MATHEMATICS

essential for learning, essential for life

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Edited by K. Milton, H. Reeves & T. Spencer

Mathematics: Essential for Learning, Essential for Life

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Preface

The twenty-first biennial conference of The Australian Association of Mathematics Teachers is aptly titled *Mathematics: Essential for learning, essential for life*.

Mathematics is an old and respected discipline of knowledge which has proved itself to be an essential component of school, college and university curricula over a long period of time and in a variety of national and international settings. This has been, and continues to be, in recognition that the content and procedures comprising the discipline have cultural, aesthetic, intellectual and utilitarian values which enrich the lives and pursuits of individuals, communities and nations, be it by making contributions to other disciplines of knowledge or applied to professions and trades.

This conference has brought together teachers, mathematicians, teacher educators and researchers from all states and territories of Australia and from other countries to demonstrate a variety of facets and perspectives of teaching and learning mathematics and to give living proof that mathematics is definitely essential to both learning and life, particularly and generally.

This publication, comprising papers presented at the conference as keynote, major presentations, and seminar or workshop offerings, shows the range, nature and quality of the work which many of our colleagues have been prepared to share with us all. We are grateful to them for this.

Editors: Ken Milton, Howard Reeves, Toby Spencer

Review process

Presentations at AAMT 2007 were selected in a variety of ways. Keynotes and major presentations were invited to be part of the conference and to have papers published in these proceedings. A call was made for other presentations in the form of either a seminar or a workshop. Seminars and workshops were selected as suitable for the conference based on presenters' application of a formal abstract and further explanation of the proposed presentation.

Seminar and workshop proposals that were approved for presentation at the conference were also invited to submit a written paper to be included in these proceedings, with the possibility of the paper being subjected to peer review. Papers that requested peer review were scrutinised blind by at least two reviewers. Papers that passed this review process have been identified in these proceedings as "accepted by peer review". Papers that were submitted to the proceedings but did not request peer review were accepted as suitable for publication by the editors.

The panel of people to whom papers were sent for peer review was extensive and the editors wish to thank them all:

Judy Anderson	Rhonda Faragher	Kay Owens
Mary Barnes	Vince Geiger	Cath Pearn
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Janette Bobis	Marian Kemp	Thelma Perso
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Judith Falle	Steve Nisbet	Michael Wheal

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Keynotes

Mathematicians, mathematics and mathematics teaching: Personal perspectives[+]

Susie Groves
Deakin University

How can we enrich the mathematical experiences of our students? This paper attempts to explore ways in which a consideration of the work of contemporary mathematicians, and the applications of contemporary mathematics, can be used to enhance the learning and teaching of mathematics. It also looks briefly at Hanna Neumann's contribution to mathematics education.

Introduction

> Mathematics is sometimes thought of as a great entity, like a tree, branching off into several large chunks of mathematics which themselves branch off into specialised fields, until the very ends of the tree are reached, where the blossoms and the fruit are found. (Tao, 1992, p. 65)

The tree and branch analogy for mathematics and its disciplines is probably a familiar one for most people, even in the wider community. However, I suspect few people think about the blossoms and the fruit — the place where new mathematics *grows* — and that even fewer think about the people who create this new mathematics — the mathematicians.

At the time when I was asked to choose a title for this Hanna Neumann Memorial Lecture, Terence Tao had just been awarded the Fields Medal — the equivalent of the Nobel Prize in mathematics. At about the same time, one of my doctoral students, Linda Darby, was telling me that one of the major differences she had found, in her work with teachers of mathematics and science, was the much more frequent use of narrative in science teaching than in mathematics (Darby, 2005). Yet, there are so many stories to tell — not just the ones from the history of mathematics books — and so many ways to use them.

In this paper, I will try to use some of these stories to illustrate what I believe are some important ways in which mathematics teaching can be enhanced by their use.

I am honoured and delighted to have been asked to present this Hanna Neumann Memorial Lecture. While Hanna needs no introduction, and I am sure someone else will introduce her life and work at the lecture, I will, nevertheless, begin by saying a little about Hanna herself.

Hanna Neumann

Hanna Neumann was born in Berlin in February 1914. She lived a remarkably full and prolific life until her unexpected death in November 1971 while on a lecture tour in Canada. Hanna married Bernhard Neumann in Cardiff in 1939. She had left Germany in

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[+] Invited paper

1938 to join Bernhard, who had left Germany five years earlier to escape the Nazis. Together they had five children, two of whom, Peter and Walter, also became well known mathematicians, while two others, Daniel and Barbara, completed degrees in mathematics, with Barbara also teaching mathematics.

Hanna completed her D Phil at Oxford in 1943, writing her thesis largely by candlelight in a caravan. She was often seen transporting her two small children around Oxford in a side-car attached to her bicycle. After an active academic life in Britain, which unfortunately meant that she and Bernhard frequently needed to live apart, Hanna and Bernhard were both offered prestigious positions at the Australian National University (ANU). Bernhard was asked to set up and head a research department in mathematics, while Hanna's position, which she took up in 1963 a year after Bernhard came to Australia, soon became a Chair in Pure Mathematics and head of the department of Pure Mathematics in the undergraduate part of the university.

I was fortunate to know both Hanna and Bernhard from early 1970 when I started as a doctoral student in pure mathematics at the Australian National University, under the supervision of M. F. (Mike) Newman and Laci Kovacs. In their comprehensive obituary, which pays tribute to Hanna's life and work, Newman and Wall (1974) describe Hanna as "enthusiastic, inspiring, energetic, firm, tactful, sympathetic efficient, patient, shrewd, humble, peace-loving, courageous, gracious …" (p. 1). For me, each of these words evokes vivid memories of Hanna.

However, this is not intended to be an historical or personal account of Hanna's life. Rather, I would like to explore, however briefly, some of her views about, and contributions to, mathematics education. Newman and Wall (1974) capture not only the joy that Hanna found in mathematics, but also her frustration with curriculum and pedagogy that failed to reveal what she saw as the nature of mathematics — especially the fact that "doing and thinking mathematics can be joyous human activities" (p. 11) — and its many applications to a wide range of areas in everyday and professional life.

