracy skills in understanding and interpreting the information in the article.

The most humorous sentence, which gave rise to the title of this paper, is the following: "Smith's Classic Crinkle Cut Original Potato Chips now have 17 per cent less salt than in 2006, Unilever (maker of Flora margarine) has removed more than 250 tonnes of salt from their spreads, while Kellogg's has removed a similar amount from their breakfast cereals since 1997." Why would one turn from reporting a 17% reduction in salt to reporting a drop of 250 tonnes? Already at this point, without any need for calculations, critical thinking (Tier 3 of the hierarchy) comes to the fore. Joel Best (2008) in his book *Stat-spotting* suggests that people use different mathematical formats depending on which "will make the most powerful impression. Often, packaging statistics effectively means making a problem [or solution] seem as large as possible" (p. 65). Percentages can be impressive but perhaps the writer of the press release worried that 17% did not sound very large. As Best says, "absolute numbers may be preferable when percentages or proportions seem less impressive" (p. 65).

[20] The complete text of the article is reproduced in Appendix A.

Certainly for most people, 250 tonnes seems like a very large amount of salt. The question for the quantitatively literate student is, “How can we make a comparison of these two numbers?” The answer is, “We can’t because, for example, we don’t know the total number of tonnes of food produced by Unilever or Kellogg’s.” One feels that for both companies the number must be incredibly larger than 250 tonnes. A similar issue arises for Smith’s Chips if one wished to know how many tonnes of salt had been removed from its chips. The basic numeracy involved is about parts and wholes and without the “whole,” both percentages and absolute numbers can be meaningless. In fact it could be that the 17% reduction in salt in Smith’s Chips is much more impressive than that of Unilever or Kellogg’s if 250 tonnes turns out to be 1% of their total tonnage of food production. By middle school, alarm bells should ring when students read sentences like this one in any curriculum area.

The sentence about Australians’ estimated salt consumption and recommendations, however, provides opportunities for gaining practice at calculating percents and applying percent understanding in context. Four fast food products are noted, each with the percentage of the recommended daily intake of salt that they contain. Since 4 grams is the recommended daily amount, students can be asked to calculate the number of grams of salt in each product, with answers ranging from 4 g to 7.76 g. These are quite easy calculations because they only require multiplication of 4 g by the decimal equivalent of the percentage, e.g., 194% translates to 1.94. This follows because if 100% is equivalent to 1.0 in decimal form (in this case each representing the “whole” of 4 g of salt), then using the proportion

$$\frac{100}{194} = \frac{1.0}{X}$$

leads to

$$X = \frac{1.0 \times 194}{100}$$

$$= 1.94$$

The calculations become more interesting, however, if one were to consider the actual amount of salt in Vegemite or Smith’s Chips today and ask how much there was in 1974 or 2006. This requires data collection at a supermarket, where one finds that Vegemite today contains (in most jars) 3450 mg of *sodium* per 100 g of Vegemite. Is sodium the same as salt? No, a gram of salt has 0.4 g of sodium in it; or for each mg of sodium in a product, there are 2.5 mg of salt. This can be confusing since all of the numbers in the article related to salt but products in the supermarket are labelled with sodium content. The 3450 mg of sodium per 100 g of Vegemite becomes (3450 × 2.5)/1000 grams of salt or 8.625 g of salt per 100 g of Vegemite; hence Vegemite is about 8.6% salt. If Vegemite has reduced its salt content by 13% since 1974, how much sodium (or salt) did it have per 100 grams in 1974? This problem is not as easy for some students to resolve. If 3450 mg is 87% of the 1974 sodium content per 100 g, then it is a matter of solving the proportion,

$$\frac{3450 \text{ mg}}{X \text{ mg}} = \frac{87}{100}$$

to obtain

$$X = \frac{3450 \text{ mg} \times 100}{87} = 3966 \text{ mg}.$$

The same calculation can be done with 8.625 g of salt or the answer for sodium can be multiplied by 2.5/1000[21]. Either way, in 1974 there were 9.91 g of salt per 100 g of Vegemite; that is, the product was nearly 10% salt.

Although one can look at the 8.6 g of salt in 100 g of Vegemite and be shocked that it is over twice the recommended intake per day, it is very unlikely that a person will indulge in a whole 100 g jar of Vegemite in a single day! Hence the normal serving size becomes an issue and some useful estimation problems can be devised for students in terms of how many grams of Vegemite they put on their toast in the morning. The issue of serving size is what raises alarm bells in relation to the fast food products discussed at the end of the newspaper article. Although one cannot imagine using up one's daily salt allowance on Vegemite, it is certainly possible with a single meal of one of these products.

In terms of the numeracy involved related to percentage, two types of problems have arisen so far:

$$194\% \text{ of } 4 \text{ g} = \square$$

and

$$87\% \text{ of } \square = 3450 \text{ mg.}$$

The first is usually solved by multiplication and the second by division after the percents are changed to decimals (e.g., 0.87) or fractions (e.g., 87/100). The third type of percentage problem arises if for example we have two measurements of sodium content for a product sold in "ordinary" and "salt-reduced" forms and wish to determine the percentage reduction for the salt-reduced product. Soy sauce for example is another high-salt content product and one company makes its usual product with 6510 mg of sodium per 100 g of soy sauce. The salt-reduced product contains 4010 mg of sodium per 100 g of soy sauce. What is the percentage reduction in sodium in the salt-reduced product? To answer this question requires two steps. Firstly, the reduction must be calculated, 6510 mg – 4010 mg = 2500 mg. Secondly, the third type of percent question must be answered:

$$\square \% \text{ of } 6510 \text{ mg} = 2500 \text{ mg.}$$

Using a proportion,

$$\frac{X}{100\%} = \frac{2500 \text{ mg}}{6510 \text{ mg}}$$

or

$$X = 38.4\%.$$

The problem is usually thought of as a division problem but thinking of it as a proportion can help some students by reinforcing the part–whole concept in this type of percent problem.

Data collection and analysis

Considering this last example about salt reduction opens up the possibility of students collecting data from the supermarket for foods that are manufactured as "ordinary"

[21] Another way to solve the problem is to think of the percent values and use decimals:
0.87 of X is 8.6%
i.e., $0.87 \times X = 8.6\%$

so $X = \dfrac{8.6\%}{0.87\%} = 9.9\%$

products and as products with less salt. Students could be set the task of carefully defining the way data will be collected to be sure reductions are meaningful (for example, the brand and product name must be the same). After collecting the data, decisions need to be made as to what is important to enter in a software package for analysis. The *TinkerPlots* software provides a "data cards" format that lends itself to students beginning to enter data and deciding on attributes they wish to add later. Figure 1 shows data cards for two different products, with the salt reduction looking to be very different between the two. A plot of *Original_Salt_Content* vs *Revised_Salt_Content* (see Figure 2) for 29 foods looks as if there may be different trends in some of the products. The tomato paste had "no added salt," whereas the chunky pasta sauce was only "salt reduced." Students may decide to add an attribute, *Type_Reduction*, to distinguish these two features in future analyses (see Figure 3). Clicking on this attribute colours the icons to produce Figure 4, which shows the difference between "no added salt" and "salt reduced" foods. Clicking on the one food that appears with the cluster of "salt reduced" products shows it to be the "no added salt" variety of dry roasted macadamias.

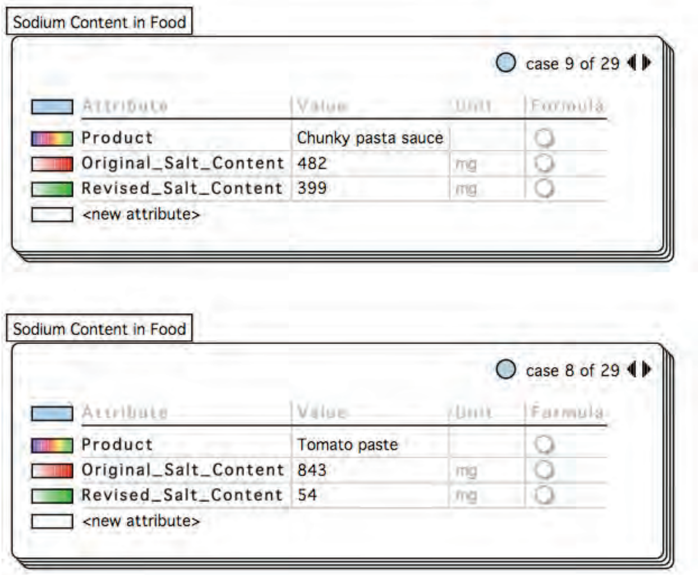

Figure 1. Data cards for two salt-reduced products.

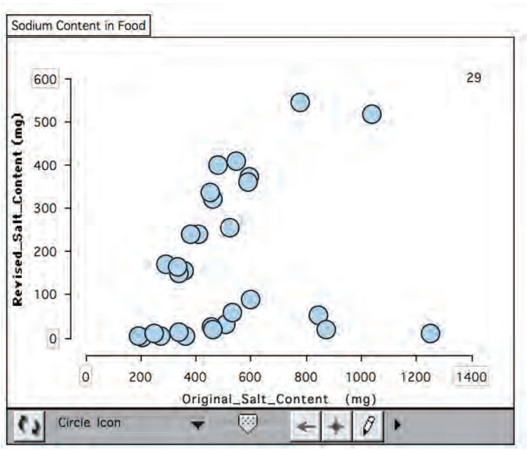

Figure 2. Scatter graph of original and revised salt content of 29 foods.

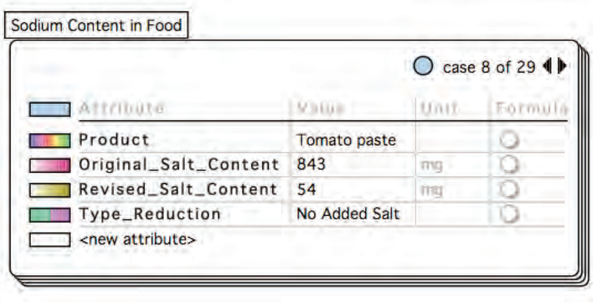

Figure 3. Addition of a new attribute.

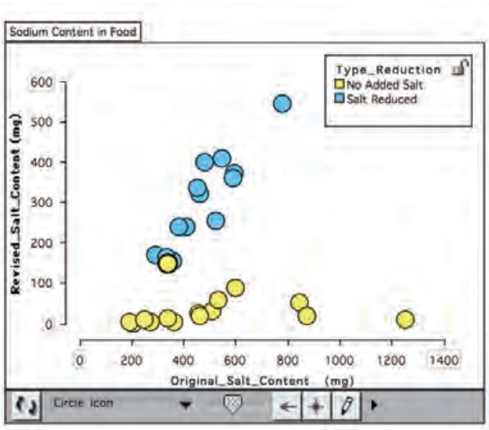

Figure 4. Scatter graph colour-coded by type of salt reduction.

Although Figure 4 shows two clusters of products related to the type of salt reduction, students can use their knowledge of percentage to define a new attribute *Percent_Reduction* to give this value for each food. Figure 5 shows how to define this attribute by clicking in the formula column of the data card as shown on the left and choosing appropriate attributes from those listed in the formula box. The data card on the right shows the application of the formula to the product on the data card.

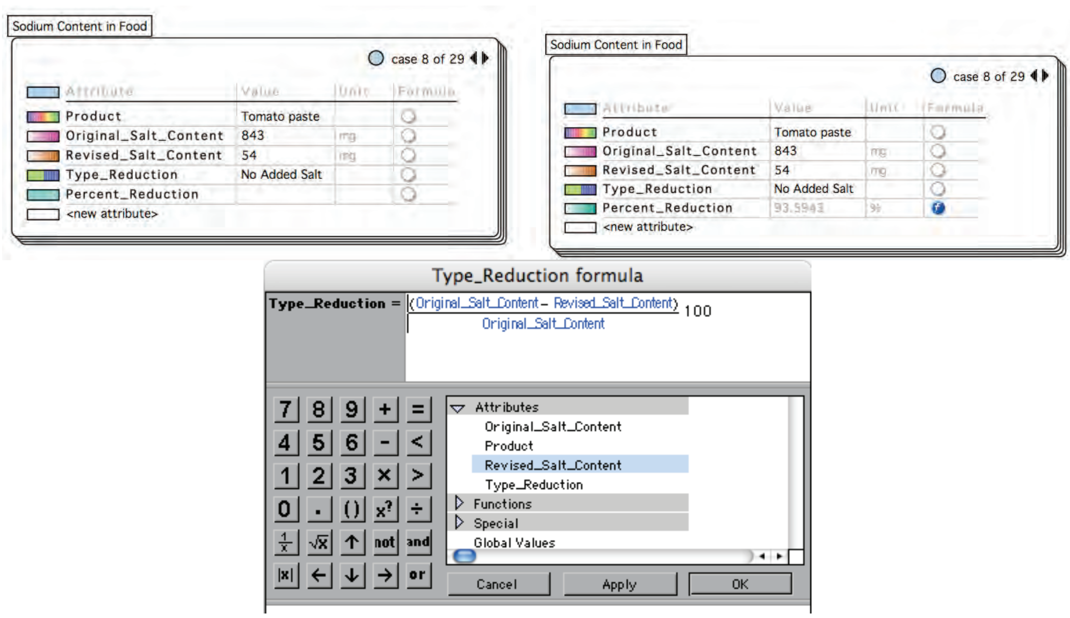

Figure 5. Defining the Percent_Reduction attribute.

Figure 6 shows a plot of the percentage reductions, coloured by *Type_Reduction* and Figure 7 shows the plot separated by the *Type_Reduction* attribute with the average percentage reduction for each type of food labelled. Such a representation may help students realise why it is important to read labels and distinguish between the two types of salt-reduced foods. It also should prompt them to question the information provided by the mean reduction in Figure 6, as this plot illustrates well a situation where the arithmetic mean is not a meaningful reflection of the data set.

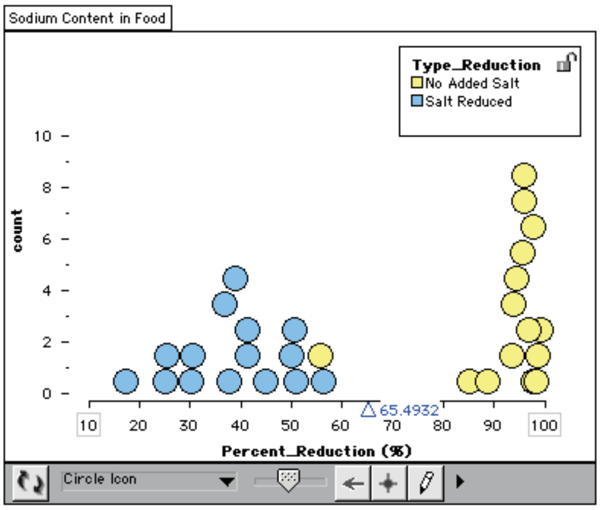

Figure 6. *Dot plot of percent reduction for 30 products.*

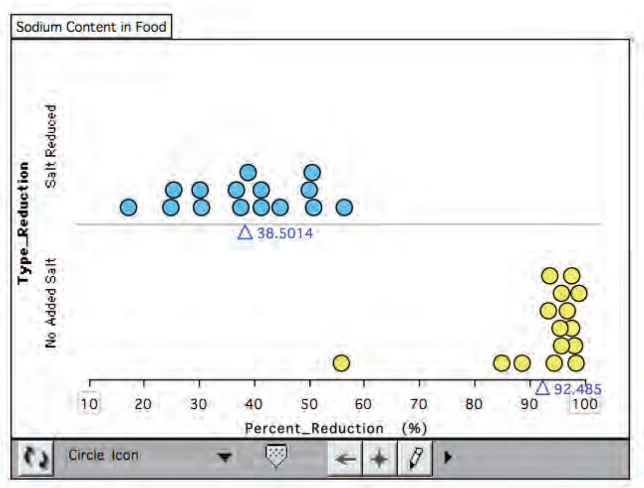

Figure 7. *Separated dot plot with mean percent reduction for the two types of salt reduction.*

Students might also consider whether some kinds of foods have one type of reduction or the other. The foods in the data set can also be categorised in five groups: Nuts, Sauce, Soup/stock, Spread, and Tinned Food. This allows a comparison across categories of foods based on original salt, reduced salt, percentage reduction, or type of reduction. Figure 8 shows the new attribute and Figures 9 and 10 show some comparisons that can be made. Sauces have the highest original salt content per 100 g of food and nuts the least. In the salt-reduced products (of both types) Soups and Stock are the highest whereas Nuts are still the lowest. Considering the percentage reduction attribute and colouring by *Type_Reduction* shows the influence of the two types on the mean value, as seen in Figure 10.

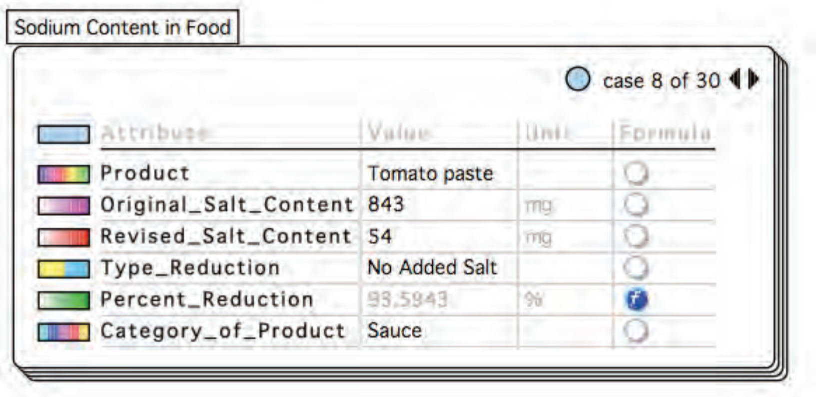

Figure 8. Addition of a further attribute.

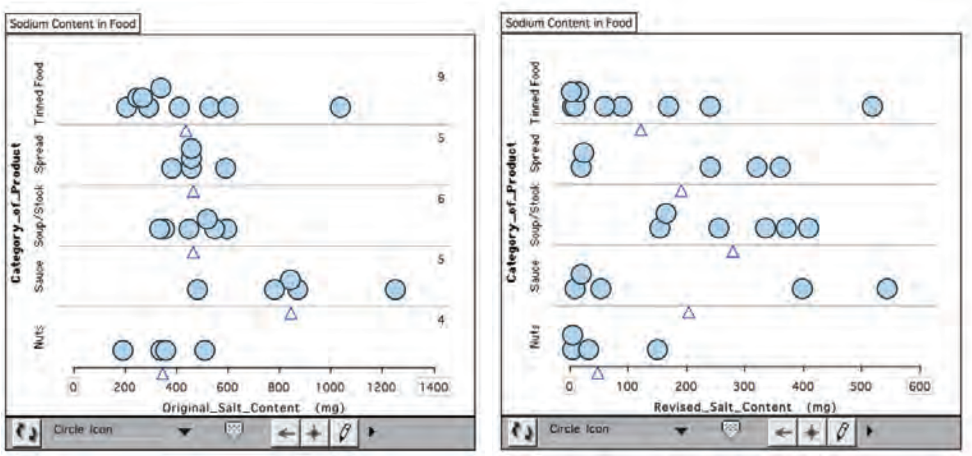

Figure 9. Comparing Original and Revised salt content by Category_of_Product.

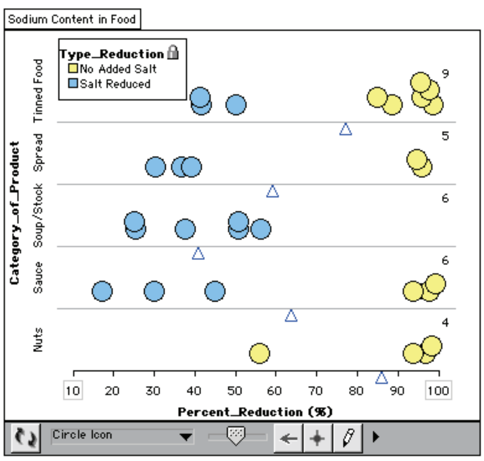

Figure 10. Percent_Reduction by Category_of_Product.

Soy sauce is an outlier in the data set in terms of salt content. Figure 11 shows its influence on the plots and means for the two salt content attributes. Soy sauce has been "hidden" on similar graphs shown earlier. In terms of percentage reduction, however, soy sauce is not an outlier, illustrating how important the distinction is between actual numbers and percentages. Its reduction of 44.8% is seen in the sauce row of Figure 10.

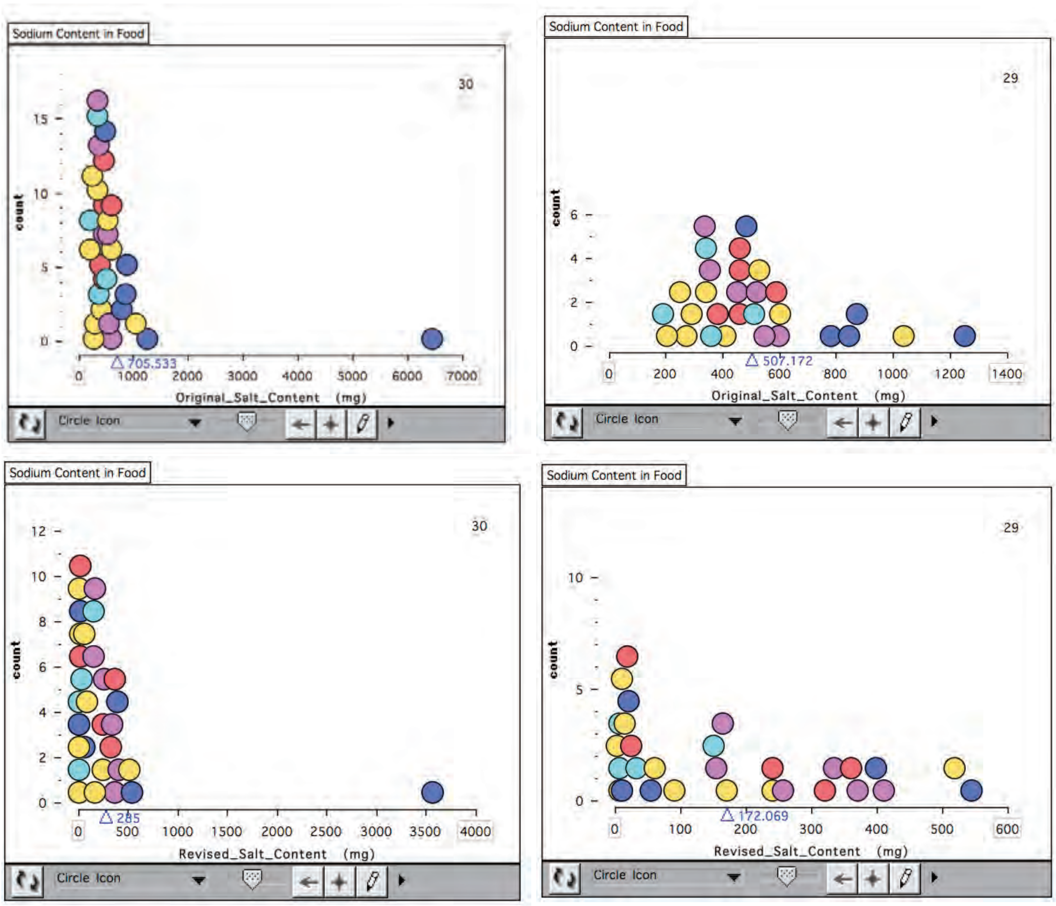

Figure 11. The influence of soy sauce on mean Original_Salt_Content and mean Revised_Salt_Content.