While still in Britain, Hanna attempted to introduce more modern pure mathematics to her university course, which she described rather scathingly as spending so much time "on enabling students to solve problems — or perhaps: so much more care is taken to turn out students not worried by an integral or a differential equation" (cited in Newman & Wall, 1974, p. 6). Almost 60 years later, this complaint could perhaps still be made of many mathematics courses in Australian schools and universities!

During her time in Manchester from 1958 to l 963, Hanna had the opportunity to develop and teach courses more in tune with her view of what mathematics is really about. She was able to use concrete examples to help students access abstract ideas in her undergraduate algebra courses, as well as illustrate to students that areas of mathematics, other than calculus, have applications in real life.

In Australia, Hanna soon became involved in secondary school mathematics. She gave lectures and in-service courses for teachers in and around Canberra. She was active in the Canberra Mathematical Association, critiquing the proposed new syllabus and working hard to persuade the community that mathematics is not something to be feared. Her involvement led her to be elected as one of the foundation Vice-Presidents of the Australian Association of Mathematics Teachers (AAMT) in 1966, the first President of AAMT being Bernhard Neumann. She gave lectures to school students in Canberra and was an enthusiastic supporter and contributor to the ANU-AAMT National Summer School for talented high school students, which was started in 1969.

Hanna published over 30 papers in mathematics, as well as a highly regarded pamphlet on probability for teachers (Neumann, 1996) and an article on teaching

undergraduate students, published after her death in the *Australian Mathematics Teacher* (Neumann, 1973).

In my opinion, Hanna's life and work show that mathematicians and mathematics teachers can work together productively to enhance the teaching of mathematics in schools — something that is sometimes obscured by events such as the recent US "Maths Wars". Hanna's vision of school mathematics, and mathematics in general, was that of a joyful activity (which is necessarily done by real people), rather than the rote learning of rules and procedures. She had a vision of contemporary mathematics having a significant place in the curriculum, together with the need to show the wide range of applications of pure mathematics to our students. Even now, 35 years after her death, the realisation of Hanna's vision is as important a goal as it was in the late 1960s.

We will now look very briefly at the life and work of another mathematician, Cheryl Praeger, who not only had strong connections with Hanna Neumann, but whose work allows us an opportunity to explore a relatively simple application of pure mathematics.

Cheryl Praeger

Cheryl Praeger was born in Toowoomba in 1948 to parents who, having come from poor backgrounds with no chance of going to university, encouraged their children to do so (MacTutor History of Mathematics archive, 2006). While completing her B Sc at the University of Queensland, Cheryl spent eight weeks at the Australian National University on a summer research scholarship (after her third year), working on a problem suggested to her by Bernhard Neumann, which resulted in a published paper (Praeger, 1970). After completing her undergraduate course, Cheryl went to Oxford on a Commonwealth Scholarship to study for her doctorate in *finite permutation groups* under the supervision of Peter Neumann, Hanna and Bernhard's son. In 1973, Cheryl was appointed to a three year postdoctoral fellowship at ANU, with her time there overlapping briefly with mine as a doctoral student.

Cheryl has been at the University of Western Australia since 1976, where she has been Professor of Mathematics since 1983. She is a Fellow of the Australian Academy of Science, former president of the Australian Mathematical Society, and a member of the Order of Australia. Like Hanna Neumann, Cheryl exudes joy in her work as a mathematician, looks for the applications of pure mathematics, and takes a keen interest in mathematics education. Her work on combinatorial designs led her to explore their application to experimental layouts for agricultural experiments, in order to help statisticians understand the symmetry groups involved. She has over 300 publications, including four books, over 250 research papers in mathematics, and a number of articles about mathematics education or for teachers.

One of Cheryl's most popular lecture topics, aimed at school level, has been on the mathematics of weaving. One of her three papers on this topic is published as part of *The Hanna Neumann Memorial Lectures at ICME 5* (see Praeger, 1986). In this paper, Cheryl describes how several problems in weaving materials can be solved using mathematics. She firstly explains how a real woven fabric can be modelled with the use of binary matrices representing a *fundamental block* (or smallest repeating block in the pattern). A problem in weaving is to determine which patterns produce fabrics that *hang together* — that is, which patterns produce fabrics where the warp and weft threads do not have subsets that can completely lift off the fabric. It can be shown that fabrics will not hang together exactly when the binary matrix representing the fundamental block of the pattern can be transformed by rearranging its rows and columns, in such a way, as to get a matrix of the form

$$\begin{pmatrix} X & 0 \\ J & Y \end{pmatrix}$$

where *0* is a matrix with all entries zero, and *J* is a matrix with all entries one.

However, finding out whether or not the matrix representing the fundamental block of the pattern can be transformed in such a way would take far too long, when the size of the matrix is large. In her paper, Praeger describes different algorithms that can be used to decide whether or not woven fabrics represented by a particular fundamental block would hang together, as well as the way in which the factorisation of binary matrices can be used to set up a loom to weave the fabric, represented by such a fundamental block.

Applications such as these go along way towards answering the perennial student question of "Why are we learning this?" even though we do not expect every student to become a weaver of fabric!

This focus on the applications of pure mathematics is also taken up by another Australian mathematician of world renown, Terence Tao.

Terence Tao

Readers of most Australian newspapers would be aware that in August 2006 Terence Tao became the first Australian mathematician to be awarded the Fields Medal (see, for example, Cauchi, 2006). In further recognition of his outstanding work, Terence Tao was also named South Australia's Australian of the Year for 2006 (see, for example, Novak, 2006) and elected as a Fellow of the Royal Society in May 2007.

Aged 31, Terence Tao has been a full professor of mathematics at UCLA since 2000, and has long been regarded as likely to win the Fields Medal, which is awarded every four years (UCLA College, 2005). He has over 80 published papers and has worked, with more than 30 collaborators, on an amazingly wide range of mathematical problems.

One of the highlights of Terence Tao's work has been his work with Ben Green on a classical question regarding prime numbers (for a simple explanation of the result, see Ross & Polster, 2007). Tao is also credited with having extraordinary abilities in assembling and managing world-class teams of mathematicians to work on problems.

Terence Tao was born in Adelaide in 1975. In April 1983, when Terence was seven years old, the Adelaide *Advertiser* carried an article entitled *Tiny Terence, 7, is High School Whiz* (cited in Clements, 1984). At that time, Dr M. A. (Ken) Clements, who is now at Illinois State University, was undertaking a bible college course in Adelaide after having "given up" mathematics education following eight years in the Faculty of Education at Monash University, and many years as a teacher of secondary school mathematics. Nevertheless, when asked by Terence's father to assess his son's mathematical abilities, Ken, who had a long-standing interested in mathematically gifted students, found he could not refuse. This began a relatively long association between Terence and Ken. Terence, at age seven, was exceptionally gifted. He was spending the equivalent of two days a week at high school studying Year 11 mathematics and physics, and the rest of his time at his local primary school, being happy and well accepted in both places. In 1986, at age ten, Terence became the youngest participant in the International Mathematical Olympiads, winning a bronze, silver and gold medal in 1986, 1987 and 1988 respectively. He graduated with a bachelor and master's degree from Flinders University at age 17 and a Ph D from Princeton at age 20, joining the UCLA Faculty that year.

After several years working as a missionary in India, Ken Clements (whom I have known for over 30 years) returned to Australia and joined Deakin University. In 1992, Terence Tao published his first book (Tao, 1992), a monograph commissioned by Ken Clements and Nerida Ellerton to form part of the study materials for the Deakin University unit *Teaching Mathematics Through a Problem Solving Approach.* The manuscript for this book was prepared in early 1991, when Terence was 15 years old. The monograph, which was used for a number of years by various Deakin staff, including myself, in a number of units about problem solving, was reprinted by Oxford University Press in 2006.

Regarding the applications of pure mathematics, Terence Tao has this to say:

> Mathematicians often work on pure problems that may not have applications for 20 years — and then a physicist or computer scientist or engineer has a real-life problem that requires the solution of a mathematical problem, and finds that someone already solved it 20 years ago… When Einstein developed his theory of relativity, he needed a theory of curved space. Einstein found that a mathematician had devised exactly the theory he needed more than 30 years earlier.
> (UCLA College, 2005)

It is easy to believe that pure mathematics in general, and number theory in particular, are all very well as genteel pastimes, but have no use in the real world. In fact, nothing could be further from the truth. In the chapter *Examples in number theory*, Tao (1992) writes: "Basic number theory is a pleasant backwater of mathematics. But the applications that stem from the basic concepts of integers and divisibility are amazingly diverse and powerful" (p. 8). To illustrate this, we will take a tiny glimpse at the way in which mathematics has been used in recent history in the design and breaking of codes.

Public-key cryptography

Codes are used when we want to transmit (secret) messages which are in danger of being intercepted. Alan Turing, who was one of the most influential figures in the development of the electronic computer, was also the presiding mathematical genius at Bletchley Park during World War II where he, more than any other person, made possible the breaking of the Enigma codes. The operations of the mathematicians, linguists and other scientists at Bletchley Park formed the basis, some years ago, for a spellbinding British documentary series *The Secret War*, which some of you may have seen, when it was shown on ABC television.

It is easy to think of codes and code breaking as only being of use in war-time, but codes are also essential when we want to store and transmit electronic data — something which happens now in every facet of life, from banking to telecommunications, to the storage of personal information on citizens, to purchasing products on eBay, all of which we want to make sure is secure. (You will all have seen messages regarding the security status of web pages you are viewing — for example, alerting you that you are leaving a page that supports encryption.)

In traditional coding systems, the key to *encode* a message can also be used to *decode* it. Therefore, this (single) key must be kept secret and only made available to people with whom you are communicating — hence, the importance of breaking codes such as the Enigma code in World War II.

In everyday modern life, however, there is such a vast amount of data handling which involves the need for security, that it certainly would not be practicable to use

traditional codes, which, apart from anything else, need the key to be agreed on between the users of the code.

It turns out that the ease with which primes can be multiplied together, compared with the enormous difficulty of factorising a number when we do not know its factors, can be used to create a new type of coding or *encryption* which is called *public-key cryptography.*

In public-key cryptography, the key for decoding a message cannot be deduced from the key for encoding it — at least not in a feasible amount of time. This allows people to send secure messages, electronically, to a destination which can publicly advertise the encoding key — hence, the name public-key cryptography. All of these systems depend on the fact that, while it is theoretically possible to deduce the decoding key from the encoding key (for example, as a last resort, by encoding every possible message of the same length as the one intercepted and seeing which message encodes to the message received), it is not feasible to do so in the time available.

A useful metaphor is to think about a traditional cryptosystem as a safe with a combination lock that, when you know the combination, allows you to both lock and unlock the safe, while in a public-key system the safe has two separate combination locks: one that locks the safe and another that unlocks it. Such codes are also sometimes called *trapdoor codes*: one where it is very easy to do something (encode a message) but very hard to undo it (decode the message).

One of the earliest public-key cryptosystems, the so-called RSA system, was devised by Ted Rivest, Adi Shamir and Leonard Adleman (see, for example, Hellman, 1978 for a description of several systems, including RSA). The RSA system depends on the difficulty of factorising large numbers, as well as making use of modular arithmetic, and some fairly elementary results from number theory. The system can be described, briefly, as follows:

Setting up the system

Select two large primes, p and q, each about 100 digits long. (These primes will remain secret.)

Let $n = p \times q$. (The number n will be made public, but knowing n will not make it possible for you to determine p and q because of the difficulty of factorising a number.)

The *Euler function* $\varphi(n) = (p - 1)(q - 1)$ is the number of integers between 1 and n that are relatively prime to n; that is, the number of integers whose only common factor with n is 1. The *Euler function* $\varphi(n)$ has the property that for any integer a between 0 and $n - 1$,

$$a^{1 + k.\varphi(n)} = a \bmod n .$$

Choose a random positive integer $E < \varphi(n)$, such that E is relatively prime to $\varphi(n)$. E, like n, will be made public: together n and E make up the *public key*.