We conclude by returning to questions similar to that in the title of this paper, “Which is bigger, a reduction of sodium content of soy sauce by 44.8% or having 3560 mg per 100 g left? Which matters most to your health?” To answer these questions requires the three tiers of quantitative literacy: (i) an understanding of percent, (ii) an understanding of the meaning of percent and percent reduction in the context of salt in food products, and (iii) critical thinking to make a decision about how this understanding will influence your eating habits.

Notes

The entire data set as collected for this paper is reproduced in Appendix B but it is more meaningful if students collect their own data. More information on salt and fast foods can be found in the AWASH report (The Secretariat of the Australian Division of World Action on Salt and Health, 2009).

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Appendix A (The Mercury Newspaper, Hobart)

Friday, February 6, 2009

Mercury, p.15

Salt-rich diet gets a lashing

DANNY ROSE

AUSTRALIA'S food and grocery industry has defended its record on salt in the wake of damning research that highlights unhealthy sodium levels found in everyday items.

The Australian Food and Grocery Council says the industry is doing its bit to educate people about a healthy diet, while food companies have been working quietly for years to cut salt from their products.

"Vegemite started reducing its salt content in 1974 — it now has 13 per cent less than the original recipe," says AFGC chief executive Kate Carnell.

"Smith's Classic Crinkle Cut Original Potato Chips now have 17 per cent less salt than in 2006, Unilever (maker of Flora margarine) has removed more than 250 tonnes of salt from their spreads, while Kellogg's has removed a similar amount from their breakfast cereals since 1997."

Current estimates are that Australians consume around nine grams of salt each day — well over the maximum of six grams and more than double the recommended four grams.

People with a high-salt diet place themselves at risk of hypertension (high blood pressure), increasing the risk of heart disease in later life and, according to a recent Australian study, Alzheimer's.

Ms Carnell said industry was also responding with the introduction of Daily Intake Guide labelling on the front of food packaging — a more user-friendly system than nutrition information boxes on the back of products.

"It is generally recognised that eating too much [salt] can contribute to high blood pressure and other illnesses," she said.

"That is why the food and beverage sector has long been working to educate consumers on how to better construct a more balanced diet."

Her comments follow the release of new research which shows how a person can exceed their recommended daily salt intake in just one meal at any of the major fast food outlets.

Subway's most salt-laden offering — a spicy Italian sub — came in at 100 per cent of the recommended daily intake, while a Classic Half Chicken meal from Red Rooster topped the list at almost double (194 per cent).

KFC's Zinger Double BBQ Bacon and Cheese Burger with large chips provides 191 per cent, followed by Hungry Jacks Whopper Double Beef and Cheese burger plus onion rings and a large chocolate shake (153 per cent).

AAP

Appendix B

Table 1. Entire data set collected for this paper.

Product	**Original_Salt _Content (mg)**	**Revised_Salt _Content (mg)**	**Type_ Reduction**	**Percent_ Reduction**	**Category_ of_Product**
Fountain Tomato Sauce	871	20	No Added Salt	97.7038	Sauce
Rosella Tomato Sauce	1250	10	No Added Salt	99.2	Sauce
Soy Sauce	6458	3560	Salt Reduced	44.8746	Sauce
Tinned tomatoes	270	6	No Added Salt	97.7778	Tinned Food
Four Bean mix	250	10	No Added Salt	96	Tinned Food
Corn kernels	205	3	No Added Salt	98.5366	Tinned Food
Mushrooms (tinned)	340	15	No Added Salt	95.5882	Tinned Food
Tomato paste	843	54	No Added Salt	93.5943	Sauce
Chunky pasta sauce	482	399	Salt Reduced	17.2199	Sauce
Chicken stock	521	256	Salt Reduced	50.8637	Soup/Stock
Beef stock	450	335	Salt Reduced	25.5556	Soup/Stock
Peanuts	360	6	No Added Salt	98.3333	Nuts
Beef strogonoff mix	780	545	Salt Reduced	30.1282	Sauce
Dry roasted macadamias	340	150	No Added Salt	55.8824	Nuts
Cashews	510	32	No Added Salt	93.7255	Nuts
Butter	460	320	Salt Reduced	30.4348	Spread
Sunflower spread	380	240	Salt Reduced	36.8421	Spread
Baked beans	290	170	Salt Reduced	41.3793	Tinned Food
Peanut butter	458	25	No Added Salt	94.5415	Spread
French Onion Soup	335	165	Salt Reduced	50.7463	Soup/Stock
Chicken Noodle Soup	355	155	Salt Reduced	56.338	Soup/Stock
Meadowlea Original	590	360	Salt Reduced	38.9831	Spread
Butter	460	19	No Added Salt	95.8696	Spread
Pink Salmon	600	90	No Added Salt	85	Tinned Food
Gravox Supreme	595	371	Salt Reduced	37.6471	Soup/Stock
Vegetable stock	547	410	Salt Reduced	25.0457	Soup/Stock
Spam	1036	518	Salt Reduced	50	Tinned Food
Spaghetti	410	240	Salt Reduced	41.4634	Tinned Food
Mixed nuts w green pistachios	190	6	No Added Salt	96.8421	Nuts
Red Salmon	530	60	No Added Salt	88.6792	Tinned Food

THE COUNTING ON NUMERACY PROGRAM

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The Counting On program began in NSW in 1999 and was designed with a twin focus. The first was to support the professional learning of teachers in identifying and addressing the learning needs of students in the middle years who have difficulties with the early mathematical concepts and skills. The second was to improve the mathematical outcomes of these students. It has a strong theoretical and research base and has undergone major changes which have been evaluated, most recently in 2007 and 2008. This paper will outline the program, and examine the gains in teacher professional learning and student learning outcomes.

Background

The middle years of schooling are widely regarded as the last two years of primary school and the first two to three years of secondary school (typically 9–14 year olds). The transition from primary to secondary school can be problematic for several reasons besides the natural growth of young people entering adolescence, such as differences in school curricula, different organisation of the school sites, and different pedagogical practices. The Victorian Middle Years Research and Development Project (MYRAD) (Centre for Applied Educational Research, 2002) review of research into mathematics in the middle years reveals considerable disengagement with, and underachievement in, mathematics on the part of lower secondary students. A number of these students are excluded from effective mathematics study in the middle years and beyond because of a lack of understanding of, and proficiency with, early school mathematical knowledge. As a response to this widespread concern, New South Wales (NSW) incorporated the Counting On program into its State Numeracy Plan 2006–2008 (NSWDET, 2005).

The Counting On program targets the needs of a group of under-performing middle years mathematics students. In brief, the program consists of a teacher professional learning component and a student learning component. The teacher learning component is a process of building capacity for diagnosis, remediation and alternative pedagogical strategies among teachers, while the student learning component is a ten week intervention consisting of a range of new and resource supported mathematical pedagogical strategies and learning activities. These are delivered by classroom teachers during their usual teaching program to targeted underperforming middle years students.

The program, with this twin learning focus upon students and teachers has continued to change in response to regular formal external evaluation. For example, the initial Counting On program in 2000 was designed for only first year secondary school students (Year 7) who had not achieved specific New South Wales Stage 3 mathematics syllabus outcomes by the time they commenced secondary school. It was later extended to include the feeder primary schools and students across all the middle years.

Theoretical basis

The research base for the program can be found in the Counting On Numeracy Framework (Thomas, 1999) which was an extension of work by Cobb and Wheatley (1988), Beishuizen (1993), Jones, Thornton, Putt, Hill, Mogill, Rich and van Zoest (1996) and relates to the Count Me In Too Learning Framework in Number (LFIN) (Wright, 1998; Wright, Martland & Stafford, 2000).

This theoretical base was supported by an increasing research base provided by the regular Counting On evaluation studies. After a pilot study involving nine schools was evaluated by Mulligan (1999), the Counting On program began in 2000 with 40 schools, more than 600 students, 120 school teachers and 40 district mathematics consultants. Further evaluation reports on the Counting On program were conducted in 2000, 2002, 2003, 2007 and 2008 (Perry & Howard, 2000, 2002a, 2003; White 2008, 2009 in press). During 2001, Counting On was implemented in 76 primary, four central and 75 secondary schools across NSW, involving more than 1400 students, 321 school teachers and 40 district mathematics consultants. The 2002/2003, Counting On programs involved three high schools per district and two feeder primary schools in each of the 40 districts. This pattern continued until 2007 when the program underwent a major revision which was implemented in 122 schools across the state grouped into 30 clusters with each cluster supported by a mathematics consultant. The 2007 revised model included a simplified assessment instrument, the inclusion of Newman’s Error Analysis, a revised Counting On CD, the formation of School clusters, a facilitator’s conference, and a facilitated professional development model. In 2008 there were further changes made in response to the recommendations and findings of the CO 2007 Evaluation report (White, 2008). Unlike the major changes of 2007, the 2008 changes were minor adjustments and concentrated on strengthening the learning communities and upon the training and support of the school program facilitators.

Teacher learning

The teacher learning process has changed and evolved in response to the feedback from regular evaluation reports. Initially NSW was divided into 40 districts and the program was managed by a program officer based at Ryde whose role included training District Mathematics Consultants. The consultants then trained teachers from program schools or others as requested. With the introduction of 10 regions, the nature and composition of the consultancy teams together with the role for each consultant changed and was determined by the regions.

The teacher learning process also changed with a two-day facilitator conference being seen as a way to deliver the program to teachers. The facilitators were volunteer teachers who would manage the implementation of the Counting On program within their schools. The facilitators would form learning communities within their school that included staff from neighbouring schools. Regions were asked to support each learning community at the facilitator conference and through the program period with a numeracy consultant. Thus the change from district to region also changed the nature of the professional learning. While initially a “train the trainer” model may have been a fairly accurate description of the process, the change to regions in 2007 made it no longer appropriate. A better term would be to call it a “facilitated model” of teacher professional learning. While a “train the trainer” process had an expectation of a consistency in delivery with everyone trained and then delivering the training in the same way, a facilitated model allows for more variability within regards to how the program is implemented because the expectation is that the school facilitator will shape the program to meet local needs. The change meant that whereas cascade models of train the trainer may suffer from “dilution” as the process moves from level to level, in contrast a facilitated model has the potential to be both better and worse than the original facilitator training. Thus the choice of school facilitator became even more important to the success of the program.

The program officer based at Ryde now organised the facilitator conference, coordinated the program and provided support through a CD that contained a written explanation about the program, video material related to the framework, an interactive interface linking the LFIN to video explanations of the framework and samples of student responses. It also included additional material and learning objects on fractions, decimals and percentages. The material on decimals is under licence to the University of Melbourne. The program officer also provided financial support for teacher relief or for purchasing resources and a CD containing all the presentations from the facilitator conference.

The program is successful in building capacity within the teachers. For example the 2007 evaluation report stated the

> …adult participants on multiple instruments were consistently positive concerning the opportunities for professional learning and collaborative sharing offered by the program. This collaborative professional learning lead to gains in mathematical pedagogical knowledge, mathematics content knowledge, diagnostic assessment and planning, and the knowledge and use of new activities and resources. Many respondents commented upon how they now focused more strongly on the individual and upon their responses and misconceptions (White, 2008, p. 76).

The 2008 evaluation report echoed these sentiments and both reports made reference to the effect of the program beyond the target group of students, where the activities and practices of teachers were applied across all the mathematics stages within the school and in some cases across other key learning areas.

Student learning

The Counting On program requires the identification of targeted students. Schools are identified as a result of student performance on state-wide testing programs and are invited to join the program. The identification of students within the invited schools relies on an assessment instrument based on the LFIN (see Table 1). It consists of six questions covering place value, addition, subtraction, multiplication, division and word problem tasks and is administered by the class teacher to the whole class.

Table 1. Learning framework levels of conceptual development in place value and multiplication and division (Perry & Howard, 2001b, p. 412).

Place value		Multiplication and division	
Level	Descriptor	Level	Descriptor
0	Ten as count	0	Unable to form equal groups
1	Ten as unit	1	Forming equal groups
2	Tens and ones	2	Perceptual multiples
3	Hundred as unit	3	Figurative units
4	Hundreds, tens & units	4	Repeated abstract composite units
5	Decimal place value	5	Multiplication and division as operations
6	System place value	6	Not used

The whole class results were used to group the students as expert (correct working and answers to 5 or 6 items and clear understanding of correct number concepts needed to solve the problems), intermediate (some correct working and answers and some understanding of number concepts needed to solve the problems but still not fully developed or consistent) and target (few or no correct working or answers and evidence of misconceptions in working and answers). The target group was then interviewed and their levels recorded. They were retested on the same test and interviewed at the finish of the program. The 2008 results for 1213 students included 954 primary students (78.6%) and 259 secondary students (21.4%). The largest groups were Year 5 (42.2%) and Year 6 (31.0%). These results are briefly summarised below.

Place value

Table 2 shows the change in levels for place value. A majority of students have improved by one or more levels (57.3%), with a sizeable group improving two levels (9.6%). A small group of students improved by three and four levels and there are some who declined by one or two levels.

Table 2. Difference in place value levels.

Difference	Frequency	Percentage Frequency
–2	4	0.3%
–1	26	2.1%
0	488	40.2%
1	557	45.9%
2	116	9.6%
3	19	1.6%
4	3	0.2%
Total	1213	100.0%

The descriptive statistics record an increase in the mean from 1.29 for the initial level (SD = 0.919) to 1.97 for the final level (SD = 1.002). Using a paired sample t-test, the results indicate that the improvement in the student place value learning outcome levels at the start and finish of the ten week Counting On 2008 program was statistically significant.

Multiplication and division

Table 3 shows the majority of students have improved by one or more levels (59.7%), with a sizeable group improving two levels (14.4%). A small group of students improved by three and four levels and there are some who declined by one, two or more levels.

Table 3. The difference in multiplication/division levels.

Difference	Frequency	Percentage frequency
–4	1	0.1%
–3	2	0.2%
–2	5	0.4%
–1	26	2.1%
0	455	37.5%
1	452	37.3%
2	175	14.4%
3	70	5.8%
4	27	2.2%
Total	1213	100.0%

The descriptive statistics record an increase in the mean from 2.69 (SD = 1.367) to 3.58 for the final level (SD = 1.228). Using a paired sample t-test, the results indicate that the improvement in the student multiplication/division learning outcome levels was statistically significant.

Mathematical word problems: Newman's Error Analysis

Only one of the two questions involving Newman's Error Analysis (NEA) in the assessment instrument was recorded for each student. Table 4 shows that the majority of students have improved by one or more levels (56.6%), with a sizeable group improving two levels (15.6%), a small group of students who improved by three and four levels as there are some who decline by one, two or more levels.

The descriptive statistics record an increase in the mean from 2.52 for the initial level (SD = 1.096) to 3.37 for the final level (SD = 1.254). Using a paired sample t-test, the results indicate that the improvement in the student outcomes for mathematical word problem levels at the start and finish of the ten week Counting On 2008 program was statistically significant.

Table 4. The difference in Newman's Error Analysis levels.

Difference	Frequency	Percentage Frequency
–4	3	0.2%
–3	6	0.5%
–2	14	1.2%
–1	52	4.3%
0	452	37.3%
1	385	31.7%
2	189	15.6%
3	79	6.5%
4	27	2.2%
5	6	0.5%
Total	1213	100.0%

Discussion

In a short program such as this, it is unrealistic to expect that all students will register an immediate improvement. Students were targeted because they have been struggling for some time with their mathematical and literacy levels and have often developed negative judgements of their own ability. To improve 1 level on either the LFIN or NEA scale in such a small time frame is quite remarkable and points to educational significance. There is every reason to expect that these gains will continue as the students build upon their success and a longitudinal study of these students would be of interest but is beyond the scope of this paper.

There are many reasons for a student's lack of progress or in some cases a regression in the levels. The 2007 evaluation report explored reasons for the regression and listed factors such as the use of different assessors, poor initial teacher understanding of the LFIN and NEA, misdiagnosis, student resistance to assessment and teacher confusion with the different levels for LFIN and NEA. This was not explored in the 2008 evaluation report (White, in press) except in a cursory manner.

Conclusion

The Counting On program successfully promotes both teacher and student learning. The data revealed a statistical and educationally significant improvement existing in student learning outcomes in all three specific areas of place value, multiplication/division, and mathematical word problems involving the first two areas. Student learning outcomes improved because teachers through the support and resources of the program had the opportunity to think, plan and reflect on their teaching. The teachers and gained greater knowledge of their students, more strategies that catered for individual differences, greater mathematical content and pedagogic knowledge which produced a wider range of classroom strategies and a greater use of concrete materials, and increased collegiality and sharing of ideas and resources.

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NEWMAN'S ERROR ANALYSIS: DIAGNOSIS TO PEDAGOGY

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The Counting On program was designed to support the professional learning of teachers in identifying and addressing the learning needs of students in the middle years who have difficulties with the early mathematical concepts and skills. The program has undergone major changes such as the inclusion of Newman's Error Analysis in 2007. Newman (1977, 1983) defined five specific reading skills crucial to performance on mathematical word problems. In this paper data will be examined arising from the Counting On evaluations focussing on student outcomes and how teachers have used NEA as a remedial and general classroom pedagogical strategy.

Background

The Counting On program conducted by the New South Wales Department of Education and Training (NSWDET) was implemented in 1999 to address the needs of students who are excluded from effective mathematics study in the middle years and beyond because of a lack of understanding of and proficiency with the early school mathematical knowledge.

The Counting On program has a twin learning focus upon student and teacher learning. The initial program was designed for first year secondary school students (Year 7) who had not achieved specific New South Wales Stage 3 mathematics outcomes by the time they commenced secondary school. It was later extended to include the primary schools and the middle years (9–14 year olds). The research base for the program was provided through the Counting On Numeracy Framework (Thomas, 1999) an extension of work by Cobb and Wheatley (1988), Beishuizen (1993), Jones, Thornton, Putt, Hill, Mogill, Rich and van Zoest (1996) and relates to the Count Me In Too Learning Framework in Number (LFIN; Wright, 1998; Wright, Martland & Stafford, 2000).

This theoretical base was supported by an increasing research base provided by the regular Counting On evaluation studies (Mulligan, 1999, Perry & Howard, 2000, 2002a, 2003; White 2008, 2009 in press). In 2007 the program underwent a major revision and was implemented in 122 schools across the state grouped into 30 clusters with each cluster supported by a mathematics consultant. It was based on the previous models but included changes designed to simplify and encourage further and ongoing involvement of schools. Features of the revised model included: a simplified assessment instrument; the inclusion of Newman's Error Analysis (NEA); a revised Counting On CD; formation of School clusters; a facilitator's conference; and a facilitated professional development model. It is the inclusion of NEA that is the focus of this paper.

Newman's Error Analysis

In Australia, Newman's Error Analysis (NEA) was promoted by Clements and Ellerton (1980, 1982, 1992; Ellerton & C1ements, 1991, 1996; Marinas & Clements, 1990) during the 1980s and 1990s although there were others (e.g., Watson, 1980). This initial momentum declined and its inclusion in the Counting On program in 2007 was via an unusual path. Clements became a professor at the University of Brunei Darussalam and was heavily involved in a national professional learning program for primary teachers titled Active Mathematics In Classrooms (AMIC; White & Clements, 2005). NEA was one aspect of this program. The AMIC program was reported in the primary journal, *Square One*, for the Mathematical Association of New South Wales. An article on NEA from *Square One* (White, 2005) was selected and added to the teacher reader section of the NSWDET website in 2006 which created a renewed interest by teachers. In 2007 it was added to the Counting On program.

The reasons for the inclusion of Newman's Error Analysis (NEA, Newman, 1977; 1983) in the 2007 and 2008 programs were to assist teachers when confronted with students who experienced difficulties with mathematical word problems. Rather than give students "more of the same" involving drill and practice, NEA provided a framework for considering the reasons that underlay the difficulties and a process that assisted teachers to determine where misunderstandings occurred and where to target effective teaching strategies to overcome them. Moreover, it provided excellent professional learning for teachers and made a nice link between literacy and numeracy.

NEA was designed as a simple diagnostic procedure. Newman (1977, 1983) maintained that when a person attempted to answer a standard, written, mathematics word problem then that person had to be able to pass over a number of successive hurdles: Level 1 Reading (or Decoding), 2 Comprehension, 3 Transformation, 4 Process Skills, and 5 Encoding. Along the way, it was always possible to make a careless error and there were some who gave incorrect answers because they were not motivated to answer to their level of ability.

There have been adaptations to NEA, two of which will be briefly described. The first was by Casey (1978), who modified the interview procedures used by Newman (1977). In a study of the errors on word problems made by 120 Grade 7 students in a single high school, the interviewers were required to help students over errors. If a pupil made a Comprehension error, the interviewer would note this and then explain the meaning of the question to the pupil, and so on. So, in Casey's study, a pupil could make a number of errors on the one question thus making it difficult to compare Casey's interpretations with Newman's.

The second adaption was by Ellerton and Clements (1997), who used a modified form of the Newman interview method to analyse the responses by students in Grades 5 through 8 to a set of 46 questions. All responses, both correct and incorrect, were analysed. A correct answer which, after analysis, was not deemed to be associated with an adequate understanding of the main concepts, and/or skills and/or relationships tested by a question, would be associated with a Newman error category, even though the answer was correct. Ellerton and Clements' modification led to the adoption of a slightly different definition of "careless" error from that previously given by elements.

While there are other theoretical approaches available to teachers, NEA is one of the easiest to use and adapt and has proven popular among teachers for the ease of the diagnostic features. What is also surprising is how NEA has been used by teachers as a problem solving strategy for students as well as a pedagogical strategy. In the next section, data from the 2007 and 2008 evaluation reports (White, 2008, 2009 in press) will examine the student learning outcomes and the teacher uses of NEA.

Student learning

The 2008 program was implemented in 99 schools across NSW. An assessment instrument based on the LFIN, covering place value, addition, subtraction, multiplication, division, and word problem tasks was administered by the class teacher as a whole class schedule. The assessment schedule was closely linked to the learning framework and the results were used by the teacher to identify the student target group. The target group were then tested at the start and finish of the program. The teachers were asked to record the results of the target group assessment process involving a minimum of five students per class on an *Excel* spreadsheet supplied to them. The spreadsheet recorded the initial level on the LFIN and NEA for the targeted students before the program was implemented and again following ten weeks of targeted activities.

In 2008 data was collected from 74 schools with 55 primary schools, 16 secondary schools and 3 central schools. There were 1213 students with 954 primary students (78.6%) and 259 secondary students (21.4%). Only one of the two questions involving Newman's Error Analysis in the assessment instrument was recorded for each student. The NEA scale from 1 to 5 was used, and a category 6 was added to represent those who could complete the word problem successfully.

Table 1 below displays the initial and final NEA levels and indicates an improvement in the overall levels from the initial to the final student assessments.

Table 1. The initial and final Newman's Error Analysis levels

NEA Levels	Initial Level Frequency	Percentage Frequency	Final Level Frequency	Percentage Frequency
1	196	16.2%	51	4.2%
2	452	37.3%	234	19.3%
3	399	32.9%	477	39.3%
4	101	8.3%	220	18.1%
5	37	3.1%	134	11.0%
6	28	2.3%	97	8.0%
Total	1213	100.0%	1213	100.0%

Table 2 below shows that the majority of students have improved by one or more levels (56.6%), with a sizeable group improving two levels (15.6%). There is a small group of students who improved by three and four levels as there are some who decline by one, two or more levels.

Table 2. The difference in Newman's Error Analysis levels

Difference	Frequency	Percentage Frequency
–4	3	0.2%
–3	6	0.5%
–2	14	1.2%
–1	52	4.3%
0	452	37.3%
1	385	31.7%
2	189	15.6%
3	79	6.5%
4	27	2.2%
5	6	0.5%
Total	1213	100.0%

The descriptive statistics record an increase in the mean from 2.52 for the initial level (SD = 1.096) to 3.37 for the final level (SD = 1.254). Using a paired sample *t*-test, the results indicate that the change in the student outcomes for mathematical word problem levels at the start and finish of the ten week Counting On 2008 program was statistically significant.

The 2008 data collected for the pre and post program student learning outcomes indicated that a statistically significant change existed in student learning outcomes between the start and the completion of the program involving mathematical word problems. In a short program such as this, it is unrealistic to expect that all students will make changes of NEA levels. These targeted students have been struggling for some time with their mathematical and literacy levels and have developed judgements of their own ability. To improve one level could involve the improvement of reading or comprehension, which is quite remarkable in such a small time frame.

Teacher use

The 2007 evaluation reported the majority of teachers were strongly positive about the inclusion of NEA into the program. Many told of how it had been adapted across subjects and different stages. Teachers reported NEA as an understandable, easy to use, framework and a process for uniting numeracy and literacy and there were requests for further teacher professional learning involving NEA. It was observed that there was a divide between primary and secondary teachers. NEA appears to resonate more easily with primary teachers and with the issues of "numeracy across the curriculum" and "every teacher being a teacher of literacy." Primary teachers were able to use it to analyse NSW Basic Skills Test errors and develop strategies to improve their students' literacy needs. In the secondary school, the resonance was not as high, with some teachers regarding NEA as an issue that was not their concern. The report stated a

> typical comment extract was *The inclusion of NEA has been extremely beneficial in providing teachers with new insights into where and why the students break down in solving word number problems. The workshops we have provided have indicated that a number of secondary mathematics teachers find it difficult to embrace this process.* However this response does not represent all secondary teachers, as is evidenced by the following comment: *One head teacher has adopted/adapted it to assist senior students in Stage 6 mathematics* (White, 2008, p. 12).

The 2008 evaluation report described how teachers had extended the use of NEA beyond a diagnostic tool to a pedagogical and remedial tool. All students were expected to work through the NEA levels for all mathematical word problems. In a whole class setting, students worked aloud in order to scaffold the learning of those struggling with one of the levels. Teachers also used it as a problem solving approach, seen in this response: “The Newman’s error analysis and follow-up strategies have helped students with their problem-solving skills, and teachers have developed a much more consistent approach to the teaching of problem-solving. Not only has it raised awareness of the language demands of problem solving, but through this systematic approach, teachers can focus on teaching for deeper understanding” (White, in press, p. 42).

Conclusion

The program was a success in changing some teacher and student learning outcomes involving NEA. Although limited by the lack of a control group, the data revealed a statistically and educationally significant change existing in student learning outcomes between the start and the completion of the Counting On program involving mathematical problem solving using word problems. As well, NEA is being used by some teachers as a remedial classroom strategy and as a wider classroom pedagogical strategy. Thus this paper suggests that NEA may provide a powerful diagnostic assessment and teaching tool for mathematics classrooms.

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MATHEMATICS EDUCATION IN MODERN INDUSTRIALISED SOCIETY: APPROACHES FROM BIOLOGY[22]

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The main function of learning and memory appears to be dealing with novel information, or problem solving. This involves the input of discrete bursts of information through the senses to the human central nervous system and the interaction of these bursts with long-term memory through the processes involved with attention and working memory (short-term memory). Mathematics, in dealing with novel information, appears to utilise remembered information in a very selective and fluctuating way, in accordance with knowledge requirements of the prevailing culture and with historical practice. It may be necessary, therefore, to re-evaluate mathematics teaching in order to more fully incorporate knowledge of a biological system which acts naturally in problem solving across subject areas.

Introduction

Mathematics has become a major component of curricula in institutionalised education in many industrialised countries, and is taught as a stand-alone subject with strong links to subjects allied with science, technology and economics. Although major elements of modern mathematics are derived from a wide variety of sources dating back through the centuries, the development of mathematics as a teaching subject in modern institutionalised education has been essentially through teaching practice influenced by the requirements of trade-based economics.

Some models of instructional design (here called teaching models) have incorporated recent research in psychology and education and are well-developed for use in teaching within subjects such as mathematics. There may be issues, however, in the categorisation of human culture or knowledge into the subjects currently taught, and this includes mathematics and the divisions within it. Studies of integrative biology, and this includes neuroscience, indicate that similar physical and biochemical processes are involved in learning differing subjects and there may be many close associations between such processes across many subjects. Such studies indicate also that there is

[22] This article is a simplified version of a more comprehensive and technical paper, with a larger set of references, to be published in the near future. The references used in this paper and the websites listed in footnotes on succeeding pages have been selected to give interested readers access to the work of some prominent researchers, perhaps as a starting point for their own research.

considerable growth of neuronal connectivity, and resulting concept formation, prior to an individual undertaking institutionalised education, or schooling, regardless of the subject categories that may or may not incorporate these concepts during schooling. Mathematics may benefit from embracing these essentially biological ideas.

Mathematics education in modern industrialised society

The development of mathematics has followed industrial and technical development and associated economic change as well as a demand for people with education in numeracy or mathematical literacy, even if this is confined to simple mathematics calculations and the making or auditing of measurements (OECD[23], 2003). In modern insitutionalised education, the subject of mathematics is described, typically, in the Program for International Student Assessment (PISA) of the Organisation for Economic Co-operation and Development (OECD), as a combination of *quantity*, *space and shape*, *change and relationships*, and *uncertainty* (OECD, 2007). These categories relate the application of ideas considered as mathematics to the solving of complex problems in a variety of differing real-world areas, such as taxation and commerce, land measurement and astronomy, and to measurement of change more generally.

The development of modern mathematics, however, including the needs-based mathematics attempted in numeracy advocacy, has followed cultural and historical practice, rather than consideration of cognitive processes. A typical concern is the lack of integration of mathematical with real-world knowledge, for example, where a child that is taught to subtract 354 from 600 is not taught to solve similar real-world problems expressed in words, with units of measurement. Some attempts at re-evaluation of mathematics appear to accommodate knowledge of cognitive process by assisting students in learning to think about their own mathematical strategies and mental operations, but there has been little re-evaluation that incorporates an understanding of knowledge of cognition obtained from studies in integrative biology.

Mathematics and models of instructional design

Mathematics, particularly as presented in modern institutionalised education, can be seen as an artificial construct wherein certain ideas, metaphors or analogies, and the concepts developed around them, have been segregated from the wider body of knowledge taught in other subjects or pertaining to real-life situations (e.g., Lakoff & Núñez, 2000). Mathematics knowledge, like all knowledge, appears to be an accumulation that relates largely to the particular culture in which it is developed. In modern industrialised society, mathematics knowledge is transmitted from generation to generation largely through institutionalised education, such as in schools and universities, and relies on person-to-person interaction and knowledge stored in external information stores, such as books and computers. Effective teaching models can determine the degree of success of such knowledge transmission.

The historical development of teaching models for mathematics was based partly on meeting the demands of such trans-generational transmission of knowledge through

[23] For OECD research publications on the Numeracy Network see:
http://www.oecd.org/searchResult/0,3400,en_2649_37455_1_1_1_1_37455,00.html

sharing information about teaching practices and obtaining normative practical knowledge (Sylwester, 1995). Such teaching models, however, in adopting socio-cultural assumptions and beliefs, have ignored biological explanations of cognition that have been available for some time. In addition, these historical models have been aligned only rarely with the controls and trials of evidence-based, scientific studies (Lyon, 2005).

Recent teaching models for mathematics, however, have incorporated studies of human cognition as well as historical practice. In some such models, learning and memory are described in terms of an information processing system that involves long-term memory (LTM), which effectively stores information more-or-less permanently, and a working memory (WM), sometimes called short-term memory, which performs the cognitive tasks associated with consciousness, but which is extremely limited in both capacity and duration. This view of learning and memory assumes that information is stored in LTM, as schemas or chunked combinations of elements, after that information has first been attended to and then processed in WM (e.g., Sweller[24], 1994). Attention and WM can be viewed as mechanisms that allow the determination, within the brain, of novel sensory information and its interaction with schemas from LTM (e.g., Postle, 2006).

Integrative biology and a closer examination of human cognition and mathematics

Cognition in terms of physical and chemical processes

In recent years some scientists have investigated human cognition by assuming that physical and biochemical processes are the only basis for learning and memory (e.g., Lakoff & Núñez, 2000), and such studies have allowed closer examinations of the functions and processes described as human thinking. Such examinations, for example those of Susumu Tonegawa[25] and colleagues (e.g., Nakazawa, McHugh, Wilson & Tonegawa, 2004), have involved studies of biochemistry and genetics of genetically-altered or knockout-gene animals, as well as a variety of measurement and imaging techniques, such as functional magnetic resonance imaging (fMRI) (see technical summary in Geake[26] & Cooper, 2003). These and other similar studies, here included under the banner of integrative biology, have been used to analyse mathematical tasks by breaking down complex learning processes, often through studies of dysfunction (e.g., Dehaene[27], 2004).

Concepts, schemas and mathematics

Such close examination of human cognition, particularly with respect to mathematics, indicates that there may be physical and chemical parallels to the elements and schemas

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[24] For the research interests and publications of John Sweller and colleagues see: http://education.arts.unsw.edu.au/staff/staff.php?last=sweller

[25] For the research interests and publications of Susumu Tonegawa and colleagues see: http://tonegawalab.org/

[26] For the research interests and publications of John Geake and colleagues see: http://www.une.edu.au/staff/jgeake.php

[27] For the research interests and publications of Stanislaus Dehaene and colleagues see: http://www.unicog.org/main/pages.php?page=Stanislas_Dehaene

described in cognitive psychology. For example, Alan Snyder[28], Director of the Centre for the Mind at Sydney University, and colleagues (e.g., Snyder & Mitchell, 2001; Snyder, Bossomaier & Mitchell, 2004), have examined evidence indicating that detailed input information is stored, particularly during early childhood, as discrete packets in the brain, and that links within and between each packet are retained when the information is stored. In circumstances where some of these links are inhibited, the remaining links form a network called a concept. The inhibition process can be described as rather like creating a pattern with skyscraper lights, by turning off some of the lights and leaving on others, where the resultant pattern is analogous to a concept. Concepts can be linked together in part or in full as meta-concepts.

Some such concepts can be categorised as mathematics knowledge, and brain scans have shown that there are, indeed, particular neuronal pathways or assemblies activated during tasks described as mathematics. For example, Stanislaus Dehaene and colleagues (e.g., Dehaene, 2004; Dehaene, Molko, Cohen & Wilson, 2004) have described mathematics networks or assemblies activated in remembering of number order, numbers of items in spatial arrangements or calculation using number lines. They have shown also that when these networks or assemblies are lesioned, a variety of numerical deficits occur depending on the lesion location.

Mathematics co-opts multifunctional concepts

Concepts that are considered as mathematics may result from simple input information. An addition symbol, for example, requires only the input of sufficient discrete visual information to differentiate spatially regions of light or dark, or differing colours, as different points or dots of information. Lines and shapes, such as those in the addition symbol, correspond in simplistic terms to the concepts as described by Snyder et al. (2004). These concepts assist in the negotiation of our environment, but may be co-opted for the abstracted information referred to as mathematics.

Dehaene and colleagues (e.g., Dehaene, 2004) have shown that concepts based on spatial conceptualisation, in fact, may be formed prior to the integration of such concepts as language formation, and this includes formation of the language elements seen in mathematics notation. Elizabeth Spelke[29] and colleagues (e.g., Barth, La Mont, Lipton & Spelke, 2005) have shown that some animals, as well as non-numerate people, and even human babies from as young as six months of age, can attend to the spatial difference between one, two and three objects, as well as groups of more than three objects, from both visual and aural input. It is clear also that some concepts considered as mathematics by humans, such as those used in monitoring the position of moving objects, are concepts also for many life forms, and other primates share the human ability to differentiate spatially one, two or three objects, as do such animals as rats, pigeons, parrots and dolphins (e.g., Dehaene, 2004).

A categorisation of memories as specific to the subject of mathematics is only one possible categorisation of separate types of perceived and learned pattern formation or concept in long-term memory. There remains, however, a strong connection between visual pattern and number, and mathematics is permeated by culturally-defined

[28] For the research interests and publications of Alan Snyder and colleagues see: http://www.centreforthemind.com/publications/publications.cfm
[29] For the research interests and publications of Elizabeth Spelke and colleagues see: http://www.harvardscience.harvard.edu/directory/researchers/elizabeth-spelke

metaphors through which number is made to correspond to spatial position (e.g., Lakoff & Núñez, 2000). Dehaene and colleagues (e.g., Hubbard, Piazza, Pinel & Dehaene, 2005) have suggested, in fact, that there is potential overlap of regions of neural circuitry that are involved in spatial representation and that are crucial for abstract representation of quantity. Such associations may include also the conceptual combination of memory pathways that are used in both language and mathematics and that correspond to written or other visual symbols and sounds.

Brain-wide processes are involved in learning mathematics

For any memory there is a complex interlinking of many distributed neuronal pathways in the brain, and the normal ongoing plasticity of the nervous system ensures that new connections can constantly be grown. This implies that, while specific brain regions are a key component in the activation of a memory, they may be only a link in a brain-wide storage mechanism. Many brain regions may be activated during processing of information in any subject and this has been shown clearly in processing of number and number relations (Dehaene et al., 2004) and equi-partitioning in integer arithmetic (Snyder & Mitchell, 2001). There may be, however, brain-wide activation common to learning processes in both mathematics and other subjects. Lakoff & Núñez (2000) refer to this in terms of conceptual metaphors, cross-domain mappings that preserve inferential structure and which are essential for linking conceptualisations generally, but which serve also for linking mathematical or other subject categories.

Mathematics education and cognitive processes

Mathematics uses an already operational system

Integrative biology has indicated that a child is not taught to do mathematics, but rather is provided with opportunities for adaptation of an already operational system (Sylwester, 1995). Close examination of human cognition indicates that mathematics and reading are human constructs, an adaptation of the capacity for planning for muscle movement (competitive pre-motor scenarios; e.g., see Calvin[30], 2004). These scenarios allow for the interaction of novel information with long-term memory, without any action necessarily, and this capacity has been conserved because it contributes ultimately to survival (e.g., Dehaene, 2004). Parts of what has been conserved are the intrinsic brain pathways from which such scenarios develop, for example, the primary knowledge domains of Geary[31] (2005) or the values of Edelman[32] (1987, 1989). At least some of these intrinsic pathways are attuned to learning mathematical knowledge.

The human ability to form memory associations as concepts and to build long-term memory, for example, by forming motor scenarios without any subsequent motor activity, has been an important component in the building up of cultural knowledge (Calvin, 2004). Importantly, however, some memory processes in one individual can be activated through observation of another individual engaged in a motor task, and such

[30] For the research interests and publications of William Calvin and colleagues see: http://williamcalvin.com/
[31] For the research interests and publications of David Geary and colleagues see: http://web.missouri.edu/~gearyd/articles_math.htm
[32] For the research interests and publications of Gerald Edelman and colleagues see: http://www.nsi.edu/index.php?method=Author&q=edelman&page=scientific_publications

processes may be significant for human learning (e.g., Sweller, 2004). Motivation may play a significant role in such learning and goal-directed activity, particularly in relation to reward, may be useful in developing teaching models for mathematics problem solving.

Describing mathematics in terms of cognitive concepts

Although modern mathematics, in order to solve problems, is concerned with relations among patterns that fit ultimately real-world phenomena, relations among neuronal patterns may not correspond directly and uniquely to such real-world patterns. There may be some incongruence, therefore, in modern teaching subjects with regard to the patterns of association recognised and the patterns of neuronal networks (concepts and meta-concepts) used when operating within those subjects. For example, presentation of the written instruction, “What is the answer when 2 is added to 4?” implies that the operation 2 + 4 will be performed by the brain and the answer will be written or spoken as 6. During such exercises, however, brain scans are more likely to reveal a sequence of activations, of varying degree, of the visual subsystem in some brain regions, of the verbal subsystem in others, and of the quantity subsystems and other systems as required (e.g., Dehaene, 2004).

Snyder and colleagues (e.g., Snyder & Mitchell, 2001) support the notion that the algebraic and algorithmic processes taught in mathematics may not correspond to the processes that they are designed to activate, and Baars[33] (1995) has proposed that humans use heuristic, or “best guess” processes and analogies, rather than algorithmic processes, in dealing with environmental input. Although several capacities have been described for the brain, for example, problem-solving, decision-making and action control, Baars (1995) considers that one of the strengths of human cognition may be in remembering and cross-analysing patterns observed from the real world, arguably an intrinsic mathematics capacity. The teaching models used in mathematics, therefore, may be enhanced by understanding the links between patterns of real-world information and the neural activation patterns of that information that develop from such cross-analysis.

Mathematics and neural dynamism

The situation with regard to understanding mathematics and its relationship to memory patterns is made more complex in that such patterns are dynamic and subject to constant change through such processes as growth, competition and inhibition in neural pathways (e.g., Edelman, 1987, 1989). The build-up of neural patterning may vary considerably between individuals over time, even with similar input and output information, and there may be considerable variation, therefore, in cognitive structures between individuals, even if there are commonalities in brain activation. Teaching models used in institutionalised education may need to consider such variation between individuals, even though this variation may not give rise to variation in real-world patterns of behaviour (Geake & Cooper, 2003). There are, however, few normative databases of cognitive development partly because, within individuals, simple memory activations may involve widespread brain regions and also because there may be less

[33] For the research interests and publications of Bernard Baars and colleagues see: http://vesicle.nsi.edu/users/baars/

activation for well-learned processes with any increased neuronal efficiency (Geake, 2004).

Neural dynamism does not operate only during storage of discrete information in long-term memory, but also in sequencing of memories (e.g., Postle, 2006) and in the linkage of emotions and chemical reward with memories (Panksepp, 1998). Neural patterns that develop with such intrinsic and dedicated flexibility act to adapt each human to a range of environmental inputs, including input classified as mathematics. It may be useful to reconfigure mathematics teaching in order to match it more closely to modern knowledge about dynamic and concept-driven, rather than static and subject-driven, human cognitive development.

Mathematics and non-institutional learning environments

Mathematics educators may need to embrace also research that shows that concept formation occurs prior to an individual's entry to learning institutions, regardless of the subject categories that may or may not incorporate these concepts during schooling. Obvious examples of this are the concepts required for communicating in a native language. There is considerable evidence to suggest that neural networks, such as those in long-term memory, begin to act dynamically as soon as there is input to the nervous system and early memories reflect the world experienced prior to school (OECD, 2003; OECD, 2004). Snyder and colleagues (e.g., Snyder et al., 2004) have suggested that human infants see the world more literally than adults by accessing lower levels of information processing, some of which are inhibited in the concepts used in adulthood. It may be important, therefore, to develop teaching models for pre-school infants to allow sufficient stimulus for development of the appropriate lower levels of information processing needed for concepts taught later in an institutional setting.

A considerable number of concepts that develop prior to schooling may relate to mathematics, with such concepts developing in a home/parent environment (e.g., see OECD, 2004). Spelke and colleagues (e.g., Xu & Spelke, 2000) have shown that natural number addition and subtraction is universal and pre-linguistic. For example, a 5-month-old child can perform non-symbolic comparison and addition despite any lack of symbolic knowledge. Some such learning in the home/parent environment may be incidental, gained through such things as observation, repetition and social interaction.

In contrast, mathematics concepts taught at school may not be the same as those held already in pre-school children, and such taught concepts may function effectively as a second language and become hard work (e.g., Dehaene, 2004). Such pre-school concepts may be inhibited by the concepts taught at school, and school subjects, including mathematics, do not always build on the long-term memory (LTM) available prior to such education, but build a completely new, and relatively unrelated LTM that may act to inhibit other memory components (see e.g., OECD, 2004). Teaching models may be more effective if they build on the concepts that exist prior to schooling, for example by incorporating universal non-symbolic abilities in order to enhance symbolic learning in mathematics.

Conclusion

Even though there is an assumption in the teaching of mathematics that it is mathematics that is being learned, many of the neural processes involved may be common to many subject categories. These processes include the association of symbols, spatially, temporally or spatiotemporally, from a variety of sensory inputs, as well as processes that are involved in automatically reinforcing or inhibiting patterns of neural connectivity. The goal of mathematics teaching is to optimise and facilitate mathematics concept-building and align this concept-building with the requirements of a curriculum. This approach may be enhanced by examining ways in which the build up of long-term memory relies on concepts held already, for example by examining information input in pre-school, and by considering cultural experiences. It may be useful, therefore, to spend some teaching time in introducing and establishing concepts that may be useful across more than one subject, as this seems to be largely ignored in subject-based teaching models.