Since the person setting up the code knows the secret primes p and q, they also know the value of $\varphi(n) = (p - 1)(q - 1)$, but this remains a secret for the public. So, for the person setting up the code, it is easy to find the inverse of E modulo $\varphi(n)$; that is, the number D such that

$$D . E \equiv 1 \bmod \varphi(n),$$

that is, the number D such that

$$D . E = 1 + k.\varphi(n) \text{ for some integer } k.$$

This number *D* also remains secret.

Summing up this stage, we have the following:

- secret: p, q, $\varphi(n)$, D;
- public: n, E.

Encryption

The first step is to represent any message as a sequence of integers. (There are many simple ways to do this, but we will not discuss them here.) Each message then needs to be split up into blocks of digits, each being a number less than *n*. Each block is then encoded separately.

Let *P* be a block in our "message"; that is, an integer between 0 and *n* – 1.

Now let

$$C = P^E \bmod n$$

that is, we raise *P* to the power *E* and find the remainder when you divide by *n*.

So, *C* is the encrypted or coded message corresponding to the original message *P*, and *C* is the message that is transmitted, by whatever (possibly insecure) means we are using.

Decryption

To decode the message *C*, we find *P* by calculating

$$P = C^D \bmod n.$$

Why does this work? Since we have

$$C = P^E \bmod n,$$

we get

$$\begin{aligned} C^D \bmod n \ &= P^{E.D} \bmod n \\ &= P^{1 + k.\varphi(n)} \bmod n \\ &= P \bmod n, \text{ since } 0 < P < n. \end{aligned}$$

Does RSA work?

When the RSA code was developed, it was estimated that it would take a million years to factor the 200 digit number $n = p \times q$, using the fastest known (computer) algorithms known at the time. Of course, computers get faster and faster so methods, such as these, produce codes that fail, in time. New cryptosystems are constantly being developed to meet this important need for secure storage and transmission of digital information. It is perhaps interesting to note that, unlike the way we usually think of mathematics as representing facts and certainty, it is theoretically possible to break such cryptosystems, but they are, for a time anyway, practically intractable.

Public-key cryptography has led to a dramatic upsurge of interest in techniques for factorisation of numbers, and number theory in general.

While the explanation given here appears difficult, it is possible to illustrate it relatively easily using small numbers; e.g., by choosing *p* = 5, *q* = 11, *E* = 7 and *P* = 2 (see Hellman, 1978, p. 27 for more details). Doing such worked examples is possible in

secondary mathematics classes, but even primary children can be exposed to some of the general principles involved and the fact that factorisation, and the use of divisibility tests, have significant contemporary applications.

We will conclude our examples of the applications of mathematics by considering the role of mathematics in one branch of art and design.

M. C. Escher and George Pólya

The famous Dutch artist M. C Escher lived from 1898 to 1972. There are numerous examples of Escher's work, most of which is highly mathematical, available in books, on T-shirts, the "fifteen puzzle", coffee mugs, jigsaw puzzles, to name just a few.

One of Escher's major preoccupations was with producing space-filling designs or tessellations of the plane. He derived much of his inspiration for this in his first visit to the Alhambra in Spain in 1922, where he studied the truly stunning Moorish mosaics on the walls and floors. However, unlike the Moors, who were forbidden to use "graven images" and who therefore only made tiling patterns with geometric shapes, Escher attempted to completely cover the plane (i.e. create tessellations) with shapes that represented objects, such as animals or birds.

I was lucky enough to visit the Alhambra in 1996. Not only was I totally overwhelmed by the tiles and patterns, but it was easy to see how Escher had been influenced by the designs and used these as scaffolds for his plane tessellations.

Ernst (1994) discusses Escher's early (and largely unsuccessful) attempts at producing space-filling designs, based on recognisable objects. He also alludes to some of the mathematical ideas underlying Escher's work and mentions links with crystallography.

Schattschneider (1990), in her magnificent book *Visions of symmetry: Notebooks, periodic drawings and related work of M. C. Escher*, describes, in great detail, Escher's "route to regular divisions" and the role that mathematics played. Escher's early unsuccessful attempts at regular divisions of the plane, using recognisable objects, were produced during the five years after his first visit to the Alhambra in 1922. In 1936, Escher revisited the Alhambra in Granada and La Mezquita in Córdoba. He used his collection of detailed sketches of the geometric designs he had seen to provide the scaffold for his interlocking designs.

> The geometric tilings copied in the Alhambra yielded many new insights to Escher, who found himself not just tinkering with the polygon shapes to derive new motifs, but also trying to discover the distinct ways in which individual figures could interlock with adjacent copies of themselves. (Schattschneider, 1990, pp. 19–20)

While this was the beginning, Escher was still struggling to produce the types of designs he desired. He described his explorations as follows:

> I saw a high wall and as I had a premonition of an enigma, something that might be hidden behind the wall, I climbed over with some difficulty. However, on the other side I landed in a wilderness and had to cut my way through with great effort until — by a circuitous route — I came to the open gate, the open gate of mathematics. From there, well-trodden paths lead in every direction, and since then I have often spent time there. Sometimes I think I have covered the whole area. I think I have trodden all the paths and admired all the views, and then I suddenly discover a new path and experience fresh delights. (Escher, cited in Schattschneider, 1990, p. 21)

The "open gate of mathematics" came to Escher via his half brother B. G. (Beer) Escher, a professor of geology, who, on being shown Escher's work, recognised that he was applying some form of two-dimensional crystallography and provided him with a number of papers published in *Zeitschrift für Kristallographie* between 1911 and 1933. Of these, the one that had a profound influence on the work of Escher was Pólya's (1924) paper that classified the 17 plane symmetry groups (also known as the *wallpaper* groups). While this result — that there are just 17 different essentially different ways to repeat a design in the plane — was known, Pólya provided an illustration for each of the 17 possible tilings, including four of his own design (Schattschneider, 1987, 1990). Escher copied Pólya's article in full and studied the illustrations to understand their geometric structure, and how "these tilings could be colored with a minimum number of colors in a way that was compatible with the symmetries of the tiling" (Schattschneider, 1987, p. 295). Pólya and Escher corresponded, but Pólya left the correspondence behind when he departed from Zurich in 1940. Schattschneider found Escher's notebook with the single word "Pólya" written on the cover in a museum in The Hague in 1976. She sent her photograph of it to Pólya, who then told her of his correspondence with Escher (Schattschneider, 1987).

Given Escher's huge popularity, it is not surprising that almost all students are fascinated by his work. They may also have heard about Pólya, often referred to as "the father of modern problem solving". At the same time, much of the mathematics that is so superbly illustrated by Escher's work occurs frequently in school mathematics curricula. For example, in Victoria, the *Victorian Essential Learning Standards — VELS* (Victorian Curriculum and Assessment Authority, 2005) refers to various aspects of symmetry, transformations and tessellations at every level from Level 2 to Level 6. As Mottershead (1977) so beautifully illustrates, the learning of geometry can be tremendously enriched by exposing students to Escher's work, and the underlying mathematical concepts.

Conclusion

In this paper, I am not trying to advocate adopting a history of mathematics approach to the teaching of mathematics, but rather to encourage us all to attempt to contextualise the mathematics that we teach, to show the links to its applications and the connections between various aspects, and to reveal the people, who create the mathematics, to our students. For most students, it is a surprise to hear that mathematics is actually created by people — some of whom are alive, or have only recently died.

While none of the examples used here are new they, nevertheless, illustrate some ways in which contemporary mathematics can be, and is, used in real life. We talk a lot about the importance of making mathematics relevant for students, but often this is understood to mean that the mathematics should be used by students in everyday life — a hard ask that often leads to a trivialisation of the mathematics we teach. Of course, assembling such examples, and adapting them to an appropriate level to suit students in one's class, is not easy. However, we do not need to develop examples for every topic. Instead, I believe that we should take the opportunity, every now and then, to look at some genuine applications in some depth. Such examples also allow us to make connections between different areas of mathematics (for example, prime numbers, factorisation and modular arithmetic, in the case of public-key cryptography). They also show the essential nature of mathematics and just how widespread its applications are (for example, the "open gate of mathematics" that enabled Escher to develop his wonderful designs for regular divisions of the plane).

Most of all, this paper attempts to persuade teachers of mathematics to convey to their students the joy of mathematical discovery — something that is often forgotten at a time when there is a real crisis in the number of students wanting to pursue higher studies in mathematics and, when, it is becoming increasingly difficult to find teachers for those students who do wish to pursue such studies.

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Developing spatial mathematics[+]

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Spatial mathematics is ordinarily neglected in K–6 education. I suggest two compelling reasons for restoring the mathematics of space to the curriculum. First, developing spatial mathematics provides many opportunities to cultivate mathematical habits of mind. I exemplify several habits of mind drawing on the work of children and their teachers. These include generalisation and its cousin, proof, as well as learning to seek invariance and to see relations. Second, developing spatial mathematics provides many opportunities to establish connections with related forms of thinking, such as those of modelling in science and of representing data in mathematics.

Introduction

Spatial mathematics is comparatively neglected in the K–8 mathematics curriculum. Most emphasis is placed on number and arithmetic, and when there is a pedagogical urge to go beyond arithmetic, the role of generalised arithmetic as the foundation of algebra receives the most attention. Yet, attention to spatial mathematics can yield rich rewards for a general mathematics education, partly because the mathematics of space has a long and rich heritage that ought not be neglected in mathematics education (Lehrer and Chazan, 1998), and partly because developing the mathematics of space is an ideal incubator for developing mathematical habits of mind (Goldenberg, Cuoco & Mark, 1998, p. 3). Mathematical habits of mind are practices that are important to the everyday conduct of mathematics but that go beyond mere instrumentalism to capture something important about mathematical knowing — what makes knowing mathematics different than, say, knowing other important things, such as how to tell stories. When one learns to reason in a manner consistent with a mathematical habit of mind, one is also learning something about what it means to think like a mathematician. Furthermore, neglect of the heritage of the mathematics of space is not simply neglect of some properties of shape or some obscure theorems about form. The real danger is the loss of connection between mathematics and other forms of mathematical and scientific endeavour.

In the sections that follow, I first describe some candidates for mathematical habits of mind, and then go on to exemplify these habits of mind in the work of children throughout the early and middle years of schooling (K–6 span). I conclude, again, with examples from children's work, with connections between spatial mathematics and other ways of knowing, especially in social and natural sciences.

[+] Invited paper

Developing mathematical habits of mind

Any suggestion of a definitive list of mathematical habits of mind would betray some mixture of hubris and ignorance. The tact I take is to suggest some candidates that I believe are accessible to children. The treatment of these habits of mind is genetic, in the sense intended by Piaget (1970), in that the form and content of knowledge is determined by its developmental history. Hence, my orientation is to focus on forms of reasoning that could, conceivably, serve as seeds for later development. When I suggest a habit of mind, I am considering its embryonic form, not its mature form in current mathematical practice.

Generalisation

When we make generalisations in everyday thinking, we are referring to dispositions and propensities. In mathematics, we up the ante. Dispositions are replaced by certainties. Certainty is grounded in understanding of the structure and behaviour of the mathematical system. Hence, a generalisation is a claim about the functioning of a mathematical system. Some generalisations are so well understood, that they merit an additional claim: One can explain the system with sufficient detail and clarity to merit certainty about one's claims. Stated another way, one can prove. Although the tight relation between proof and explanation is well understood in the mathematical community, it is often misplaced in school mathematics. This has been the subject of a number of studies, all of which suggest the need for a tighter linkage between proof and explanation. As I suggested previously, a sensible approach to this problem is not to begin frantic reform in the years of secondary schooling. Instead, we need to think of gentle introductions to generalisation and knowing. I propose a few forms of introduction to generalisation that have shown to be fruitful pathways for children.