In terms of cell connectivity, there may be little distinction between knowledge labelled as mathematics or another subject, or non-subject knowledge, and this has caused some dispute among educational theorists (e.g., Dehaene, 2004). Any differences between subject-specific and generalised knowledge may lie, however, in the means of access to long-term memory (LTM) that are activated by input information. This is not to say that teaching of information in subjects is not valuable, but that such information needs to be incorporated into LTM in context with information with which it is connected, or with which it may need to be connected. In this way retrieval through attention and working memory processes may include such contextualised information. This may include information about specific procedures and strategies, not as separate elements as suggested in some educational literature, but as elements totally integrated with as many strong links as possible across the knowledge base in LTM (Postle, 2006).

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OTHER PAPERS

ENRICHING AND DIFFERENTIATING THE MATHEMATICS CURRICULUM AT MELBOURNE HIGH SCHOOL

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Melbourne High School (boys only) is one of two select-intake government schools in Victoria where students achieve outstanding results across all studies. As the coordinator of the High Achieving Student Program I have had the privilege of working with the mathematically elite in the school at Years 9 and 10 for many years. I have taken on the challenge of putting into place a program with my Year 10 maths class where the performance levels of distinct groups of students were identified, appropriate instruction delivered and suitable assessment made. This is broadly called differentiating the curriculum and I saw this to be important not only to provide greater assistance to students with difficulties but also to enhance the performance of the high achieving students in the class who would otherwise be bored by the delivery of the regular curriculum. This paper describes how the maths courses at Years 9 and 10 offer high achieving students across all classes the opportunity to work with challenging content-based materials aimed at enriching the experience of students who opt to take such a challenge; some examples of such materials are provided in the paper.

Introduction

Melbourne High School (boys only, Years 9–12) is one of two select-intake government schools in Victoria. Melbourne High School's curriculum is derived from the fact that at Year 9, students arrive after satisfying the standards of the entrance examination at a number of different albeit a high set of different levels. The school is rated as number one or two in the state — beside our sister school, Mac.Robertson Girls' High School — at Victorian certificate of Education (VCE) Year 12 level, but the aim of the school is to encourage a well rounded education of mind, spirit and body for every student. Breadth of study and co-curricular involvement is central to the whole program. The intention is to nurture well rounded, mature and socially responsible young men.

Gifted education research suggests that highly able students can complete school work at four times the rate of an average student so giving more time to follow their passion in their interest areas. Mathematics forms an essential part of the key interest area for a large number of our students and so the demand is high for experiences of a higher level such as competitions, extension groups and enrichment materials.

Philosophy behind extending students

Melbourne High School embraces a philosophy emphasising "more than just marks" with a focus on extension, enhancement and active involvement in the school's extensive co-curricula program to foster depth of knowledge rather than acceleration. The level of performance of students in the mathematics faculty means additional enrichment material designed to apply mathematical concepts to more complex tasks is used, rather than accelerating students into next year's course. Students are required to think and perform on deeper, broader levels of investigation. The idea of learning next year's mathematics course using a textbook and passing a test, based on repeating a sequence of solution routines is not undertaken at Melbourne High School.

For most students, our experience is that such acceleration does not allow enough time for their thinking to mature. It shows that the students can repeat the solution steps in routine, simply expressed questions with a limited number of steps. What it does not show is an understanding of the core principles that underpin each concept studied. Students are able to progress quite happily through to Year 10 with this approach but when it comes to the mastery of complex mathematics topics in senior years, this learning approach is found wanting. This is what I call the *performance wall* — students with a superficial level of understanding of the concepts are able to respond to questions of the type covered on that level but when they encounter questions at higher levels which involve assembling a number of concepts, students are challenged to show their true understanding of the workings and meanings of the mathematical methods.

In educating the students in Australia's schools, it is the responsibility of the curriculum structures to accommodate individual differences in the performance of our students in some way. In practice, it is the mathematics faculty in each school that needs to match student potential with suitable levels of study. Some education research suggests that approximately 10% of our students can be classified as being gifted and talented according to their area or areas of expertise. If we add another 15% of students as fitting into a high achieving student group then about a quarter of the nation's students are in need of extension or enrichment materials in the mathematics curriculum. It is the contention of this paper that it is the responsibility of the mathematics faculties in *all* schools to put into place a wide range of experiences to go some way towards satisfying the needs of this top quarter of our students.

Higher order thinking and the use of suitable tasks

When it is said that extension and enrichment tasks need to incorporate aspects of higher-order thinking it is important to understand what this means. The application of a higher-order thinking regime requires students to interpret information and ideas which are not presented in a routine way. In contrast to this, lower-order thinking is characterised by the simple recitation of memorised, rehearsed routines; this all too commonly becomes the focus of the mathematics curriculum — a system that consists of a series of skill-based tick boxes of which each student needs to show mastery. With a crowded curriculum, mathematics is seen as the time-hungry subject which gets a large amount of the instruction time in the school day and is often is seen as the easiest target to give up time for new curriculum initiatives. With an erosion of instruction time comes a need to cover the basics and a necessity to meet the mathematical outcomes rather than examine a range of applications, challenging questions and open-ended scenarios. Indeed the reporting system proposed by earlier versions of the Victorian Curriculum standards framework became a checklist with hundreds of outcomes that needed to be measured and reported against — an administrative nightmare.

In the author's view, an outcomes-based model alone is inappropriate to the laterally thinking group of gifted and high achieving students in our classes. These students are treading water, finding mathematics a boring and an unfulfilling repetition of facts and algorithmic routines. They need a challenge and they need the opportunity to think, to combine whole pieces of information together, be presented with a problem that requires them to gather their ideas together, to organise an approach, to find a range of solutions, to test those solutions and validate them, and finally to select the best ones and explain their choices. This places these students at the cutting edge of what problem solvers do; this is the exciting part of what mathematics can and should be.

Differentiating the mathematics curriculum in the secondary school

As teachers we are responsible to deliver a curriculum suitable to the needs of our students but this can become unwieldy, difficult to administer and almost impossible to resource — it is not easy and all too often becomes too hard to deal with. Often the instruction is aimed at the middle ground with assistance for underachieving students and little for the high achieving students. Students who can successfully complete the class tasks may be directed to help other students having difficulty or given more work of the same type — not really an incentive to finish quickly.

A model that I have used in my Year 10 mathematics class this year has allowed me to address the needs of the high achieving students while also providing time to work with the students who need more assistance. I am the form teacher of a Year 10 mathematics class and so developed a good working relationship with the boys.

I used a diagnostic test to establish a baseline of student performance on a holistic level. It gave me a profile of the abilities of all students in the class from the first day. The test was based on a broad range of skills and topic areas covered by all students at the end of their Year 9 studies. The class was divided into three groups of roughly the same size. I was most concerned with being able to get the lower achieving group in a positive frame of mind rather than being ostracised and feeling turned off through the process.

I met with the bottom third of the students and indicated that they would get the focus of my attention in class. A number of boys said that they had never done very well with their mathematics work and we agreed that their goal for the year should be to form a better understanding of the material that they previously had not understood. I had to handle this part of the process with great care as I did not want the students to be labelled in a negative sense either by themselves or by their peers. I packaged the aim of work for the year as being linked to improved performance prior to choosing mathematics subjects for the VCE Years 11 and 12. This process formed an agreement based on trust with the boys. In my experience, boys like to know where they stand in an honest positive sense. It took me a number of discussions with the boys to put a positive spin on the results.

One student did not respond well to this approach, however, as was conveyed to me indirectly by other communication channels. I spoke to his parents and gave them positive feedback and spoke to him a number of times to clarify the situation. This resulted in a positive working relationship being established. It is the establishment of trust between the teacher and the student that is crucial in the education process.

I started the year working through the topic area of number, consistent with the general syllabus and when the associated achievement test was administered it was obvious (while not surprising) that the students who had scored well on the diagnostic test had mastered the content areas examined in the initial diagnostic test. In fact some students completed the test in a very short time and asked if they could be given work of a more challenging standard.

I wrote some advanced work based on the topic area of study and got those high achieving students to sit together in a group in the classroom. An illustrative example of a task of this kind for linear algebra is shown in Figure 1. The students' were required to address the advanced tasks by working together cooperatively. During some lessons I had the luxury of having a spare room adjacent to the larger classroom and I organised for the high achieving students to work together in the smaller room. Because the rooms were adjacent, I was able to spend some time in each room keeping an eye on both groups, providing examples on whiteboards in both rooms. At the same time I gave my teaching priority to the middle and bottom thirds of the group, consistent with my clearly articulated approach after the first diagnostic test. In this way I felt that I had all major parties satisfied. The high achieving students were working together as a cooperative group to answer challenging non-routine questions based on the topic of study and the middle and lower groups had my attention to master the basic skills according to the textbook exercises. All work completed was handed in and marked. This was not hard to do and it meant that the high achieving students took the process seriously. It also allowed me to check that each student was able to cope with the more difficult material.

One issue that arose concerned the need to ensure that the high achieving group had met baseline expectations in regard to the basic skills being covered. To do so, I still required them to complete the harder questions in the exercises in the textbook, consistent with the expectations for the rest of the class. It was important to make sure that all students had covered questions for the key content areas, as these were likely to be a foundation for further work and also because they would be covered in the semester exams. All students in the class needed to show me that they had completed the required questions in the textbook according to the syllabus in preparation for the mid-year examination.

In subsequent topics I used a largely multiple-choice diagnostic test to monitor the performance of the high achieving student group. This was quick to mark and gave me the information that I needed in a direct and efficient way. I needed students in that group to prove that they had mastery of the key concepts to be studied before giving them more challenging work. My experience is that some students might excel in some topic areas but not in others, so that a practice of automatically keeping this group together throughout the year might in fact do some students a disservice. Accordingly, at the completion of each topic the students in the class completed a common test in line with school expectations.

Linear Algebra Sheet 1

1. Example: If $a + 2 = 7$, find the value of $3a - 12$ without solving for a first.

If $a + 2 = 7$
then
$3(a + 2) = 3 \times 7$
$3a + 6 = 21$
$3a + 6 - 18 = 21 - 18$
so $3a - 12 = 3$

Note how the value for a was not found in this example.

2. If 6a = 2a + 10, find the value of $13 - 2a$ without solving for a first.

3. If $4p + 8 = -12$, find the value of the following without solving for p

   (a) $8p$ (b) $2p$ (c) $2p^2$ (d) $p(p + 4)$

4. (a) If $\frac{1}{a+1} = -5$, find the value of $\frac{1}{a+6}$ without solving for a.

   (b) If $\frac{1}{a+1} = 5$, find the value of $\frac{1}{a-1}$ without solving for a.

5. If $\frac{1}{2a+3} = \frac{2}{3}$, find the value of $\frac{1}{4a+3}$ without solving for a.

6. Find the value of $a + 4b$ if $\frac{a}{-b} = 4 - \frac{3}{b}$.

7. If $\frac{x}{y} = \frac{a}{b} = \frac{1}{5}$, find the value of (a) $\frac{x+3a}{y+3b}$ (b) $\frac{x+ak}{y+bk}$

8. If $\frac{p}{q+r} = \frac{2}{5}$ and $p + q + r + 1 = 11$, show that $\frac{r-p}{q} = \frac{2p}{q+r}$.

9. If $\frac{p}{q} = \frac{2}{5}$, find the value of

   (a) $\frac{p-q}{q}$ (b) $\frac{p+q}{q}$ (c) $\frac{p-q}{p}$ (d) $\frac{p-q}{q}$

Figure 1. Sample of an enrichment class task used with the high achieving group.

School enrichment policy

Melbourne High School is committed to an extensive enrichment and extension program across all subject areas, allowing students of a high calibre to explore concepts and issues initially raised in the school setting and to take those notions of interest to further ends. As a matter of school policy, all faculties are required to place extension units on the school portal and teachers are required to encourage their students to undertake work with these enrichment tasks. Students are expected to access these units from home, download them and work on them either alone, in groups of students or with the assistance of their teacher. Students gain recognition for working on these extension units through the teacher's reporting process as well as having their names printed in the *Students High Achieving News*, released at the end of first term each year. This curriculum news publication celebrates the achievements made by students across all areas of school life for the previous year.

I saw the release of mathematics extension units from Years 9 and 10 as an opportunity to involve high achieving students from all classes not just the students in my class on a topic-based level. In my previous role as Head of Mathematics at Melbourne High School I took on a whole-level responsibility to all classes working across Years 9 and 10.

I convened meetings with staff, gave out the tasks and solutions and outlined the program, highlighting the fact that the students were responsible to access the tasks and that staff could use them according to the appropriate teaching dynamic in their class. Some staff used the materials in class with groups of high achieving students while other teachers collected the tasks at a negotiated time after the students had worked on them.

To illustrate these extension opportunities, the appendix shows the questions from two enrichment sheets that were offered to Year 9 students during the second semester last year at Melbourne High School.

Resources

Resources to support an enriched and differentiated curriculum are best collected over time from a variety of sources, both local and overseas. There is not space in this paper to provide extensive detail, but two particular kinds of resources are worthy of special mention.

Mathematics competitions

A number of mathematics competitions provide a variety of useful materials for students. These include at least the following Australian materials:

- Australian Mathematics Competition (http://www.amt.edu.au/eventsamc.html)
- Mathematics Challenge for Young Australians, including Enrichment Stages (http://www.amt.edu.au/mcya.html)
- University of NSW International Competitions and Assessments for Schools (ICAS) (http://www.eaa.unsw.edu.au/about_icas/mathematics)
- The BHP Billiton School Mathematics Competition (http://www.mathscomp.ms.unimelb.edu.au)

Useful reference materials

There are many publications intended to challenge successful students, some of them deriving from mathematics competitions and others from experience working with advanced students. At Melbourne High School, we have found the following to be especially valuable for this purpose.

- Atkins, W. J., Edwards, J. D., King, D., O'Halloran, P. J. & Taylor, P. J. *Australian Mathematics Competition, Book 1, 1978–1984*. Australian Mathematics Trust. (Information available at http://www.amt.canberra.edu.au/book01.html)
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- Atkins, W. J & Taylor, P. J. *Australian Mathematics Competition, Book 4, 1999–2005*. Australian Mathematics Trust. (Information available at http://www.amtt.com.au/)
- The Canadian Mathematics Competition series of publications, *Problems, Problems, Problems*. Published by the University of Waterloo, Canada. (Information available at http://cemc.uwaterloo.ca/books.html)
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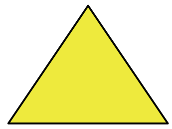
Conclusion

Experience over many years with boys at the selective Melbourne High School has reinforced our views that the curriculum can be differentiated to ensure that all students are catered for adequately in an environment focused on mathematical enrichment for many. While it is important to ensure that all students gain a proper grounding in mathematics, making good use of available published resources allows the school to cater well for the more sophisticated students, for whom the regular mathematics curriculum may be insufficiently challenging.

Appendix: Sample enrichment sheets

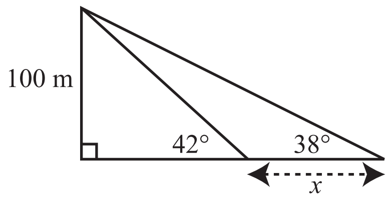
Year 9 Maths Enrichment Semester 2 — Trigonometry

1. Mark on the equilateral triangle the edge length of 10 cm.



   (a) Find the height of the triangle using
      (i) Pythagoras theorem in exact form (using surds) and expressed correct to two decimal places.
      (ii) Find the height of the triangle using trigonometry expressed in exact form and expressed correct to two decimal places.
      (Hint: $\sin 30° = \cos 60° = \frac{1}{2}$, $\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$)
   (b) Find the area of the triangle also expressing your answer in exact form and in decimal form to 2 decimal places.

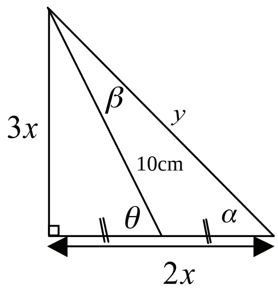
2. Find the length marked as *x* in this diagram showing full working.



3. Using a calculator express each value correct to 4 decimal places and so show that the expressions are equal
   (a) $\sin 105° = \frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}$ (b) $\sin 225° = -\frac{\sqrt{2}}{2}$

4. Using a calculator show that the trigonometric identity: $\sin^2 \theta + \cos^2 \theta = 1$ is true for the following angles (note that $\sin^2 \theta = \sin \theta \times \sin \theta$).
   (a) $\theta = 30°$ (b) $\theta = 45°$ (c) $\theta = 60°$

5. (a) Find the values of the lengths *x* and *y* in the diagram expressed in exact form, using surds.
   (b) Find the angles θ, α, β in the diagram expressed in decimal form correct to two decimal places and in degrees and minutes.



YEAR 9 MATHS ENRICHMENT SEMESTER 2 — PROBABILITY

1. Sam rolls two dice. Find the probability that:
   (a) the sum of the dice is at least 9; and
   (b) the difference between the number on each die is 3.
   (c) (i) Sam invites William to play a game for which William wins if the difference between the numbers on the dice is at least 3. Show that this game is not fair.
   (ii) Suggest how this game could be made fair — explain your answer.

2. David bought these four dice:

COLOUR	NUMBERS ON FACES
RED	1, 1, 1, 2, 2, 3
BLACK	1, 1, 2, 2, 3, 3
BLUE	1, 1, 2, 2, 3, 4
GREEN	1, 2, 2, 3, 3, 3

   (a) Find the probability that when each die is rolled the number 1 shows uppermost.
   Pr (red shows 1) = __________ Pr (black shows 1) = __________
   Pr (blue shows 1) = __________ Pr (green shows 1) = __________
   (b) He rolls the dice in different coloured pairs (red & black, red & blue, etc.) and writes down the result. Find the probability that the number 1 shows uppermost on both dice for each roll.
   (c) (i) He rolls each die once. Find the total number of ways that the number 1 can show on all four dice.
   (ii) Find the probability that when the dice are rolled once they will each show the number 1.
   (d) Find the number of ways that when the dice are rolled the numbers landing uppermost add up to 5.
   (e) (i) Find the probability that when the dice are rolled the numbers showing uppermost add up to 6.
   (ii) Mervyn buys four coloured dice with the numbers {1, 2, 3, 4, 5, 6} on their faces. Find the probability that when he rolls them the numbers showing uppermost add to 6.
   (iii) Which set of dice (David's or Mervyn's) gives the better chance of giving a sum of 6? Explain your answer.

TEACHER PARTICIPATION IN THE AUSTRALIAN MATHEMATICS COMPETITION PROBLEM FORMATION

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The Australian Mathematics Competition (AMC) has been running for 34 years. It has grown to the extent that around 500 000 students sit the paper each year. This paper gives a brief outline of a teacher's participation on the Problems Committee of the Competition and describes some examples of opportunities for teachers to make use of competition questions.

Introduction

The Australian Mathematics Competition[34] (AMC) has been running every year for 34 years now (Australian Mathematics Trust, 2009a). In recent years about half a million students per year from 35 countries have taken part in this written competition. A core element of the AMC comprises the problems themselves, for which the AMC Problems Committee is responsible. This committee comprises academics, mathematicians and practicing mathematics teachers.

As a member of the AMC Problems Committee for 30 years, I have acquired considerable knowledge and skills that have been helpful to me in my teaching. This has certainly been the most enlightening committee I have ever been involved with and I never cease to learn more each year. This paper is to share some of my learning from this experience with the next generation of mathematics teachers.

Committee processes

The Chair of the AMC Problems Committee asks each panel member to design fifteen questions that go into the mix for consideration at a meeting fourteen months before the actual date of the Competition. Questions designed by the Committee members are generally innovative and interesting. The panel meets over a weekend and considers the questions from many perspectives, including: language; clarity; distracters; previous statistics; diagrams; balance; gender; discrimination; cultural bias; ranking; marking; and classification.

[34] Detailed information about the nature and operations of the AMC, including sample questions, is available at Australian Mathematics Trust, (2009a).

The AMC aims to have problems with a modern everyday theme. One way of attending to this is to check that the syllabus in each state is considered in the classification of each question into bands of five. Information is obtained from suitable people in each Australian state to ensure that students who enter the competition encounter questions that are consistent with their curriculum, at least as far as the earlier (and easier) questions are concerned.

There are some questions that overlap, i.e., are included in more than one AMC paper. We try to stimulate interest, to tap into student curiosity, to procure discussion afterwards and to enable schools to build a rich item bank of challenging questions. Once a paper is formed, it is sent out to state and overseas moderators, who provide us with a rich and varied set of comments to enhance or even replace questions.

There is a third and fourth stage of moderation, as well as a solution book development stage to provide the best quality assurance for each question. We gain a good deal of knowledge and many ideas from studying the discrimination information (measured with biserial correlations) contained in the solutions booklets of previous years. Inspection of these will reveal that these have high values in many of the fairly early questions in papers. In recent times the AMC has changed the format of the last five questions (numbers 26 to 30) from multiple choice to an open-ended format comprising a response with an integer between 1 and 999.

The paper is printed in at least three languages; also a glossary is provided in some nations. South Pacific schools are active and eager participants in the competition. Sometimes the data from a question will surprise us and provide teachers with information that may enhance the teaching of specific topics.

In general terms, it seems that students are improving in many areas of mathematics. The AMC is the one competition that has the support of all the professional associations. These are the groups involve in the teaching, in the enrichment role, in the course development and in fostering interest in mathematical learning. These groups are aware that our competition is the foundation upon which is built the *Mathematics challenge for young Australians*, the enrichment series (named after mathematicians such as Euler, Gauss, etc.) and ultimately the AMO, APMO and IMO. Details of the *Challenge* are available online (Australian Mathematics Trust, 2009b).

The AMC works with and through schools in the identification of latent talent. A number of students who have gone on to further enrichment activities and indeed careers of a mathematical nature, often recall their keenness and feelings of achievement from the obtaining of a Certificate or Prize in the AMC.

Continually throughout the world, more professional groups, more competition committees, etc. are forming links and basing their activities on the AMC. This was acknowledged at such forums as the World Federation of National Mathematics Competitions (WFNMC) in Cambridge in 2006. Information about this international group is accessible from the AMC website, listed in the references at the end of this paper.

We are always open to new ideas and encourage teachers to put themselves forward as moderators or problem-constructors. The creativity of the panels and moderators is such that new ideas for questions are always forthcoming, but we are always interested in hearing from mathematics teachers who may like to become involved in the future.

Examining data

Competition data are available for questions. These data from previous competitions can be studied by the Committee to help with the design of questions, but can also be examined by teachers to shed some light on student thinking. Here we consider two examples.

Intermediate Question 5

If $\sqrt{x+1} = 3$, then $(x + 1)^2$ equals
(A) $\sqrt{3}$ (B) 3 (C) 9 (D) 27 (E) 81

Discrimination for this question was very high, with a biserial coefficient of 0.45 for students in Year 9 and 0.46 for students in Year 10. The biserial shows the difference between performance on a particular question of the top 25% and lowest 25% in the overall competition performance, so that a high number, as in this case, suggests that the question discriminates well between students.

A staffroom discussion on this question may focus on such things as rushed answers, not reading the question precisely enough, but more importantly the students who "have not fully got on top" of the idea of square and square root and the relationships between these. It then becomes a handy question to carry into the classroom.

Junior Question 24

The diagram shows a 5 by 5 table. The top row contains the symbols P, Q, R, S, and T. The fourth row contains the symbols P, Q, and R at the centre. The remaining squares can be filled with Ps, Qs, Rs, Ss and Ts such that no row, column or diagonal contains the same symbol more than once. The symbol that must go into the shaded square is
(A) P (B) Q (C) R (D) S (E) T

P	Q	R	S	T
	P	Q	R	

In 1990, this question was also used as Intermediate Question 17 and Senior Question 12, so that data allows for comparisons across year groups. Detailed results for this item allow for the possibility of some interesting comparisons between students in different year groups and also between boys and girls. The table below shows the proportions of students in various categories who responded correctly to the item.

Table 1. Proportions of students who responded correctly.

	Year 7	Year 8	Year 9	Year 10	Year 11	Year 12
Females (%)	26.5	33.4	42.5	46.4	53.5	57.5
Males (%)	22.8	29.0	38.6	44.1	52.4	58.8

Discussing some questions

Teachers in the mathematics staff room can use data from the competition for very productive discussions. In this section, we consider some examples to illustrate how this may happen, using four questions from the 2008 Intermediate paper. In this case, all data refer to Year 10 students.

Question 5

The digits 5, 6, 7, 8 and 9 can be arranged to form even five-digit numbers. The tens digit in the largest of these numbers is

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

The discrimination biserial for this item was 0.48.

The correct answer was alternative A; while 42 % of cohort got the right answer, other alternatives attracted a deal of interest: B (22%), E (17%) and D (12%). A staffroom discussion on this question may focus on the rationale for each wrong answer; what is behind (E)? Does the word "largest" throw some students?

Question 6

Four consecutive odd numbers add up to 48. What is the largest of these numbers?

(A) 13 (B) 15 (C) 17 (D) 19 (E) 21

The discrimination biserial for this item was 0.49.

Correct answer B (68%); of the distracters, C (12%) was the only option for which there was strong interest.

A staffroom discussion on this question may focus on coming up with different distracters to A and B; can the staff redesign the question with just 4 choices?

Can the class do the question in a group work situation as an open-ended task, and design their own question with a different number to 48?

Note the difference in responses to Questions 5 and 6.

Question 8

What percentage of y is x?

(A) $\frac{y}{x}$ (B) $\frac{x}{100}$ (C) $\frac{x}{y}$ (D) $\frac{100y}{x}$ (E) $\frac{100x}{y}$

This question has a very high discrimination rate of 0. 41.

This question was of concern as only 30% of students chose E, the correct response. Alternative C attracted 25%, D attracted 22% and A attracted 15%.

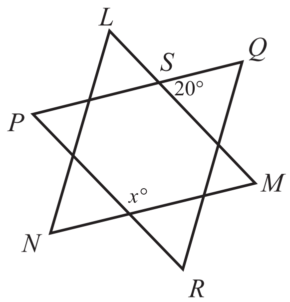
This distribution of responses has implications for a number of other areas of the mathematics curriculum. A staffroom discussion on this question may focus on how the skill associated with this item may impact on other areas of the curriculum. Teachers may note that while there is an acknowledgement of the existing confusion between A and C, 40% of students seemed not to think there was any need to consider the association of percentages with 100. This skill has implications even up to tertiary studies.

Question 9

In the diagram, triangles PQR and LNM are *both equilateral* and ∠QSM = 20°. What is the value of *x*?

(A) 70 (B) 80 (C) 90 (D) 100 (E) 110

(Note: Students are told generally that diagrams are not drawn to scale.)



The discrimination biserial for this question was 0.37. While B is the correct response, chosen by 36% of students, there were 29% selecting D and 13 % selecting E, with 11% selecting C and 10% for A. So, in this case, all four distracters seemed plausible to a large part of the population. A staffroom discussion on this question may focus on why D would be such a strong choice; the data reinforces the early questions in geometry as strong discriminators for learning and teaching applications, which may challenge where we put the topic in our syllabus. It could be a nice challenge to redesign the question with ∠QSM = 18^{o}

Conclusion

It is important to note that it is that it is not necessarily the harder questions that provide the best classroom and staffroom discussions. While the more difficult ones are very useful in the Enrichment sessions, it is often the first 10 or 15 questions in the Junior or Intermediate papers that provide a rich source of discussion.

The Australian Mathematics Trust will continue to make available solutions and statistics for the AMC in order to enhance learning and teaching, and to add to the enrichment activities going on in the classrooms and staffrooms. We are always willing to discuss new ideas with teachers, and in particular in hearing from staff in regional and rural areas. I feel I am representing all teachers of Mathematics as well as being a volunteer on the Problems Committee.

The Australian Mathematics Competition offers an important opportunity to students to engage in mathematical problem solving, but it also offers an important opportunity to teachers to use the information to understand better how students think about mathematical problems. For teachers, the experience of working with the competition is professionally enriching.

Note

Questions and data are the property of the Australian Mathematics Trust.

References

Australian Mathematics Trust (2009). *Australian Mathematics Competition aims and information*. Accessed on 4 April 2009 at http://www.amt.canberra.edu.au/amcfact.html. (a)

Australian Mathematics Trust (2009). *Mathematics challenge for young Australians*. Accessed on 4 April 2009 at http://www.amt.canberra.edu.au/mcya.html. (b)

ONE SCHOOL'S STORY: DIRTY MATHEMATICS

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This is Sandgate District State High School's story of developing a sense of ownership of its mathematics program by students and teachers and the subsequent improved outcomes for students and teachers. The development of Dirty Mathematics is a story of: identifying reasons for declining performance of students in mathematics; developing a program to increase the engagement of students; implementing this program to improve the learning outcomes of students; evaluating the effectiveness of the program against external benchmarks; constantly refining the program; meshing together concepts of authentic learning, aspects of brain-based learning theory, changing pedagogical practice, and ICT integration. The program has received both national and state awards in recognition of its success, briefly described in the paper.

Introduction

Leading up to 1999, staff at Sandgate District State High School in Queensland were becoming increasingly concerned that our Middle Schooling Mathematics Program was not meeting the needs of our students. Students had been arriving in Year 8 with increasingly diverse levels of ability, with fewer students achieving in the top percentiles. Performance in the senior school was not as good as we thought it should or could be. These overall results were historically similar to those of other schools; although similar, however, we felt that they were not good enough. A lot of students were not participating effectively in their learning — students were disengaging. The way we did mathematics at our school had an out-of-date, tired feeling, raising the question, "What should we do about this?" Through teacher observation, parent and student feedback, a detailed study of syllabus requirements and discussions with regional mathematics advisers and personnel from the Queensland Studies Authority, the major factors contributing to this decline in performance were identified as the disparity in demands between the middle school and senior syllabi, with the learning experiences provided being no longer applicable to the changed clientele.

[35] In 2009, Peter Cocks was recently transferred to Cairns State High School. During his time at Sandgate District State High School, Peter played a vital role in achieving the changes that lead to the successes of Dirty Mathematics.

Developing a plan

Teachers at Sandgate SHS were given—and accepted—the challenging task of developing a long-term strategic plan to design and implement an engaging, cohesive and seamless program for students to advance through Mathematics from Year 8 to Year 12. We researched, attended professional development activities, consulted with our cluster schools and with our community and talked, talked, talked. Our research indicated that developing a sense of ownership of their mathematics by the students was an important aspect in re-engaging our students.

How were we to give them this sense of ownership? Our response involved allowing students to make both guided and open choices about: their learning, about their learning environment (encouraging doing and discussing rather than sitting and listening—or not listening as was more often the case), about their mode of engagement and about the nature and format of the assessable products demonstrating their learning.

From this background, the principles and specifications for our program development emerged. We wanted to include the following elements into our planning:

- Explicit consideration of the productive pedagogies highlighted in the New Basics work of the Queensland Department of Education and Training (2004), begun more than a decade ago.
- Principles of brain-based learning theory.
- Provision of opportunities for:
  - accurate recall, selection and use of definitions, results and procedures;
  - appropriate selection, and accurate and proficient use of procedures;
  - application of problem solving strategies allowing students to make choices, interpret, formulate, model and investigate different contextual situations;
  - critical thinking — interpreting the world mathematically and using their mathematics to form hypotheses, test their hypotheses, draw conclusions, make recommendations, make informed decisions; using strategies for analysing, proving, evaluating, explaining, inferring, justifying and generalising;
  - students to constantly move between concrete and abstract experiences within their studies to develop deep knowledge and understanding of concepts, making connections between old and new knowledge, extending familiar concepts to develop new concepts.
- Inclusiveness — catering for the diverse range of student needs, e.g. students with special learning needs, Aboriginal and Torres Strait Islander (ATSI) and English as a Second Language (ESL) groups.
- Integration of Information and Communication Technologies (ICT), namely interactive animations involving mathematical content and processes, dynamic spreadsheets and the use and interfacing of data loggers with computers.

The final result of attempting to cater to all of these requirements simultaneously was a developmental program based on problem solving with strong contextual connections to the real world. *Dirty Mathematics* was born!

Dirty Mathematics

Dirty Mathematics begins in Year 8 with the development and solution of problems being designed as a transition from primary school to secondary school and from Year 7 to Year 9. It then moves into Year 10, focusing on transitioning students into the senior school studies of mathematics. Dirty Mathematics integrates history, people, places, spies, espionage, intrigue, murder, politics, taking risks, discussion, communication, planning, designing, creativeness, satisfaction, fun, working together and taking pride in excellence. It is centred on the four generic problem solving and project frameworks, viz. *Planning*, *Designing*, *Investigating what is…* and *Evaluating the use of mathematics in…* Problem solving is both open and closed in nature. Projects are cross-curricular and are drawn from Mathematics, Science, Study of Society and Environment, Manual Arts and Human Movements.

The topic by topic approach often implemented in mathematics programs is replaced by a thematic design in which projects are supplemented by open and closed investigations, and activities are chosen to stimulate and appeal to the enquiring mind and to provoke discussion and argument among the students. This discussion engages the students and the engagement in turn encourages learning. Choices need to be made by students.

The program features more hands-on practical activity work than we previously embarked upon, and involves getting dirty with mathematics in several ways: using instruments, playing with materials, playing with and in dirt, allowing the creativity of students to come through, getting students doing and being physically engaged to have their minds participating. Students experience how their mathematics is used as a tool in investigating local and global issues

The integration of ICT has allowed for greater creative opportunities in the delivery of the content, in planning individual and group processes and in the planning and implementation of interaction with content moving students from familiar contexts to more intellectually challenging learning and assessment opportunities. ICT has promoted independent learning and collaboration, provided an authentic context for learning and sustained engagement, scaffolded students to achieve success, allowed students to assess the extent of their learning, promoted higher order thinking skills and enhanced inclusiveness. ICT has provided a stimulus for the collection and analysis of data; creation and evaluation of possible mathematical models; justification of a model; and comparison of experimental models with theoretical models. In short, it has provoked a critical evaluation of the use of technology to:

- develop higher order thinking, e.g., the “What if…?” and *cause and effect* situations, removing the need for tedious and time consuming calculations allowing the real learning journey of hypothesis formulation and testing to be undertaken;
- encourage students to manipulate information, synthesise, generalise, explain, hypothesise and arrive at conclusions.

While Dirty Mathematics focuses on initiatives characteristic of the middle years of schooling, it takes a long-term, developmental view of enabling our students to leave school with the best possible preparation for post-secondary destinations and with expanded options. We take our students, from their arrival in secondary school in Year 8 on a journey of exploration culminating in their final studies at the end of

Year 12. Table 1 lists some examples of units from this structure. (The examples marked with * are described in more detail later.)

Table 1. Examples of units in themes across the school.

		THEMES			
LEVEL		***Planning***	***Designing***	***Investigating***	***Evaluating the use of mathematics in***
Year 8	**Examples**	A holiday within Australia	A courtyard	What is the effect of a rainforest canopy on the transmittance of sunlight?	Anthropometry
Year 9		An event	Unusual houses*****	What is good water?	CSI Sandgate: Anthropometry and forensic science*****
Year 10		An overseas trip	A container	What is the best mulch?*****	Senior school mathematics options
11 Mathematics A	**Examples**		A rainwater tank for a house in Sandgate	The dating of archaeological findings*****	
11 Mathematics B		A computer virus simulation		The relationship between neck measurement and shirt size	
11 Mathematics C		Matrix codes		Koch snowflakes	
11 Pre-vocational Mathematics		Planning a local and interstate itinerary			Developing and operating a small business
12 Mathematics A		A navigation exercise			Investing in the Stock market
12 Mathematics B			Optimised containers		Charging and discharging capacitors
12 Mathematics C			Designing a parachute*****		Markov chains applied to Monopoly®
12 Pre-vocational Mathematics			Developing a herb garden	The Australian Stock Exchange	

Since 2001 our mathematics program has been providing real world learning experiences, although it was only recently that we realised that what we were doing had been given a name: *authentic learning*. Quite unwittingly, in implementing our program we have been using the powerful concepts of authentic learning with an apparent reinforcement of our approach in developing ownership. In her seminal paper, Marilyn Lombardi (2007) clarified this concept:

> Authentic learning typically focuses on real-world, complex problems and their solutions, using role-playing exercises, problem-based activities, case studies, and participation in virtual communities of practice. The learning environments are inherently

> multidisciplinary. They are not constructed in order to teach geometry or to teach philosophy. A learning environment is similar to some ‘real world’ application or discipline: managing a city, building a house, flying an airplane, setting a budget, solving a crime, for example (pp. 2–3).

Since the identification of the original problem back in 1999, staff members have constantly re-visited the design features in Dirty Mathematics, so that our approach allows students to take ownership of their mathematical problems and their solution. Indeed, in the language of the conference theme, they produce mathematics that, from their point of view, “is mine.”

Sample tasks

Our programs have included designing appropriate tasks for our students. Here is an example concerning anthropometry and forensic science for Year 9 students:

> *CSI Sandgate*
> Your task is to determine who the missing person is, based on the finding of four complete missing bones found at an unused rubbish tip. Local police have compiled a list of ten missing people to whom the remains may belong.

In this task, students are provided with four bones (*humerus*, *ulna*, *radius* and *tibia*) and a booklet containing a description of ten missing people. The accompanying booklet contains a photograph of each person together with a brief description of the features of each person. Students have to determine a strategy or strategies to identify the missing person. Students need to discuss how the “evidence” they have could be analysed to help indentify the missing person. They need to hypothesise that there may be a relationship between the length of each of these bones and the height of a person. This relationship may differ between males and females so data needs to be collected on a gender basis, graphed, analysed using regression facilities of a graphics calculator to determine possible relationships, test these relationships, use the results of each relationship to test the validity of calculations and outcomes and finally use the relationships to identify the missing person.

Samples of four other tasks we have used are described briefly in Figures 1 and 2. (These were earlier indicated by asterisks in Table 1.)

Each task is planned using a *front-ended assessment* approach. That is, the assessment is developed at the same time as decisions are made regarding the content, procedures and process skills to be developed and the learning experiences to be provided. All the required general literacies and the subject-specific literacies are explicitly listed and integrated into the learning experiences to be provided. To assist the development of higher level thinking and problem solving skills, students are provided with a range of investigations and practical work based on the theme used for the task.

Unusual houses

(*Zorro's Secret* — The missing Walt Disney episode)
(http://www.williamlthomas.com/stories/discovery.htm)

> "That night Diego and Alejandro went to bed. Bernardo decided to pack one of Zorro's costumes just in case they needed the Fox. He entered the secret passage through the front room cabinet. He packed Zorro's clothes and proceeded back down to where he entered. So preoccupied was Bernardo with the details of packing he failed to notice the secret panel still ajar…")

Your task is to design and construct an unusual house.

Why is it unusual? Like Zorro's hacienda, your house is to have secret passages for escape in times of trouble and/or the house should have secret meeting rooms with concealed entrances.

In your constructed model these should be well concealed and not easily identifiable.

What is the best mulch — Gravel, bark chip or sugar cane mulch?

Develop an experimental strategy to answer the question.

Clearly define your control for reference.

Conduct your research, gathering data which you believe will be needed to justify your conclusions.

Present your findings in a form appropriate to your nominated audience.

Figure 1. Two examples of middle school tasks.

Designing a parachute

Task A – Modelling the effect of surface area, mass.

(a) Measure and record the time taken for a sheet of poster board to drop a certain distance. Repeat the drop at least 5 times with poster board of different dimensions but of the same mass. Record surface area against time. Display your data in tabular form. Clearly identify any constants and/or variables that you believe need to be taken into account.

(b) Develop a model appropriate to the data stating any assumptions that you may have made.

(c) Discuss the strengths and limitations of your model and the effects of any assumptions that you may have made.

Task B – Edward the sky-diving egg

Synthesise a strategy to design a parachute system that will land a raw egg safely (unbroken) onto the bitumen from a height of 15m. Validate your design mathematically clearly indicating the effect of any assumptions made and yes, validate as well with a nice drop.

The dating of archaeological findings

Data have been collected on four different skull measurements from Egyptian males living in the area of the Great Pyramids from two different time periods: 4000 BC and 150 AD. The four skull measurements (mm) have been abbreviated as follows:

MB: maximum breadth across the skull;

BH: basibregmatic height—from the base of the jaw to the top of the skull;

BL: basialveolar length—from the back of the skull to the front;

NH: nasal height—from the jaw to the base of the nose.

The data attached to this task are also available electronically for analysis from Q:\Students\Mathematics\11 Maths A\Skulls Data.xls.

Have the dimensions of skulls altered significantly over the time period 4000 BC to today?

Figure 2. Two examples of senior school tasks.

Resourcing

Resource requirements for each project are identified when the tasks are designed. If current resources are not adequate, the necessary materials are purchased or are made. Our teachers themselves are also valuable resources. To utilise their experience, ideas and creativity, Year Coordinators organise fortnightly or monthly meetings. At these meetings, teachers discuss the learning experiences that need to be provided to meet the range of student needs, provide in-service support for new staff, identify professional development needs and further requirements needed to successfully deliver our program, provide feedback to the staff for full staff discussion and make recommendations for improvements to achieve further success.

Measuring our short-term, mid-term and long-term success

Overall, the results achieved through Dirty Mathematics have been very successful. Records have been kept over a number of years to evaluate the success, or otherwise, of the program. There are many statistics available, but not space in this paper to describe or analyses these in full. Table 2 summarises some of these data, and is followed by a brief description that elaborates them.

Table 2. Summary of results over recent years.

Short-term	Improved performance of students by the end of Year 8 when compared to all baseline data measures.
	Improved performance of students identified with learning difficulties or physical impairment when compared to all baseline data.
	Improved performance of identified ATSI students in Year 8 compared with baseline data and continued improvement.
Mid-term	Improved performance of students identified in the Year 7 test as being 'under the national benchmark'.
	Continued improvement in performance of students including improved performance of the majority of students from Year 7 test to Year 9 test.
Long-term	Improvement and continued improvement in earlier cohorts – historical Year 12 exit results.
	Increased engagement of students in their learning and increased student control over their learning.
	Re-engaged staff and re-vitalised professional conversations.

In the short term, we compare each student's semester report grade against previous report grades and against two baseline measures: the state-based Year 7 Numeracy Test results and the results of the Australian Council for Educational Research's Progressive Achievement Tests in Mathematics (PAT Maths), which is administered to all our students at the start of Year 8. Over the long term, we compare students' exit results against district and state data as available through Queensland's Corporate Data Warehouse. We now have another source of data to inform our decision-making, the National Assessment Program — Literacy and Numeracy (NAPLAN) results for numeracy. In comparing data, a common scale of A, B, C, D, E has been used; a C grade is considered a passing grade on this scale.

Short-term success

At the start of Year 8 in 2006, 2007 and 2008, baseline data indicated that 54.7%, 53.5% and 42.9% of students respectively rated a C grade or higher on the Year 7 Numeracy Test. The PAT Maths scores indicated 35.9%, 38.1% and 16.7% of students were rated C or higher in 2006, 2007 and 2008 respectively. By the end of Year 8, having completed the first year of Dirty Mathematics, 80.4% of students in 2006 ($n = 271$), 90.3% in 2007 ($n = 273$) and 82.5% in 2008 ($n = 192$) gained a C grade or higher. Similarly increases in performances were noted for students identified with learning difficulties or physical impairment when compared to all baseline data and ATSI students.

Mid-term success

The Year 8 cohort in 2006 (Year 9 in 2007) showed an increase from Year 8 to Year 9 in the percentage of students achieving a passing grade. These results improved from 80.4% in Year 8 ($n = 271$) to 83.7% in Year 9 ($n = 268$). The Year 8 cohort in 2007 (Year 9 in 2008) also showed an increase in performance from Year 8 to Year 9 with the percentage of students with a passing grade improving from 90.3% in Year 8 ($n = 254$) to 92.5% in Year 9 ($n = 266$). These increases also reflect improved performance by students identified in the respective Year 7 Test as being “under the national benchmark.”

The NAPLAN numeracy results For Year 9 2008 indicate a similar improvement. The percentage of students at or above Band 6 for our school was 96.7%, comparing favourably with 90.0% for the State and 93.6% nationally. These results also show that of the 39 students at Sandgate SHS who were placed below the national benchmark in Year 7, only two students were placed in Band 5 or less with NAPLAN.

Long-term success

Our long-term success has been measured by analysing the Year 12 exit results for students who entered Year 8 in 1999 (the last year of the “old” approach), 2000 (the first Dirty Mathematics group), 2001, 2002 and 2003 (successive Dirty Mathematics groups). These students completed Year 12 respectively in 2003, 2004, 2005, 2006, 2007. These exit results are validated through the Queensland Studies Authority’s quality assurance procedures and with school performances compared through Education Queensland’s Corporate Data Warehouse figures. Over the interval 2003 to 2007, the Corporate Data Warehouse information shows a continuous improvement in the performance of our students. From being positioned in the lower to middle state band when compared to schools from across Queensland having a similar client characteristics, our students, after participating in Dirty Mathematics, now exit Year 12 consistently in the lower to middle “Higher than” state band.

Increased engagement of students in their learning and increased student control over their learning

Students now often ask permission to undertake extra investigation. All encouragement is given. For example, when doing the rainforest light study, a group of students asked if they could also examine the intensity of light reaching the floor of the forest over a certain area. “Go for it,” was the reply. This reflects an increase in confidence in both their abilities and in risk-taking.