Definition

Definition is at the heart of developing mathematical objects that can participate in larger systems of relation. Hence, definitions are important building blocks of mathematical reasoning. All too often, these building blocks are denied to children, who, instead, are asked to use mathematical objects, such as the number sequence, that have either not been defined or that are expected to be somehow apprehended directly through experience. There are some notable exceptions to this in the literature, particularly around the definition of odd and even numbers, but perhaps one reason for the notoriety of these examples is the paucity of alternatives. Thinking about space, in contrast, is more accessible to children, and here we can take opportunities to involve children in the definition of nearly any space figure. Definition of a space figure, such as square or triangle, involves children in the important habit-of-mind of generalisation (when we say square, we do not mean only this square), and it invites too closer examination of the qualities of space, such as "straight," or "angle."

Figure 1 displays a second grader's proposal of a triangle that she claimed met the consensus reached by the class that a triangle had "3 sides and 3 corners." Her class did not want to accept this figure as a triangle. Their teacher invited them to consider how they might wish to modify their definition to exclude it. For this class, what became contested was the nature of side.
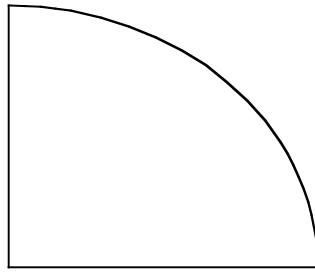
Figure 1. A drawing depicting triangle consistent with definition.

In another third-grade class, definition of a triangle suggested contest about the nature of "tips", in figures that children saw in their textbooks. This gave rise to discussion of the meaning of connected, as is evident upon inspection of Figure 2, which illustrates part of the class investigation.
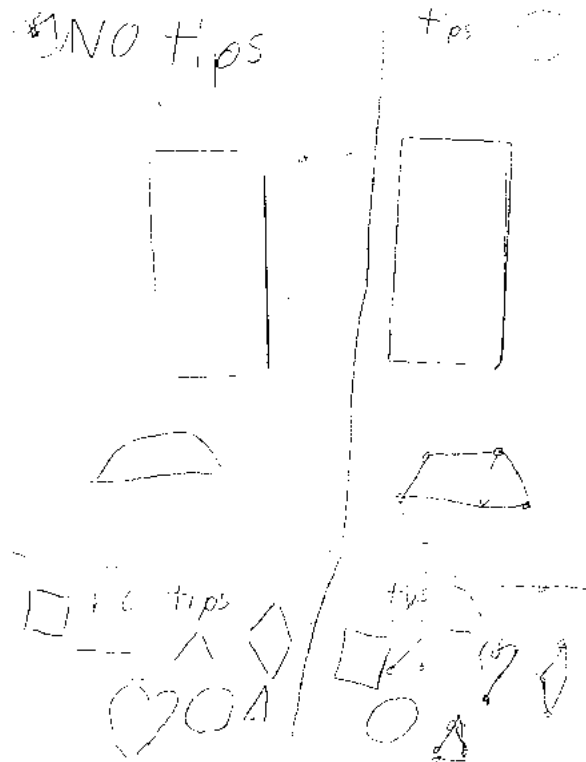
Figure 2. Investigating qualities of "tips" (connectedness).

Of course, it is never too early to begin to initiate children in definition.

Figure 3 displays the results of investigations of 3-D structures by 4- and 5-year old children. It parallels the investigation of connectedness in 2-D, conducted by the third-grade children. The lesson to be drawn is that we do not believe that definition is too complicated a matter for younger children but, instead, that it is our role as teachers to find forms that challenge children, yet remain solidly within their grasp.
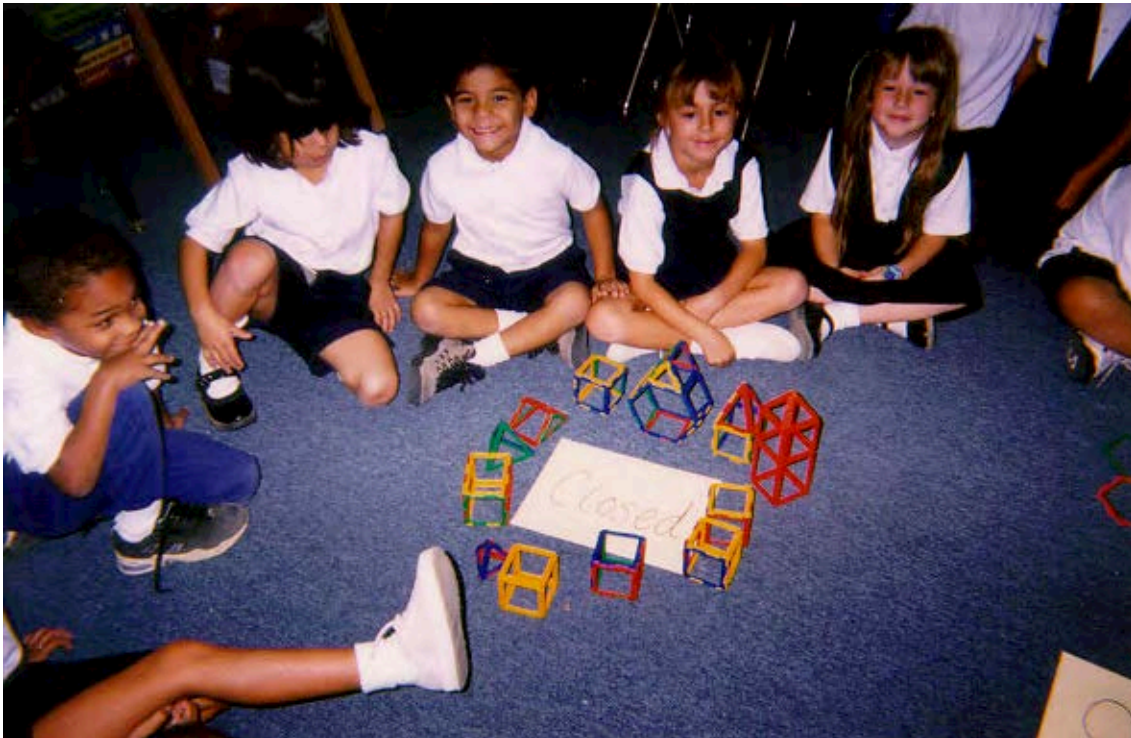
Figure 3. Kindergarten children make definitions of 3-D space figures.