Re-engaged staff and re-vitalised professional conversations

More staffroom and faculty conversations are centred on ideas for other themes or ways of extending or making the current themes more engaging and challenging. Teachers have become more willing and indeed eager to readily talk about pedagogical practices, assessment procedures, or the best way of approaching a topic. The former professional barriers of not wishing to reveal what they have been doing have been broken down. Discussions are now more robust with more teachers regularly engaging in professional dialogue. These observations provide informal evidence of the success of the program from a teacher point of view and ultimately benefit the quality of learning experience provided to the students.

Conclusion

Making the mathematics of the official syllabus into “Mathematics: it’s mine” has provided an environment of ownership and improvement for both students and teachers. We feel that over the past decade, the Dirty Mathematics program has changed the culture of mathematics in the school, and resulted in improved student and staff learning. Getting our hands dirty with mathematics has improved what we do in the short term, but has also resulted in longer-term sustained improvement, with better connections between our middle schooling and senior schooling efforts. Buoyed by the success of our work, with its focus on authenticity throughout the secondary years, we intend to continue working in this direction.

Note

The successes of Dirty Mathematics were recognised with a National Numeracy Award in 2003 and more recently in 2008 with a State Showcase Award for Excellence in the Middle School.

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GREAT MATHEMATICS CLASSROOMS: IT'S SIMPLY MATH-TASTIC!

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What makes a great classroom? What are the features and foci that great mathematics teachers apply to engage students and develop critical thinking? In this paper, we explore a mind map developed by the author that includes a number of elements that feature in any great mathematics classroom. These include: (i) meaningful learning through connections, (ii) a focus on mathematical reasoning, (iii) novelty and variety of lessons, (iv) technology integrated into the classroom, and (v) motivation through differentiation. How many of these elements are featured in your classroom?

I have been fascinated for years by the common response that occurs when I announce that I teach mathematics. More often than not, the response is one of shock, then negativity. It is particularly evident that many do not remember their mathematics with enjoyment. When I question people about their experiences in secondary mathematics classrooms, I am often regaled with tales of boredom, tedium, textbook driven lessons and a complete lack of connection between the learning and the real world. These responses have driven my determination to provide a mathematics classroom that is relevant to my students' world and challenges their thinking and application of their learning.

So just what makes a great mathematics classroom different to others? What are the features of a great mathematics classroom? Why do students learn and engage enthusiastically in great mathematics classrooms? How does deep understanding and complex problem solving develop in great mathematics classrooms?

The answers lie with many of our wonderful and engaging mathematics teachers. Their intuitive approach and clever pedagogy empowers their students to not only learn mathematics, but efficiently analyse, apply and create with their mathematical knowledge. The mind map in Figure 1 presents the author's view of some of the main features of a great mathematics classroom in the 21st century. Using the mind map as a scaffold, we will focus on each element in more detail through the paper.

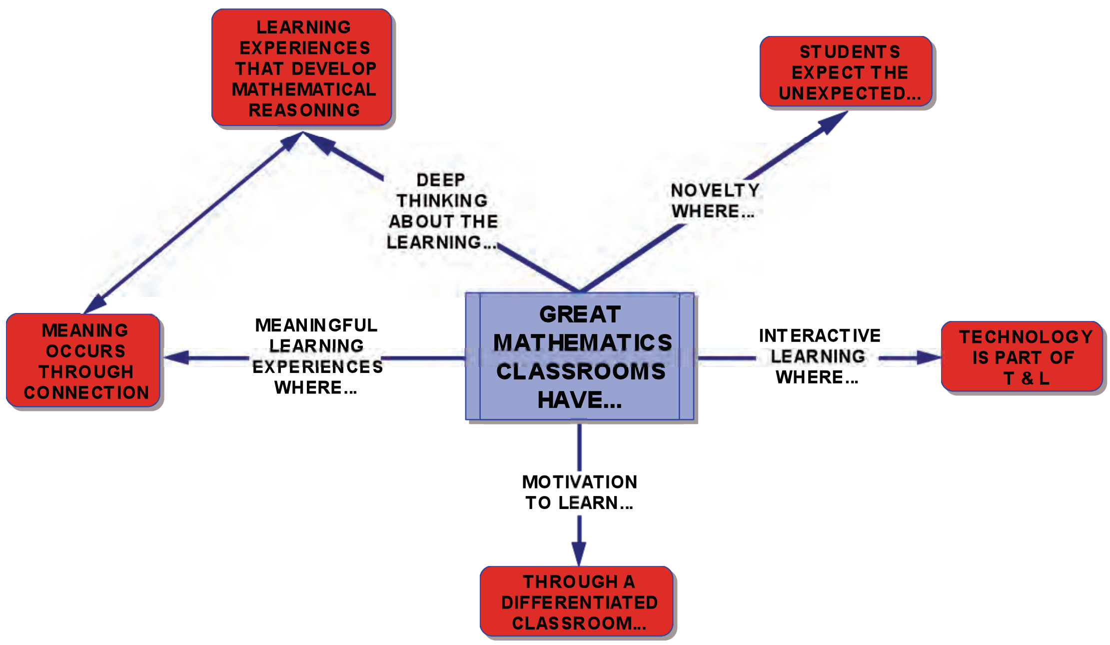

Figure 1. Mind map of great mathematics classrooms.

Meaningful learning experiences where meaning occurs through connection

As depicted in Figure 2, meaningful learning experiences are those for which there are connections for the students.

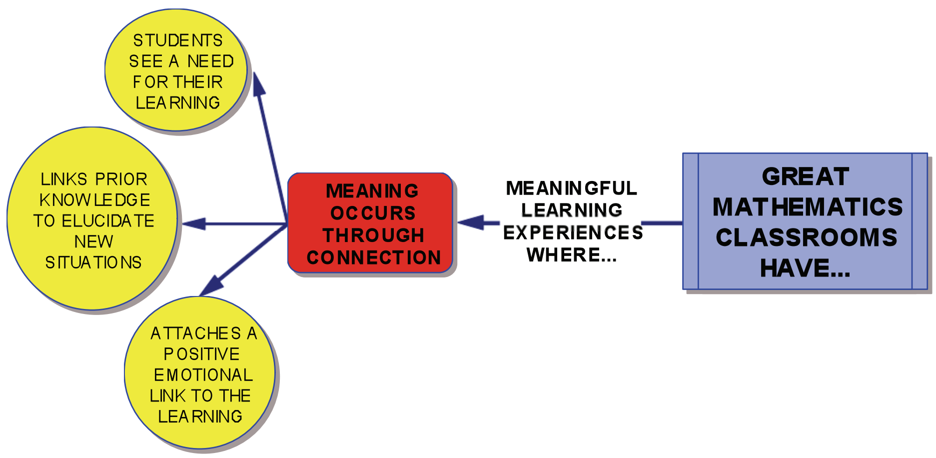

Figure 2. Meaning occurs through connection.

Attaching a positive emotional link to a student's learning

Attaching a positive emotional link to students' learning instantly gains attention. Beginning a lesson with, "Today we are going to start on algebra" will not evoke a positive emotion. In fact, it will probably create the converse due to the negative connotations that surround the term "algebra." One only needs to watch a classroom scene from *Home and Away* or any other sit-com on television. The lesson will invariably be a mathematics lesson, complete with a middle aged and very cynical male teacher, desks in rows, everyone looking extremely bored, behaviour issues everywhere and the topic for the lesson being algebra! An alternative and powerful approach would be to present a computer game and then follow this with the program for the game by asking "Can you have the computer game without the algebra?"

Linking prior knowledge to elucidate new situations

According to Maguire, Frith and Morris (1999, p. 1842), research suggests that "when new learning is readily comprehensible (sense) and can be connected to past experiences (meaning), there is substantially more cerebral activity followed by dramatically improved retention."

The teacher becomes the means by which the student can find the link between new and prior learning. When this link is created, the learning has meaning and may be retained. Episodic learning is reduced through the explicit consideration of past knowledge and experiences when confronted with a new situation or concept. Students then build on their knowledge store rather than simply accumulating information with little connection or relevance.

Students see a need for their learning

It seems that the brain's working memory asks two questions to determine if the information will be stored or saved. These are: "Does this make sense?" and "Does it have meaning?" (Sousa, 2008, p. 55). The need for relevance is greatly influenced by the student's experiences. This need will also impact on not only the acquisition of the information but also the application of the information. The teacher must be able to explain how the information will be meaningful to students to enable the learning process to begin. This can be easily developed at the beginning of any mathematical topic. Brainstorming with the class to list different occupations that would have a need or use for this particular type of mathematics develops a need and appreciation.

Learning experiences that develop mathematical reasoning

Figure 3 depicts the importance of mathematical reasoning in great classrooms.

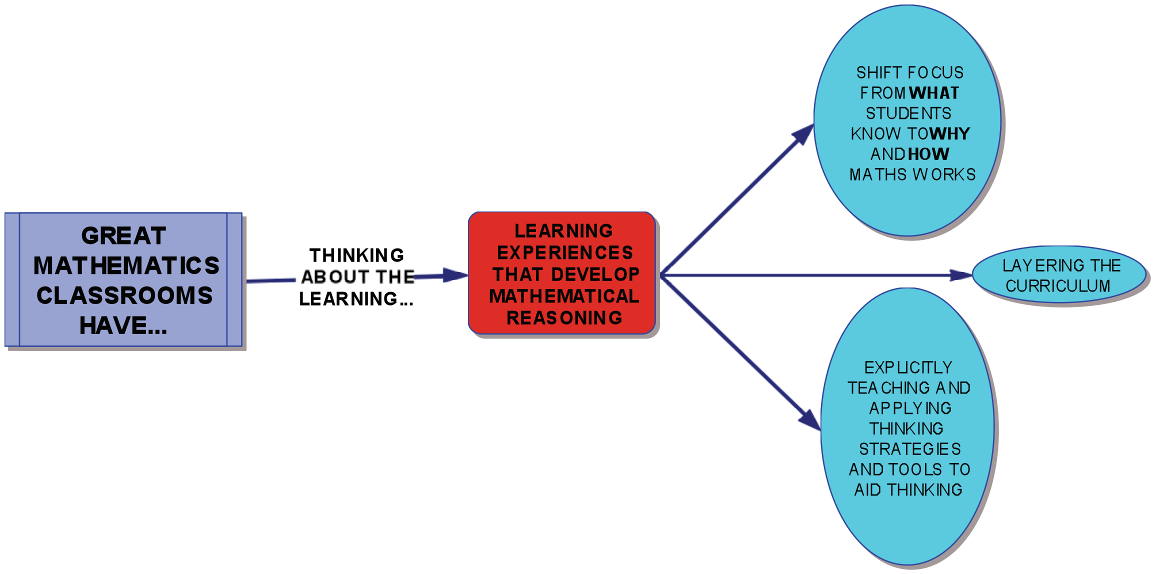

Figure 3. Three kinds of learning experiences for developing mathematical reasoning.

Shifting the focus from *what* students know, to *why* and *how* mathematics works

With the pressure of league tables, content driven curriculum and sometimes even a lack of deep understanding of a topic, teachers can resort to a teaching process that only values skills and knowledge — *what* students know and are able to do. Rote learning becomes the common learning process and superficial understanding results. The unfounded belief that time spent on reasoning and deep understanding is a waste of valuable lesson time adds to a shallow learning experience. The converse is a powerful learning process that actually promotes deep understanding and competence through the demand of *why* and *how* the mathematics works.

The teacher needs to continually weave into the lesson not only the mathematical concept involved but also metacognition — continually thinking about *what* you are doing, *why* you are doing it, *how* you are doing it and reflecting on the efficiency of the process.

Layering the curriculum to encourage higher order thinking

Nunley (2009) has developed a student-centred learning model which is based on neuroscience. Her model, consisting of three layers, enhances motivation and promotes higher order thinking. The model encourages complex learning by dividing the learning into three layers: (i) rote learning, (ii) connecting the new learning to prior knowledge through application, and (iii) critically thinking about their learning in a real life situation.

All students will benefit from the rote learning at the introduction of a skill or concept. The value will come from the teacher's knowledge of how much practice each student needs to have mastery. Moving to each layer at the relevant learning point will maintain student interest and motivation.

Explicitly teaching and applying thinking strategies and tools to aid thinking

Costa & Kallick (2009) have argued for the importance of *Habits of Mind* in constructing thoughtful classroom, business and home environments. As Anderson (2004, p. 2) notes, from a similar perspective,

> We must never abandon our content for the sake of thinking. We must have something to think about… We use a variety of thinking skills, tools and strategies to <u>act on</u> our content… Much of the literature available today tells us that these skills, tools and strategies need to be taught explicitly to students. This does not mean to the exclusion of the content, but rather taught deliberately and then applied to the content.

A great mathematics classroom infuses the learning with thinking strategies that enhance higher order thinking. Explicitly teaching thinking tools and strategies and using them as scaffolds allows students to explore their learning confidently and efficiently. It also supports the notion of metacognition in the classroom.

Novelty — where students expect the unexpected!

The brain is attracted to novelty. It searches for and responds to novelty and during the adolescent years the search for novelty becomes quite intense. Ultimately, in the classroom it is the teacher's decision whether the lesson has novelty. According to Sousa (2008), lessons that contain repetition or a predictable teaching style lower students' interest and the brain tempts itself to search for other novel experiences. Figure 4 suggests that mathematics classrooms that provide a variety of teaching and learning experiences are more likely to promote novelty and maintain student interest.

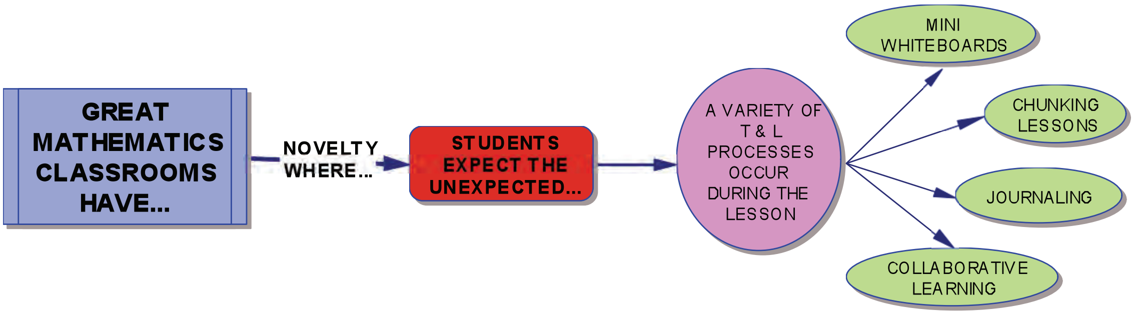

Figure 4. Some ways of incorporating novelty into lessons.

The variety of teaching and learning processes is only limited by our creativity and imagination. Figure 4 shows four examples, although many others are possible. Writing journals can help students develop their literacy skills while reflecting on their learning. Mini whiteboards are a novel but powerful tool for formative and summative assessment. Chunking the lesson considers the capacity of working memory so that a well-planned lesson would include a flow of short learning experiences followed by revision of the information learned. Structuring opportunities for collaborative learning allows students to learn with and from each other.

Interactive learning where technology is part of the teaching and learning

Over recent years, technology has provided many opportunities for the development of new kinds of interactive learning. Figure 5 depicts some of these.

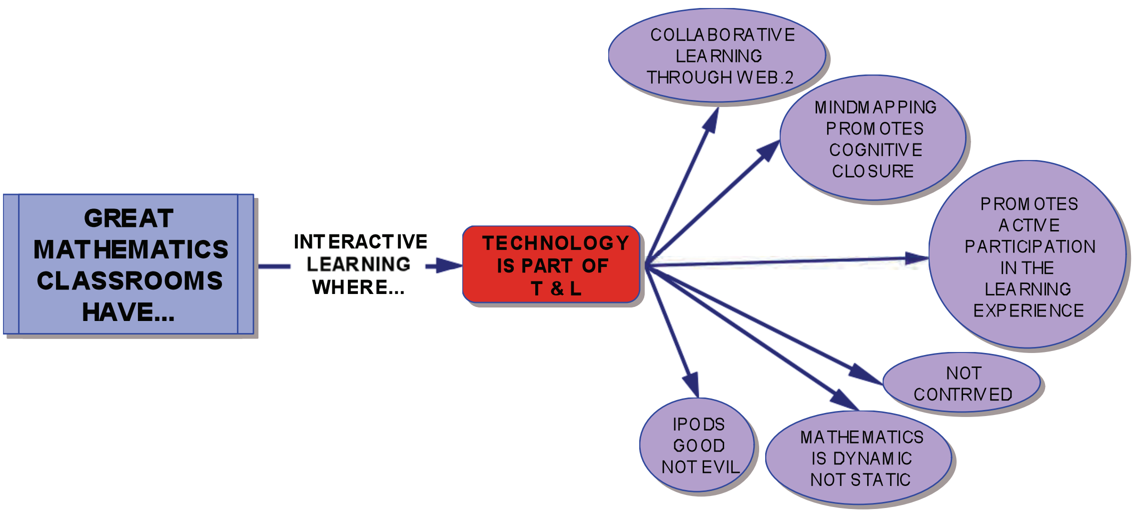

Figure 5. Technology is part of teaching and learning.

Collaborative learning through Web 2.0 technologies

Students these days expect to be able to actively participate in their learning. They also need to learn collaborative skills and Web 2.0 technologies promote this style of learning. Wikis are just one example of Web 2.0 technology that develops collaborative skills through discussion tabs and the opportunity for all members to be able to post, edit and change information. (Readers unfamiliar with these second generation uses of the Internet, also referred to collectively as Web 2.0, are referred to GoToWeb20 (2009), where there are many examples described.)

Mind mapping and cognitive closure

The end of a lesson is often a rushed affair with little consideration for what has been learned. If the end of a lesson became a cognitive experience where students focus on the learning and its meaning, the possibility of the learning being retained is increased (Sousa, 2008).

One approach to cognitive closure is the use of a mind mapping program with an interactive whiteboard. Students can add to the mind map at the end of each lesson. Working together to create a mind map allows students to discuss and make meaning of their learning, promotes collaboration and ultimately gives each student a mind map of the topic for study purposes.

Promotes active participation in the learning experience

Marc Prensky calls our students “digital natives” — multimedia is part of their lives and with the volume of interaction that they have with technology, he claims that our students “think and process information fundamentally differently from their predecessors … their thinking patterns have changed” (Prensky, 2001, p. 2). Indeed, one of the greatest dilemmas facing teachers today is that we speak an outdated language in comparison to our digital students.

The interactive world our students inhabit through technology has also made a change to their style of learning — they expect to participate in their learning. They find it difficult to engage in lessons that are teacher-directed, as they do not experience this in their world. They seek opportunities for active participation and interactive whiteboards can easily provide this, but only if the teacher makes the board interactive, and the learning is dynamic and collaborative. The use of investigations and open-ended questions can encourage active participation through a student-centred environment.

Technology is not contrived

The use of technology must be imbedded in the lesson to make it an authentic learning experience. For meaningful learning to take place, the application of technology must be genuine. Technology should be used when its application enhances the teaching and learning; the contrived application of technology will actually take away from the meaning of a lesson as the students focus on the technology and not on the learning.

Mathematics is dynamic, not static

It must be a treat for teachers to be able to use dynamic geometry software to present the measurement of angles as the measure of turn. This can help ensure that students recognise plane figures in all orientations by simply rotating the shape on an interactive whiteboard. Also, students can investigate the concept of integration as the area under a curve and actually demonstrate this through the continual creation of smaller and smaller rectangles. Technology has given teachers the opportunity to present mathematics in a dynamic and visual form. Precision is now available to all and students can create, investigate and test hypotheses through the availability of powerful programs and the interactive whiteboard.

iPods — for good, not evil!

Today's students are rarely seen without something attached to their ears. The iPod is an invaluable way of giving feedback. Teachers can record the feedback, save it as an MP3 file and students can upload the file to their iPods. Students can then access the feedback when they need it. Students delight in the fact that they have audio access to feedback at any time. Informal research from the author's students indicates that they actually do listen to the recordings where before they would often not bother to read the feedback as it was too much effort.

Motivation to learn through a differentiated classroom

Great mathematics classrooms do not assume that all students are the same, as Figure 6 suggests.

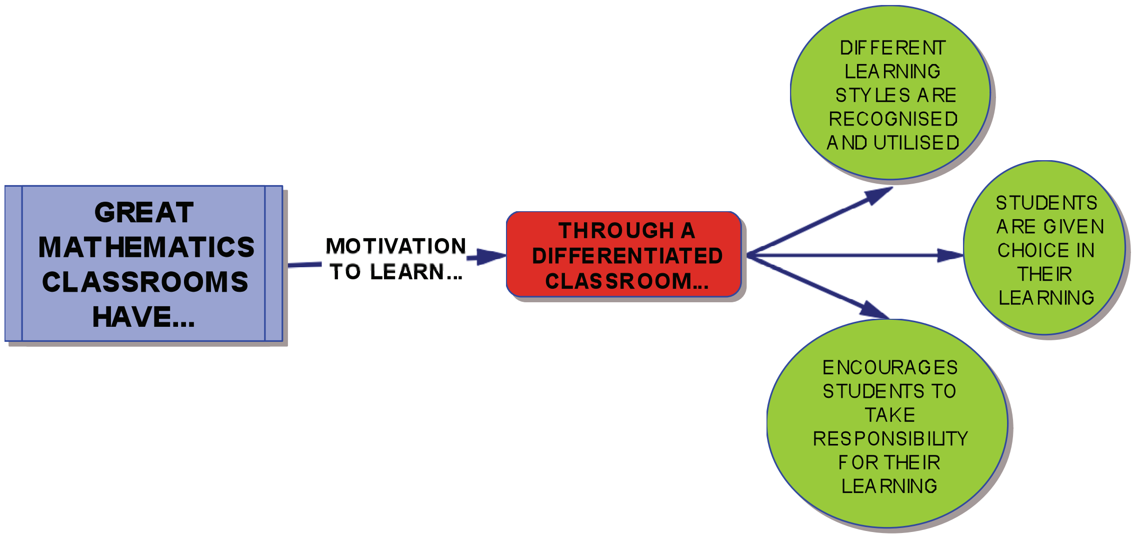

Figure 6. Motivation to learn through a differentiated classroom.

Different learning styles are recognised and utilised

Research suggests that students are far more likely to be successful in learning in mathematics if teachers are aware of their students' learning style preferences and use strategies that are compatible with these preferences. (e.g., Sousa, 2008) Learning preference audits completed by a class can provide great insight for lesson preparation. Students often appreciate the identification of their preferred learning style as many struggle to understand why they cannot learn in the same way as some of their peers. Teachers can also benefit by recognising their own preferred learning style as this often dominates their own teaching. Considering the various approaches students take in their learning and adapting the teaching and learning strategies accordingly will enhance the learning environment.

Students are given choice and take responsibility in their learning

The opportunity to experience choice in the learning process transforms any classroom. Choice is the pinnacle of student-centred learning, providing motivation and attention. Ultimately choice produces students who are accountable for their learning. Unfortunately, some mathematics teachers see the subject itself as being too rigid and sequential to allow choice. While the content may not always allow choice, however, the learning process does. Providing a selection of activities that will appeal to the various learning preferences but still focus on the same outcomes, skills or processes will augment the learning and promote an inclusive classroom. Teachers can split a set of exercise questions in a textbook into three levels and allow students to select the level that best suits their needs (though in the author's experience they tend to attempt the hardest level given such an opportunity!) The use of open-ended questions allows students the freedom to extend and create knowledge. Also, it is useful to present the work in a capacity matrix at the beginning of the topic so students are conscious of the bigger picture of the whole topic and the sequence of the concepts to create it. Students can easily see then what they need to know and how they can get there.

Summing up

Great mathematics classrooms do not just happen but rather are formed through the creativity, dedication and excellent planning by the teacher. The mathematical concept is not just the focus in these classrooms. Instead, there is a holistic approach that values all members, recognises and celebrates their differences, promotes thinking skills and tools, and creates a need for the learning of the concept. A rich and authentic learning environment provides active participation by using technology that connects and enriches the experience. Ultimately it is these classrooms that students want to be in — great mathematics classrooms!

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WHAT'S THE PROBLEM?

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Research conducted over the last two decades indicates that traditional pen and paper methods of teaching mathematics do not work particularly well for a large number of students in classrooms, yet teachers persist with these methods. Based on research conducted by the Project Zero team, this paper describes the use of open-ended problem solving to develop students' mathematical knowledge, skills, understanding and thinking as well as allowing all students to feel valued and successful. By using specific strategies and routines, the classroom becomes a whole community of learners where everyone, including the teacher, is engaged in thinking and learning.

Introduction

It is widely accepted that there is a difference between mathematics and numeracy, and that society wants students to be numerate as well as having mathematical knowledge and skills. This view of numeracy is summarised by The Australian Association of Mathematics Teachers' (AAMT) definition: "To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life. In school education, numeracy is a fundamental component of learning discourse and critique across all areas of the curriculum" (AAMT, 1998). How can this be achieved? The weekly tables test, the skills and drills with mathematical processes, the ubiquitous worksheets, a plethora of concrete materials, meticulous lesson preparation, detailed evaluation and assessment? Still there are a large number of students who struggle to make sense of the mathematics they have been taught. These are the students who seemed to "get it" yesterday but by today they have apparently forgotten it.

Perkins (1992) suggested that traditional teaching practices are back to front. He demonstrated that instead of teaching the knowledge and skills first then challenging students to demonstrate their understanding teachers should be providing opportunities for students to think, question, reflect, and explore concepts and then teach the knowledge and skills as and when they are needed. "Learning is a consequence of thinking. Retention, understanding and the active use of knowledge can be brought about only by learning experiences in which learners think about and think with what they are learning" (p. 8). This view is supported by the findings of Askew, Brown, Rhodes, Wiliam and Johnson (1997) who found that the most effective teachers of numeracy were connectionist teachers who did not see it as a necessary prerequisite that pupils should have learnt a skill in advance of being able to apply it. Indeed the challenge of an application could result in learning. The Department of Education, Tasmania (DoET) embraces the importance of thinking in the teaching and learning of mathematics-numeracy:

> Thinking in mathematics-numeracy involves problem solving, communicating and reasoning. Students use and apply mathematical skills and concepts in familiar and unfamiliar situations, test, generate proofs and hypotheses and examine them for validity and accuracy. Positive attitudes and dispositions towards mathematics learning and active engagement with mathematical tasks are integral to thinking, acting and working mathematically (DoET. 2007).

Ritchhart (1994) elaborated this when he claimed that:

> When students are given the opportunity to pursue their unique problems and solutions, your role and expectations as a teacher must shift. No longer a dispenser of facts and the ultimate authority, you must model posing problems and asking questions that engage students in thinking *mathematically* rather than recalling answers. You must begin to pose problems that will engage not only your students, but you as well. When the classroom becomes a community of learners, a climate of trust and inquiry is formed. The use of open-ended questioning, the positive recognition of efforts, and a focus on the thinking process rather than on final answers help facilitate the creation of such a climate (p. 12).

Finding the problems

So where are the problems that will challenge and engage students; problems that have a context in the real world, that are meaningful to the lives of students? There are plenty of text books and teacher resource materials available filled with mathematics problems but as noted by Kemp and Hogan (2000) many of these “real world problems” tend to be contrived, often irrelevant and somewhat meaningless within the context of the lives students lead.

There were two events that occurred at about the same time that provided the necessary resources. In 2006 the Australian Government Quality Teacher Program (AGQTP) offered a professional learning course on *Making Thinking Visible* run by Ron Ritchhart. He introduced his thinking routines; a series of routines designed to become embedded in daily teaching practice. They have few steps, are easy to teach, easy to learn, and allow students to unlock their thinking on a topic or concept and make their thinking visible to others. They provide a firm scaffold on which students can build their thinking and are highly effective in all areas of the curriculum. The second event involved a DVD and an electricity bill.

The DVD is called *Numb3rs* (CBS Paramount, 2006) and involved two brothers, one an FBI agent and the other a professor of mathematics. In each episode they work together to solve crimes using various aspects of mathematics and mathematical modelling to reach a solution. The opening title of each episode states: “We all use math every day: to forecast weather, to tell the time, to handle life. We also use math to analyse crime, reveal patterns, predict behaviour; using numbers we can solve the biggest mysteries we know” (CBS Paramount, 2006).

It was so obvious really: things we do every day are filled with mathematics and numeracy problems? Our lives can be viewed as a series of problems from every strand of the mathematics curriculum: time, distance, spatial awareness, temperature, chance, data, and quantities. This was reinforced when my son asked me to help solve a problem. He was living in a flat with two friends and they had received their first electricity bill for $287. Jake had one of those cards that you can use to pay towards your bill as you can afford it and had already paid $40 on his card. His flat mates had not yet paid anything towards the bill. However he was struggling to work out how much each of them owed. Jake's electricity bill became the first real life problem I used in the classroom. I took the bill into the classroom and using Ritchhart's *Think, Puzzle, Explore* routine asked my students to help find a solution.

The level of engagement, discussion, questioning, justifying and debating was remarkable. The students were keen to share their own family's bill-paying stories and were excited at the prospect of being able to solve my 18 year old son's dilemma. Various solutions were reached, shared, discussed, debated and discarded. From that point on a large part of my numeracy program has consisted of problems that have arisen in my own life. My students have helped me budget for dental treatment, devise reasonable time limits on my son's computer gaming, fence the vegetable garden, work out the best deals for travelling interstate, decide how much food to buy for a birthday barbecue and approximate how many books my daughter can buy in Melbourne before going over her baggage limit. The students have also started to bring their own problems from home to be explored by the whole class. They have investigated problems involving pocket money, sharing bedroom space, stacking wood, cricket scores and mixing lawn mower fuel to name just a few.

An issue of fairness and a parental dilemma

One of the problems that resulted in an astonishing amount of discussion and debate, as well as providing a basis for many and varied independent inquiries, is shown below:

> *A fair amount of time*
> Sam is in Year 11 and has a fair amount of homework each week. He plays soccer and volleyball. He loves to play an online computer game in his spare time.
> We are having frequent disagreements and arguments about how much time he should be allowed to play his computer game.
> Can you help me to work out how often and for how long should he be allowed to play his computer game so that he is still able to fulfil his other responsibilities effectively?

Using the *Think, Puzzle, Explore* routine the students recorded what they thought about this problem, what strategies they thought they might use to solve it. They then recorded the puzzles or questions they had about this problem and finally they noted things about the problem which they felt needed to be explored further. Most students recognised the problem as a variation of a common household issue and had plenty of stories of their own to share regarding arguments about homework. There was also some interesting dialogue around the concept of "responsibilities" and what that meant for a parent as well as a child. Most students decided that they would need to draw up a timetable in order to reach a solution. All the students demanded clarifying information in the form of puzzles and questions such as:

- "How much homework is a 'fair amount'?"
- "How often does Sam have to go to soccer and volleyball training each week?"
- "How long does it take to complete the computer game?"
- "How much of his weekend is taken up with sport?"
- "What time does he have to go to bed?"
- "What time does he get up in the morning?"
- "Does he go to school all day every school day or does he have some free time?"
- "What time do the training sessions start and finish?"
- "Is he allowed to eat his tea while he plays computer?"
- "Is Sam quick at doing homework or does he take ages to finish an assignment?"

Once I had provided the extra information the students wanted, they worked in pairs on the solution. Each pair produced a timetable of Sam's week, highlighting the times when they felt it would be reasonable for him to play his computer game. All the timetables were different depending on the value placed on homework, sport and computer gaming. As each pair shared their proposed solution heated debate ensued and there was much disagreement with individuals having to justify their stance on this issue. In order to support their proposals a number of students asked to explore various aspects of the problem in greater depth. One pair wanted to find out about optimum fitness levels for a variety of sports and how to achieve them. A number of students wanted to explore the research on healthy sleep requirements in children, adolescents and adults. A third group were interested in the effects of computer gaming. We worked on this problem for over a week.

There was a phenomenal amount of mathematics being used including addition, multiplication and subtraction processes, estimation and calculation with time and timetabling. The students also learned a valuable lesson about problem solving: that there can be many solutions to the same problem and these are dependent on a number of variables. In fact sometimes it is impossible to arrive at a solution at all due to the complexity of the issues involved. The students also explored the complex issues of fairness, responsibility and viewpoint, and there was a significant amount of literacy and health and well-being involved in their independent inquiries.

Conclusion

Establishing a culture of thinking in the classroom encourages originality and creativity, allows all students to "have a go" and removes the fear of "being wrong" as so many issues and problems do not have a specific solution. Ritchart (1994) has provided a powerful rationale for the use of investigations for this purpose:

> Investigations should be a regular and integral part of a mathematics curriculum that seeks to develop numeracy. They provide firsthand experiences with numbers in a meaning-rich environment. Drill and practice have their place in solidifying procedures after concepts are understood, but they should not characterize students' mathematical experiences. Closure is an important component of any lesson. Students must be given the opportunity to share their discoveries and deal directly and honestly with challenges from others. By articulating what they have learned, students link their understanding to previous knowledge (p. 13).

The students develop the confidence to share their mathematical thinking, to be challenged by their peers, and to justify their solutions. The classroom becomes a vibrant noisy place filled with discussion and debate. The students learn to value their own thinking as well as that of their peers and learn to listen to and accept alternative viewpoints. The students have time to explore concepts, to ask questions, to trial strategies and test hypotheses. They are engaged, excited learners who want to find answers and to explore new ideas. They are allowed, even encouraged, to make mistakes, as this leads to more questions and deeper exploration.

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THE LINK BETWEEN PLANNING AND TEACHING MATHEMATICS: AN EXPLORATION IN AN INDIGENOUS COMMUNITY SCHOOL

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This workshop reports on a teaching exploration at an Indigenous Community School in the Kimberley region of Western Australia that sought to use a specific way of thinking about particular content domain, partitioning, to develop focused, mathematically rich learning experiences. A set of four 90 minute lessons was planned and taught by Rebecca, building on activities suggested by Rebecca and Peter, and incorporating the pedagogical approach being advocated by the project. The successes and challenges are reported below.

Introduction

This is a report of a teaching investigation at an Indigenous Community School in the Kimberley region of Western Australia. Peter Sullivan was at the school as part of the *Maths in the Kimberleys* project, which is led by Robyn Jorgensen from Griffith University, and is an Australian Research Council project in partnership with the Association of Independent Schools of Western Australia.

A set of four 90 minute lessons was planned and taught by Rebecca Youdale, building on activities suggested by Rebecca and Peter Sullivan, and incorporating the pedagogical approach being advocated by the project. There was an interview assessment of student learning after each lesson. A report of an earlier set of lessons with a focus of the topic of subtraction is also available (see Sullivan, Youdale & Jorgensen, under review).

The purpose of the visit was to spend time in classrooms investigating which aspects of the pedagogical model were able to be implemented readily, and which were proving challenging. A subsidiary goal was to explore the mathematical strengths of the students, and any challenges in implementing a planned teaching sequence addressing a specific aspect of mathematical content.

Rebecca suggested partitioning numbers as the focus of the teaching. Rebecca had also suggested that the students seemed to be adept at money. In response Peter suggested the following as the focus ideas, with each key idea growing out of the money experience, and some suggested activities many of which are presented below: patterns in numbers to 10, 100, etc.; breaking numbers into parts; building up numbers.

In an email response, Rebecca replied:

> Your assumption about the purpose of us thinking about partitioning is spot on. I'd especially like to develop the understanding that 19-16 is the same as 19-3. The Grade 4s have lots of experience with standard place value decomposition in 3 and 4 digit numbers and also partitioning small numbers (up to about 15). I'd say their understanding and skill ranges from sound to excellent.
> The Grade 3s have just started to learn about place value in 2 digit numbers and are consolidating their "friends of 10." Some have caught on very quickly, and are demonstrating a keenness and good understanding of number patterns and small number partitions.

In the earlier report, one of the arguments made was that teaching has more chance of success if the focus is clear. It seemed that Rebecca had a clear idea of the key stages in an important topic. Recognising that an explicit intention was to build on what the students know and could do, the overall goal was agreed as being to lead students towards calculating change, and to partition numbers as a way of manipulating numbers easily.

Activity report

The following presents a brief indication of the activities that formed part of the lesson sequence. Each activity is described, and there is a reflection of the activity presented as well.

Day 1: Patterns in numbers to 10 and 100, emphasising money

The goal of this day was to establish building to 10 and 100.

Subitising with money to $10 (and later to $1)

Using various combinations of $1, $2, and $5, amount were shown for a short time and then covered. Students first whispered their answer to the person sitting next to them and then declared their answer.

> The students seemed to be extraordinarily adept at doing this accurately. Given that this involves a number of key skills in combining and partitioning numbers, it created the sense of the strong foundation on which the lesson sequence could build.

This was repeated using 10 and 20-cent pieces first, and later adding in 50-cents as well.

> The students also seemed very good with this. There were some students who could recognise $1 but could not say how many cents there are in a dollar.

Race to $10

There is a well-known game where students, starting at 0, in turn add 1 or 2, and the one who makes the total 10 is the winner. There is a winning strategy. The students played a variation, where they could add either $1 or $2 to reach the target of $10

> The students were able to play the game easily. As a review of the activity, Rebecca played this for some time with individuals to see whether they would see the pattern, and recognise a winning strategy. Only one student did, and he was asked to report on this, but there was an emerging awareness in the others.

The reporting phase raised an interesting challenge. The student who identified the winning strategy was asked to explain his reasoning. There are many advantages in students being encouraged to explain their reasoning. There seems to be a difficulty, though, in using such explanations as part of the learning of others. Not only are such explanations not instructive, but the other students seemed uninterested, thereby creating potential management problems.

Race to $1

The students then played a further variation to the game, this time adding on either 10c or 20c. The person to make the total $1 was the winner.

> This also worked well, in that the students engaged in playing the game and followed the rules, thereby (theoretically) getting valuable practice at building up to $1. In retrospect, it would have been useful to play the game the other way, starting at $1, and racing to 0 in that it would show the breaking up aspect of the $1, which is part of the challenge in calculating change, and especially the aspect counting back 90, 80, 70, etc.

How many ways to make $10

The task was posed, using $1, $2, and $5, in Rebecca's words, "Your job is to work out as many ways as you can to make $10".

> The students wrote their answers on a small whiteboard. The students seemed to understand the task and worked productively, with many students producing multiple correct answers.
> This seems to an example of the type of task that can be successful. It is complex enough to allow for multiple answers, some reasoning and problem solving is required, and it is practising a core skill toward the goal, that of ways of building to 10.

It was, though, another example where the review of the discussion was problematic. The type of question, "how did you get that" is too abstract, at least for students at this level. One possibility could be to ask students to describe one answer, then another, then ask how did they get from one to the other.

Day 2: Further emphasis on building to $1

The overall emphasis on day 2 was on building to $1 (100c).

Making $1

There were further activities subitising with money as were described earlier in Day One.

> There was more subitising to $1, this time 50c and 5c were added. Whereas it seemed that all students were fluent, even with the addition of the 50c, some started to be confused by the 5c. Rebecca even did some combinations such as $3.85, which three or four of the students were able to do, but this was difficult for the others. This confirmed the importance of building up, especially to totals other than 10 and 100.

Three silver coins

The task was "I have 3 silver coins, how much money might I have?"

> This again was the sort of task that should work. It had a range of answers, it prompted communication, it is challenging mathematics, and it addressed the overall theme. It did not work as intended, with many students including $1 and $2 in their total, making it more complex. Rebecca had explained this well. The reason for not mixing the dollars and cents was that it seemed to make the calculation more difficult. It turned out that the local word for coins or change is "silver", which highlighted the need to consider alternate interpretations of events and language at all times.

Making $1

The task "Using coins, how many different ways can you make $1" was then posed.

> The students were given a clipboard and worksheet and worked on the question in pairs. It is suitable for the students in that it is somewhat challenging, it has multiple possible answers, therefore requiring discussion and explanation. In this case the review worked well. On later inspection of their sheets, it was noted that all students got at least 5 correct responses and some more. In the review of strategies, some of the first answers were 50 + 50, 20 + 20, etc., "count by 5s". One student had worked out that there were 20 five cent coins in $1, and explained how she did it.

It seemed that such open-ended explorations worked, and even the reporting back had some value for the teacher and the individual reporting. It was the value for the others that could be considered.

Missing addends to $10, $1

Using actual money jingling in a pocket, and modelling the actions, questions such as the following were posed: "I have $10 in my pocket. If I pay $2 for something, how much do I have left?"

> This provided another take on the "building to 10s" strategy, and directly addressed the theme of giving change. The students found this straightforward and nearly all students could answer the various questions.

Questions such as the following were then posed: "I have $1 in my pocket. If I pay 30c for something, how much do I have left?"

> Some students had more difficulty with these questions, but others could do it. The purpose and intent of the question was clear.

Ping Pong

In the easy version, the teacher calls out "4" and the students call out the number (6) that is needed to make it up to 10. In the more challenging version, the teacher calls out 30c and the students call out the response to make it up to $1; occasionally the teacher says *ping* to which the students respond *pong*.

> The students could do the easy version, and it looked like they did this regularly. They found the version to $1 much more difficult.

Day 3: Further partitioning, including subtraction

The third day built on the experience of the other days and considered subtraction in various forms, more on building to 10, and introducing empty number lines as a way to record and describe intuitive strategies.

Missing addends — other amounts

This was a repeat of the earlier activity except rather than making amounts to $10, other amounts were used such as,"I have $5 in my pocket and give you $1; how much do I have left?"

> This was clearly much harder for them than the breaking up $10 that happened earlier.

One difference, one more, one less

Rebecca posed problems like the following, actively and explicitly modelling each one.

- “What is the difference between $9 and $10?”
- “Peter has $7, and I give him $1 more…”
- “Peter has $67, and I give him $1 more…”
- “Peter has $7, and $1 fell out of his pocket. How much does he have left in his pocket?”
- “Peter has $67, and $1 fell out of his pocket. How much does he have left in his pocket?”

> The students were able to answer the questions well. It would also have been possible to introduce the word “less”.

One more, one less on the hundreds board

Using a hundreds board, with rotating numbers, a set of numbers was shown, and students asked to call out, rhythmically, the set of numbers that were, in turn, one more, one less, 10 more, etc.

> The students did this well.

Make to 10 card game

This was played in groups. Students dealt out ten cards, then in turn collected sets of cards that added to ten. Players score points for each card so there are advantages in getting, for example, three cards that added to ten.

> Rebecca modelled the game and then allocated the students to play in pairs with one pair against another. A group of students coped with the mathematical demands of the game easily, although only one tried to find more than two cards to make ten. There were significant social tensions between them. Rebecca reported that other students she had watched experienced difficulty with the game. It is a valuable game, and worth repeating.

Empty number lines

The empty number line provided a good additional representation, of number calculations. In this case Rebecca, after modelling how the empty number lines worked, posed the task 8 + 5, and asked the students to show this on the small whiteboard.

> The students did not understand the process for doing this and only 1 or 2 students produced appropriate responses. It was suspected that it would be worth persevering with this representation. It seemed that the students had all the skills that were needed, for example, to break 25c into 20c and 5c, and count down first by the tens ($1, 90c, 80c) and then the fives. We have a photo of the work of a Grade 3 student, who is an irregular attendee, who did this accurately, and who erased the answer before Rebecca could see it.

Again there was a review of the activities and consideration of options for the following day. This day had included more abstract ideas, and the students experienced more difficulty with these than with the previous days’ suggestions. Further development of these ideas was needed.

Day 4: Emphasising that the two parts make a whole

The fourth day continued the theme of partitioning.

Something out of a $1

Problems such as the following were posed: “I have $1. I gave (student name) 30c. How much do I have left?”

> The students performed better at this than previously, and this was even extended to including 5c pieces. There was a wonderful exchange where the task “I spend 65c. How much did I have left?” which was reviewed and agreement on 35c as the answer. The task “I spent 35c, how much did I have left?” was posed and one of the students gave a nice explanation how these two tasks were related.

Split a number two ways, split a number three ways

Find a way to select a number. Rolling two ten-sided dice can do this. Then have the students split the number in two parts, and split it into 3 parts (e.g., 48 can be 40 + 8, or 40 + 7 + 1). The students can be asked to produce multiple solutions.

> The task was initially modelled by Rebecca. The students appeared to understand how to break 48 into two parts, 40 and 8. There was an interesting response from a student who when asked to suggest how 48 might be broken into 3 parts said 4 and 0 and 8. The students had a worksheet on which to record their answers. Some students did this well but most students found it difficult. It seemed like this was getting to the core of the challenge with partitioning, that of finding a way to split a number. Of course later they neded to split it in particular ways, but this seemed like a good first start.

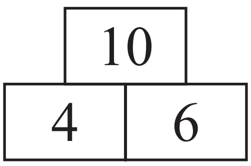
It was noted that while the students were still engaged in these challenging tasks, there were also additional distractions. The day extended the earlier concept development.

Some further suggestions

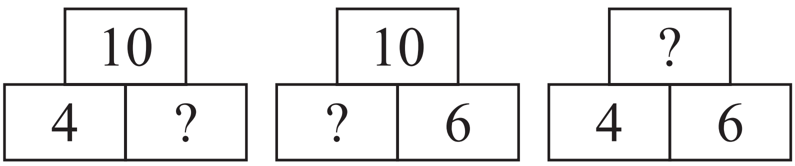
The following activities were not done, but would be reasonable suggestions for the next steps.