In later grades, we often raise the conceptual complexity of these conversations

Figure 4 again reminds us that our pursuit of definition need not remain in the realm of 2-D. Here, third-grade students, with a developmental history of this form of reasoning, defined the properties of Platonic solids, although, as several pointed out to us, their definition did not include the grounds of knowing "for sure" that there could be only five.

Figure 4 is one student's grounds for ruling out a 3-D figure constructed with rhombi (Lehrer and Curtis, 2000).

In later grades, we often raise the level of challenge by inviting speculations about generalisation of the definition, to other surfaces. Sometimes students do this work for us. For example, a fifth grade student pointed out that one of the class definitions for straight, "no change in heading when you walk," had different consequences for different surfaces. He noted that the definition was consistent with everyone's expectation for the whiteboard in the front of the room or for the floor, but he asked the class to consider what would happen "in the world." A basketball was a handy prop for the ensuing conversation, which we found especially interesting, because the student making the conjecture had a history of failure in the mathematics of number.
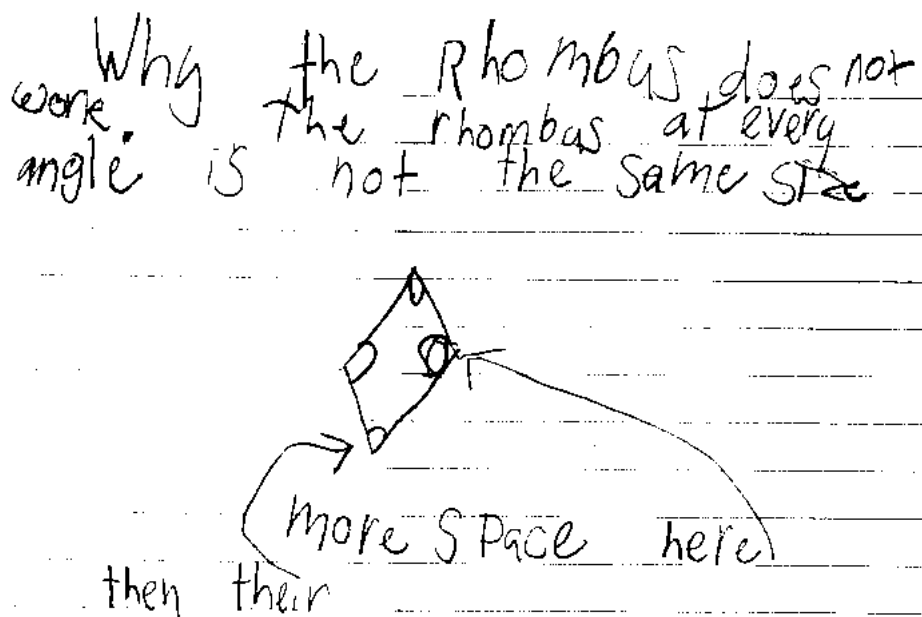
Figure 4. A third-grader explains the shortcomings of a rhombus as an element of a Platonic solid.

Knowing for sure

The flip side of a pursuit of generalisation is a pursuit of the security of the grounds for "knowing". How can we know "for sure?" Definitions, by their nature, rule out proof, because they are erected from axioms that may later be contested but which are taken as obvious during the process of definition. However, once brought into being, mathematical objects can be extended and related to other mathematical objects, and these extensions and relations bring with them insecurities. Can they be trusted, and if so, on what basis? I present two examples that illuminate how informal notions of proof about spatial relations can be profitably raised, long before the secondary years.

In the first example, third grade children had defined a representational system of 3-D objects called a "net": a composition of the faces of the corresponding solid in the plane. Nets were functional in the sense that, when folded, they re-constituted the corresponding object. During the course of their investigations, the children realised that more than one net could be constructed to represent the same object. This naturally led to an inquiry: For a given object, how many different nets were possible?

In the following, we illustrate these issues with a net of a cube, although students considered far more complicated structures and problems.

Children's first instincts about the number of nets of the cube were to guess and test, repeatedly, but as new nets emerged, their teacher pressed them to consider how they would know when they had them all. It soon became clear that physical exhaustion was not going to suffice (There were ambitious proposals for a week of generation and test). A prior question was one of equivalence: Which nets would be considered unique for purposes of counting? To resolve this question, children first decided that any net that was a flip, slide, or rotation of another would be considered equivalent. Note that this important mathematical idea does *not* need to be the exclusive province of true/false number sentences! They then invented a form of exhaustive search of the space of possible nets. Starting with a "backbone" of a column, and considering all possible