Splitting a number many ways using number building blocks

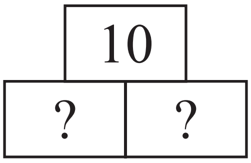
In Europe, it is common to use “building blocks” to represent combining and partitioning numbers, as follows.



So different versions of the task can be presented such as the following show:



The following can also be posed with the possibility of many possible answers being an important element.



Find the missing number

Equations such as the following were posed. The intention was to allow students to focus on breaking up the numbers in a way, which makes the calculation easier.

8 + 2 = 7 + __
88 + 12 = 89 + __
7 + __ = 5 + __ (note that there are many possible answers to this question)

Changing the numbers around

Pose questions such as: “What is the best way to add…?”

1 + 23 + 9
1 + 9 + 1 + 9 + 1 + 9
7 + 35 + 3

Broken buttons

Pose questions such as: “How can you show 57 on a calculator if the 5 button is broken?”

Student achievement

To gain a sense of the learning of these aspects of partitioning, the students were interviewed individually using one section of the Victorian *Early Numeracy Research Project* interview (Clarke et al., 2001), supplemented by some subsitising with money items. The focus of the interview was on whether or not students had mastered particular levels within the number domains: counting and addition/subtraction were used in this case.

If strong achievement is defined as where eleven or more of the 15 students answered a particular question correctly, then there was strong achievement in:

- recognising the total of two $2 coins and two $1 coins, shown for two seconds;
- counting forwards by 1;
- counting from 52 to 63;
- counting from 24 back to 15;
- stating the number after 56;
- stating the number before 56;
- counting by 10s past 100;
- counting by 5s to 90;
- counting by 2s to 40;
- calculating 9 + 4, where the 9 objects were covered, requiring counting on;
- stating the answer to 4 + 4;
- stating the answer to 2 + 19;
- stating the answer to 4 + 6.

If *satisfactory* achievement is defined as where from seven to ten of the 15 students answered correctly, then there was satisfactory achievement in:

- recognising the total of three 20c coins and one 10c coins, shown for two seconds;
- stating the answer to 8 – 3;
- stating the answer to 27 + 10;
- stating the answer to 10 – 7.

Two questions at which the group overall were unsuccessful, with three or less correct were:

- counting from 84 to 113 (most faltering at 109);
- 12 – 9.

It is noted that the *strong* and *satisfactory* sets of items represent excellent achievement by the class, at least comparable with the reference school classes in the ENRP results, and are perhaps indicative that the activities presented above were broadly successful with these students.

Reflecting on pedagogical issues

It is relevant to use the experience to reflect on aspects of the pedagogies used. While there were some very successful activities during the week, such as the open-ended tasks, the games, the challenging and mathematically rich activities, the use of home language for student-to-student discussion, there were clearly also difficulties. The following are some of the pedagogical issues identified.

Rebecca patiently probed the student thinking, and invited them to explain their reasoning. Yet this was not often successful from a whole class perspective. One example of student reasoning came from the student who explained his strategy for winning the *Race to $10* game. He gave an extended explanation, and if you knew what he was trying to say, he was correct. However, his explanation would not have informed other listeners. There were a number of other instances where an individual gave an excellent explanation that elaborated on the desired type of thinking, but not in a way that would engage the other children. The other students were not interested in such explanations, which may be partly a function of the lack of clarity of the explanations. There is a need to reflect on the best ways to conduct such whole class reviews of student work.

There seemed also to be some latent tension between students, with niggling between some children, some of whom respond violently on occasion. The reviews of work, in which the students are not engaged, seem to exacerbate the tension.

The use of *Kriol* for communication between the students enhanced student-to-student dialogue and allowed the students to articulate their thinking. There was a challenge, though, in moving this towards a need to communicate using mathematical terminology. To a short term observer, it seemed that over the years only some students were moving towards greater proficiency in spoken English, even though it seemed that they comprehend English well.

Another challenge was in posing the tasks, especially the more open-ended ones. Some students seem not to listen to instructions so it was necessary to communicate the desired activity through modelling. There was a clear tension between avoiding telling the students how to do the task, on one hand, and communicating expectations clearly on the other.

A further issue was the extent to which the students' thinking was challenged. Clearly, in the above, there were mathematically rich and challenging experiences in which the students participated well, even beyond expectations. But it was perhaps unreasonable to expect the students to do this for the full 90 minutes of each mathematics class. It seemed reasonable to plan some activities that were reinforcing known mathematics, where the goal was engagement. It seemed that activities that drew on two senses were suitable for this. These can be competitive games, including card games, or some aspect of physical activity, combined with a mathematical experience, or drawing, or story telling.

At all levels of the school it seemed that students had an orientation to calling out answers. Yet this had the effect of minimising the requirement for all students to be thinking. Rebecca had an excellent strategy of asking students to whisper their answer to a neighbour, recognising the need for the students to say the answer, while still preserving the possibility of thinking for all.

Conclusion

The learning associated with this sequence of lessons suggested that there were real advantages in planning coherent sets of experiences with the key ideas clearly articulated, and activities chosen to allow students to engage with those ideas. The notion of using the students' strength, in this case subitising with money, as the basis of the learning was also successful. There were a number of highly successful experiences, as described above, some of which were open-ended explorations. Some pedagogical challenges were identified, and finding ways of addressing those challenges will be a key focus for subsequent investigations.

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HANDS ON HEADS ON:
THE EFFECTIVE USE OF MANIPULATIVES

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The use of mathematics manipulative materials is reported to be commonplace in most primary classrooms, although their use diminishes after Year 3. In this paper we question the way in which mathematics manipulative materials are used. More specifically, where mathematics manipulatives are claimed to be used, we question their effectiveness in developing certain mathematics concepts.

Introduction

We all know of storerooms in schools where mathematics materials or manipulatives sit, unused and gathering dust. Often an inspired teacher has purchased some new material as a "pet" project and when that teacher moves from the school, that enthusiasm is not continued and the materials lie unused, often with their purpose not understood. Alternatively materials are purchased as a part of a new trend that has its day and is then forgotten; or they are purchased as a result of a skilled sales person who convinces staff that the materials will solve all of their problems in teaching mathematics. Another scenario for the purchase of materials that later lay dormant occurs when the staff is told there is a short time span in which to make choices before the end of a budget period; so catalogues are accessed and decisions made in haste. How much thought is given to the efficacy of using mathematics manipulatives? Or is it merely to spend the budget?

In a recent study (Swan, Marshall & White, 2007), when teachers were asked to justify the use of mathematics manipulative materials, vague clichés about children engaging in "hands on maths" or moving "from the concrete to abstract" were provided. Other responses referred to children being more engaged when using mathematics manipulative materials and making mathematics enjoyable or "fun" for children. Given that there is increased demand on curriculum time and that a large amount of time is devoted to the teaching of mathematics; and given the amount of time effort and money invested in mathematics manipulatives, it appears that vague clichés are not sufficient justification.

Throughout this paper comments made by teachers involved in the research period will be examined and links made to the effective or ineffective use of mathematics manipulative materials. Other aspects of this research project have been reported elsewhere (Marshall & Swan, 2008).

Research method

The research questions focused primarily on ascertaining what, where (year levels) and how manipulative materials were being used, and their perceived efficacy in enhancing the learning of mathematics. Given the "mapping" features of the required data, survey research was considered to be the most appropriate method to initially address this aim. More specifically, the representative and descriptive character of the requisite data necessitated the administration of a *descriptive* survey. This form of survey "aims to estimate as precisely as possible the nature of existing conditions," (Burns, 1997, p. 467), and "describe some samples in terms of simple proportions and percentages of people who respond in this way or that to different questions." (Punch, 1998, p. 78) Determining the *state* of the application of mathematics manipulative materials entailed finding out simple proportions and percentages of, for example, the frequency with which certain manipulative materials were being used across the full range of year levels in WA primary and middle school classrooms. Determining the *nature* of the application of mathematics manipulative materials involved identifying the views and opinions of teachers on existing conditions, such as main hindrances, and advantages and disadvantages of using them in the classroom.

A questionnaire was developed to question and collect data on the way that manipulative materials are being used, with follow-up semi-structured interviews involving a sample of teachers to probe further. Five previous examples of surveys had been used to gather data on the use of mathematics manipulative materials. Three Australian studies were carried out in primary and secondary schools in New South Wales. Howard, Perry and Lindsey (1996) presented some initial baseline data on the use of manipulative materials in secondary school mathematics classrooms; Howard, Perry and Tracey (1997) compared primary and secondary school teachers' views on the use of mathematics manipulative materials; and Howard, Perry, and Conroy, (1995) looked at the use of concrete material in Years K to 6. Hatfield (1994) surveyed the use of manipulative devices in elementary schools (K–6) in Arizona; and Gilbert and Bush (1988) studied the familiarity availability, and use of manipulative devices in mathematics at the primary level across 21 states in the USA.

The sample

Responses were received from 820 teachers in 250 schools, which represents approximately one-third of all Western Australian schools, with at least one teacher from each of these schools, but up to 15 from some. The responses came from teachers in large metropolitan primary schools, District High Schools (Years K–10), and remote Aboriginal community schools. They encompassed many religious and educational philosophies, from Catholic, Anglican, Lutheran and Islamic colleges to Montessori and alternative schools. Interviewees were chosen from those teachers who decided to reveal their identity and indicated that they would be prepared to participate in an interview or focus group. An attempt was made to include teachers from many different types of schools, and to interview some rural teachers.

Findings

A great deal of data were collected, far too much to report here. In this paper the focus on teacher comments made on the initial survey and later comments made by teachers who participated in individual or group interviews. These beliefs were grouped under several themes. There were many more beliefs as to the role of manipulatives in teaching and learning mathematics but the number of instances tailed off quite dramatically in higher grades. For example, five respondents referred to manipulatives adding variety to a mathematics lesson. Table 1 groups together the six most common themes made by respondents.

Table 1. Teacher beliefs about manipulative use.

Theme	**% stating this belief**
Engages students; heightens interest; motivates	23% (192 instances)
Concrete visualisation; helps abstract become concrete — easier to go from concrete to abstract	22% (188 instances)
Hands on learning, learning by doing, there to be touched	20% (165 instances)
Builds better understanding	15% (120)
Helps grasp/reinforce concepts	7% (61)
Appeals to all styles of learning	6% (48)

Teachers wrote these comments on the survey instrument that was sent to all schools in Western Australia. There was no opportunity to clarify what was meant when these phrases were written. From some of the teachers' comments, it was apparent that some of the respondents were not familiar with all of the materials mentioned in the survey.

In order to probe this thinking a little further and a sample of respondents were interviewed. Given a more focussed opportunity to consider and discuss the issues surrounding the use of mathematics manipulative materials, teachers' responses were still generally clichéd and vague.

Learning theory and use of mathematics manipulative materials

Much is made of learning theory in university teacher preparation courses and yet it appears that beyond a few vague clichés, theory and practice are not linked — at least in the case of mathematics manipulative materials. The systematic use of concrete objects to help children learn may be traced back, in part, to the theories and practices of Montessori, Piaget, Stern and Dienes. The assumption that they improve learning needs to be questioned in the light of current education theory. Utall, Scudder and Deloache (1997) argued, "the relation between the manipulatives and their intended referents may not be transparent to children" (p. 44). They also suggested, "if children do not connect the manipulatives they are required to use with the relevant concepts they are required to learn, then they are forced to do double duty. They must learn two separate systems" (p. 47). This is not to say the use of manipulatives is a waste of time, but that their use needs to be carefully planned and integral to the lesson. Sowell (1989) completed a meta-analysis of research studies relating to the use of manipulative materials and concluded, "The effectiveness of manipulative materials is shown most clearly in comparisons of long-term use of concrete materials" (p. 504). Teachers who invested thought into how mathematics manipulative materials would support their mathematics program in a meaningful way rather than in an *ad hoc* lesson here and there, made more effective use of mathematics manipulative materials. Choosing if, when and how to make use of manipulatives presupposes that careful thought has been given to the mathematics to be developed. Clearly, at certain times certain manipulatives are not appropriate; equally any type of manipulative may impede learning if used inadvisably. MAB or Base Ten Blocks are an example of a mathematics manipulative material that may cause more harm than good if introduced too early, or without consideration for the developmental needs of the learner.

Poor justification

Teacher comments that related use of manipulatives with making mathematics fun were of concern to the researchers. This is not to suggest that children should not enjoy mathematics lessons but the word “fun” lowers the level of manipulative use to that of a toy. Moyer (2002) noted that teachers participating in a yearlong research study expressed similar sentiments. She noted that:

> In many instances teachers indicated that the use of manipulatives was ‘fun’. Initially the term ‘fun’ seemed to indicate that teachers and students found enjoyment in using manipulatives during mathematics teaching and learning. Further analysis of the data suggested that embedded in teachers’ use of the word ‘fun’ were some unexamined notions that inhibit the use of manipulatives in instruction. Teachers made subtle distinctions between ‘real math’ and ‘fun math’, using the term ‘real math to refer to lesson segments where they taught rules, procedures and algorithms using textbooks, notebooks, and paper-and-pencil tasks. The term ‘fun math’ was used when teachers described parts of the lesson when students were having fun with the manipulatives (p. 185).

The comment made by one of the survey respondents that mathematics manipulatives are “particularly good for remedial and Aboriginal children,” suggested that mathematics manipulatives are a tool to be used to fix problems rather than to pose problems and extend more able children. It seemed, if a child is experiencing difficulty, all you need to do is “get the blocks out.”

There appeared to be a belief among some teachers that the manipulative itself would somehow miraculously impart mathematical knowledge. This thinking is inherent in the following comment and others like it. “Sometimes kids will pick up a ‘wrong’ concept from a manipulative.”

From the survey data it is not possible to clearly tell how mathematics manipulatives are actually being used in the teacher’s classroom. However, comments such as the following and other similar comments suggested that children might simply be watching a teacher demonstrate the use of a manipulative material or watch and copy the teacher. “Children always benefit from a ‘visual’ demonstration.” Several studies (Moyer, 2001; Hatfield, 1994) indicated that manipulatives were being used in teacher demonstrations rather than by children. Comments such as the one above indicated similar approaches possibly being used by teachers in the current study.

Implications

In this section some key issues regarding the use of mathematics manipulatives are explored. Possible reasons for the various behaviours are indicated, however the need for more investigation is apparent.

The use of manipulative materials decreases as the year levels increase. In Years 2 to 7, manipulative materials are still listed as being used several times a week; every couple of weeks in Year 8 (the first year of secondary schooling in most Western Australian schools); and about once a month in Year 9. Gilbert and Bush (1988, p. 467) found that "the overall use of materials decreases as grade level increases." Also Howard, Perry and Lindsey (1996) found that the use of manipulative materials in secondary classrooms was low, especially when compared to such use in primary school classrooms.

The finding that mathematics manipulative use drops off makes sense when teachers believe that mathematics learning moves from the concrete to the abstract. The implication is that manipulative materials are viewed as toys or as play things and once children reach Year 3 or thereabouts they should be given "real maths" to do, that is written, abstract mathematics. Also the controlling role of textbooks and the pressure of external testing were mentioned by teachers as factors in this phenomenon. This drop in mathematics manipulative use through the year levels is supported by the graph in Figure 1.

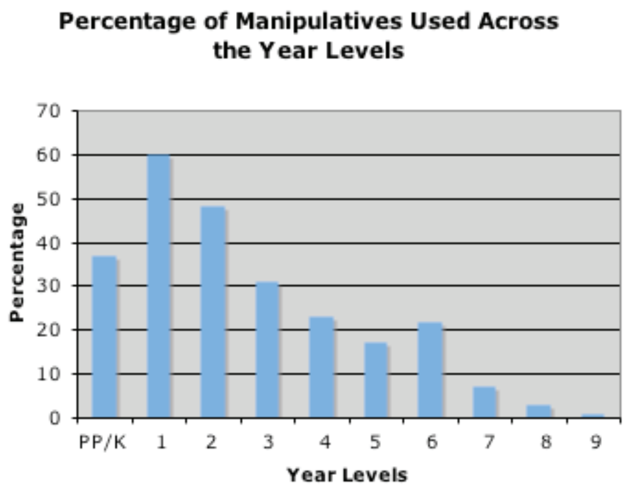

Figure 1. Percentage of manipulatives used across year levels.

Of interest are the two graphs shown in Figure 2, the first depicting the usage pattern across year levels for Unifix, typically an early childhood manipulative. The second graph on the use of MAB (Multibase Arithmetic Blocks, or Base Ten Blocks), often considered to be better suited to middle primary children, presents a scenario that indicates its major use to be in early childhood.

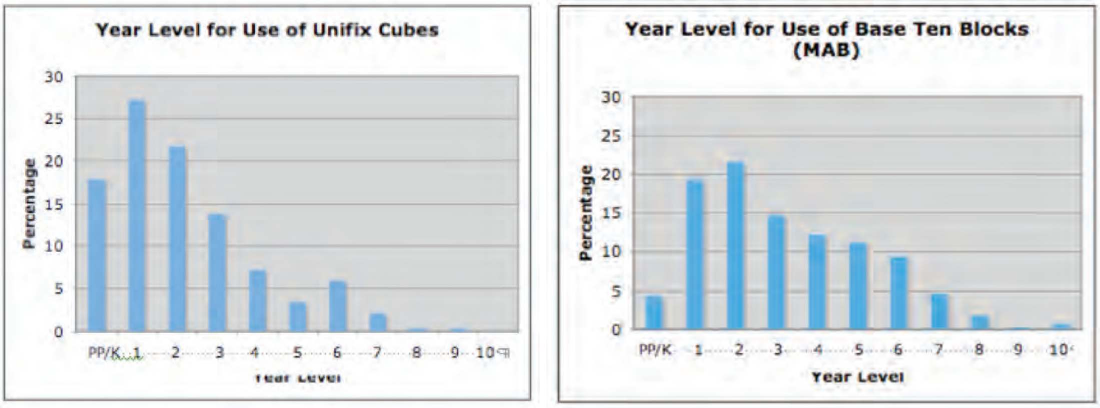

Figure 2. Year levels for use of common manipulatives.

When to use mathematics manipulatives

One of the desired outcomes of this project was the development of a “continuum” of mathematics manipulatives use across the year levels. Participants were asked to consider where eight key mathematics manipulatives were best introduced, or for which year level/s they believed them to be most appropriate. Discussion arose about the advantage of retaining some materials until the middle or upper primary years. There was a belief that some students in these grades, when faced with materials they have used in junior grades, comment that they have “already done this” or that it is babyish. However, with materials such as Pattern Blocks, the concepts being developed may be new or at a more sophisticated level, so their use across all year levels is easily justified.

Often materials such as Base Ten Blocks are introduced in the early years. These particular materials may be used to allow children to model number relations and often, to introduce formal algorithms. They are a bridge to the symbolic representation of these procedures. The recommendation is that this does not happen too early; and that this material not be introduced to the latter part of Year 3. Having Base Ten Blocks as construction materials in early years classrooms is simply not good use of such a resource.

Consider where the following mathematics materials might fit in the school. What year levels? What topics?

- Attribute Blocks
- Base Ten (MAB)
- Centifit Cubes
- Cuisenaire Rods
- Geoshapes/Polydron
- Pattern Blocks
- The Brick
- Unifix Cubes

Implementation gone wrong

In this section, a mathematics manipulative material that has received mixed reactions throughout Australia — Cuisenaire Rods — is discussed. Cuisenaire Rods were introduced to Australia in the 1960s at the height of the mathematics manipulative and "new maths" revolution. In 1953, Professor Caleb Gattegno "discovered" Georges Cuisenaire, an innovative Belgian teacher who for more than twenty years had been using a range of coloured rods to aid the teaching/learning of arithmetic to children in his middle primary school grades. Within a year, the entrepreneurial Gattegno established a production and marketing company in Great Britain to provide the material. Gattegno introduced the rods into the United States of America and in the early 1960s he visited the east coast of Australia where the acceptance and use of the rods was immediate. Along with Gattegno's extensive range of guidebooks, a plethora of materials (course guides, instruction manuals and even children's workbooks) was published. Much of this literature indicated that the introduction of the rods was really the introduction of another system. Figure 3, from Cole (1966), depicts this.

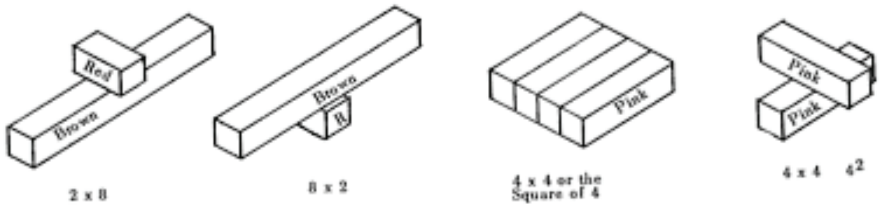

Figure 3. Introduction of Cuisenaire rods (from Cole, 1966).

Chambers (1964) wrote,

> The Cuisenaire-Gattegno approach is fundamentally different and may be summarised as follows: After a period of free play to develop familiarity with the rods and skill in handling them, the infant pupil proceeds with a wide variety of pre-number experiences to develop the concepts necessary for understanding arithmetic. Then follows, possibly in first but certainly in second grade, a study of numbers up to 10 and beyond which involve activities in exploring the number system (p. 11).

Curriculum guides were produced in several states but generally, teachers, except those in the early childhood grades, did not accept the rods. Of course, there were teachers who "mastered" the different teaching skills necessitated by the use of the rods, but by the 1970s, any impact was negligible. In fact, very little had changed in the day-to-day teaching of mathematics. The rods had been promoted as the solution for the teaching of arithmetic and the apparent enthusiasm of the teaching community for the manipulative material was evidenced by the number of local manufacturers, supported by many publishers and the huge quantities of the material that was acquired by schools. Within a generation of the introduction of Cuisenaire Rods in Australia, many teachers were left with the belief that, instead of finding a cure-all for teaching arithmetic, they had been sold "snake oil."

New knowledge and understanding of how children learn mathematics can justify the revival of this most versatile material that has the potential of providing a rich concrete experience in mathematical exploration. However, much of the jargon and "extras" are still best ignored.

Conclusion

If mathematics manipulative materials are to be taken seriously and used appropriately they need to be raised from the realm of a toy to be played with, and given the status of a tool for learning. It appears that the use of manipulatives is justified on the basis of "motherhood" statements and platitudes. If mathematics manipulative materials are not given appropriate consideration as to how they assist with learning of mathematics and how they fit the mathematics program, their potential is not met. They are not there to simply be "toyed" with. However, if thoughtful consideration is given to why mathematics manipulative materials are used and how they are used they can make a valuable contribution to children's learning. If teachers believe these materials will support children's learning they will make the effort to promote understanding of their use with parents and will overcome classroom management issues that are often cited as hindrances to their use.

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