unique nets, the students invented a method to guarantee finding all possible nets of the cube. Their proof explained, even as it guaranteed. Along the way, children developed what their teacher labelled as an algorithm, thus expanding their notion of this important mathematical idea beyond the realm of arithmetic. Figure 5 displays this accomplishment. Of course, this success instigated new questions. A third grade student posed this question: If each face of the cube was split into two right triangles, how many different triangle-based nets could there be?
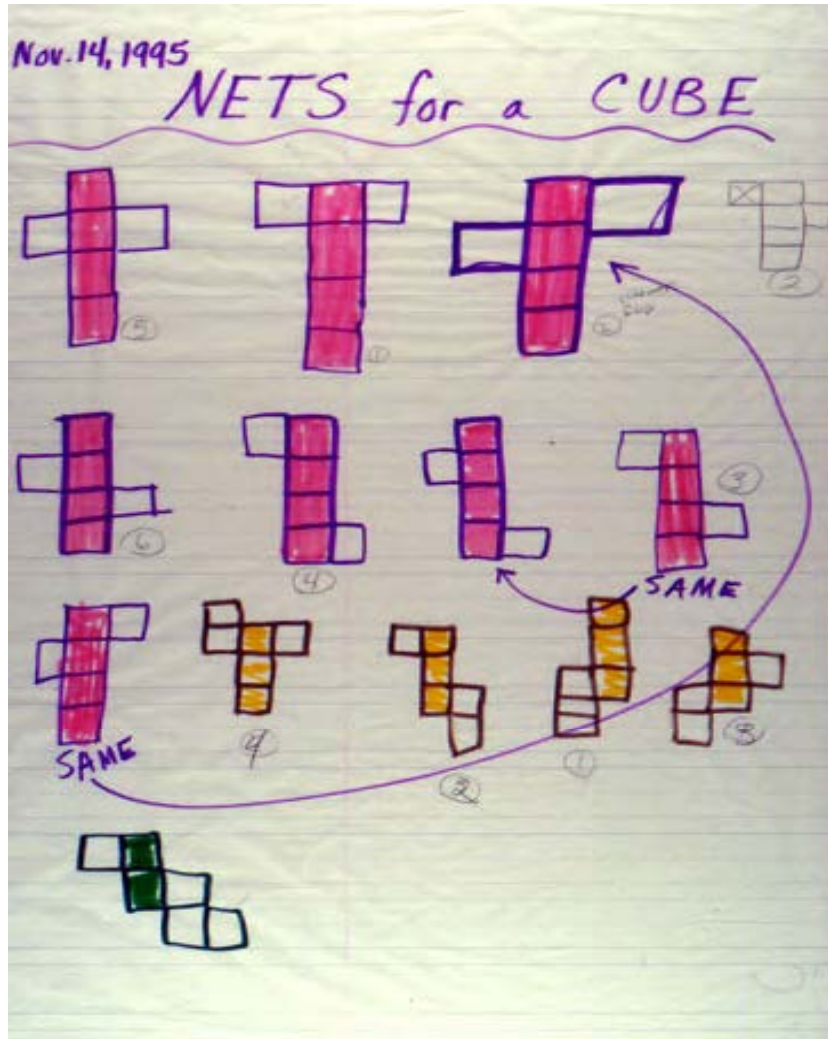
Figure 5. A proof for the number of unique nets of a cube developed by a third-grade class.

The second example occurs during an investigation of the number of diagonals in a polygon, given the number of sides. In one small group, the investigation proceeded by first constructing the number of diagonals for known polygons, such as triangle, square, pentagon and hexagon. Then, by examining the first and second order differences, students induced a pattern that they were confident would hold for any number of sides, as displayed in Figure 6. As one of the students said: "There is a pattern inside the pattern!" She was referring to the relation between the first- and second-order differences.
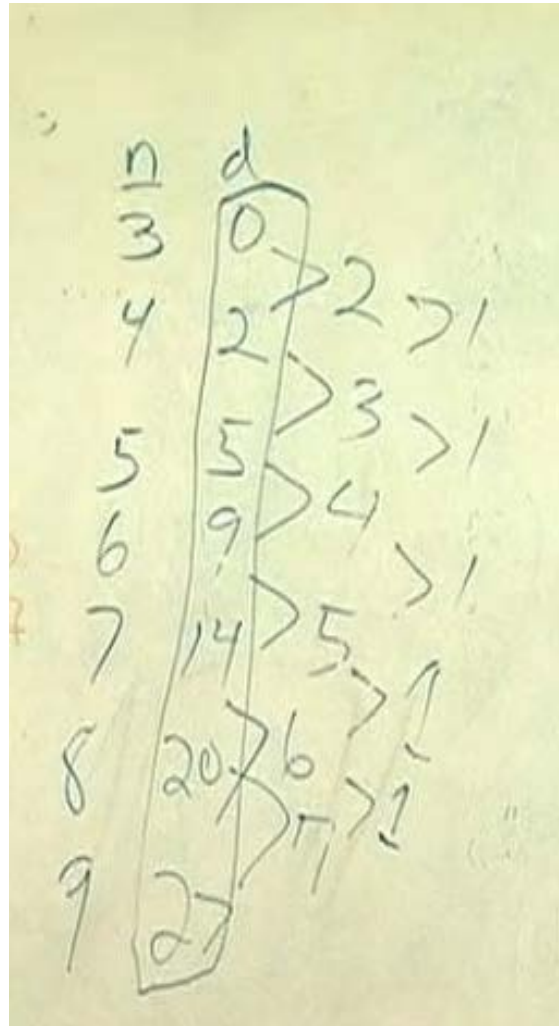
Figure 6. Exploring relations between the number of sides and the number of diagonals of a polygon.

Other students found an algebraic expression that fit this pattern: "number of sides divided by 2 times the number of sides minus 3". But, when the group that discovered the pattern, or those who had succeeded in creating an algebraic expression was pressed by their classmates to explain the basis of the pattern, all were stumped. The expression seemed to work and the reasoning about difference appeared sound, but what could account for it? A classmate proposed an alternative way of viewing the diagonals of a polygon, what we might call a "directed graph." His proposal was: what was important to consider was the number of vertices that could be "reached" from a given vertex. With this in mind, the number of vertices that could be reached was equivalent to the number of sides less the neighbouring sides (2) and the vertex itself (1), yielding the expression $(n - 3)$. One could visit from each and every vertex, yielding $n \times (n - 3)$. Since each path was directed but direction did not matter (vertex 1 → vertex 3 was the same as the path traced by vertex 3 → vertex 1), the same algebraic expression was recovered, but now it could be explained, so the basis of the generalisation was secured.

Invariance

The remaining candidates for habits-of-mind are important servants of generalisation and proof. One is the notion that when conducting an investigation, look for what stays the same when something changes. As re-consideration of some of the previous examples suggests, spatial mathematics affords many opportunities for children to see the pay off for thinking in this way. Consider, for example, the problem of defining uniqueness in the informal proof of the number of nets of a cube, or the invariance of the second difference during the investigation of the number of diagonals of a polygon. One of my favourite examples comes from a unit developed by Dan Watt that explores mathematics via artistic design. Children design quilts by considering motions of units ("core squares") on the plane, as exemplified in Figure 7.
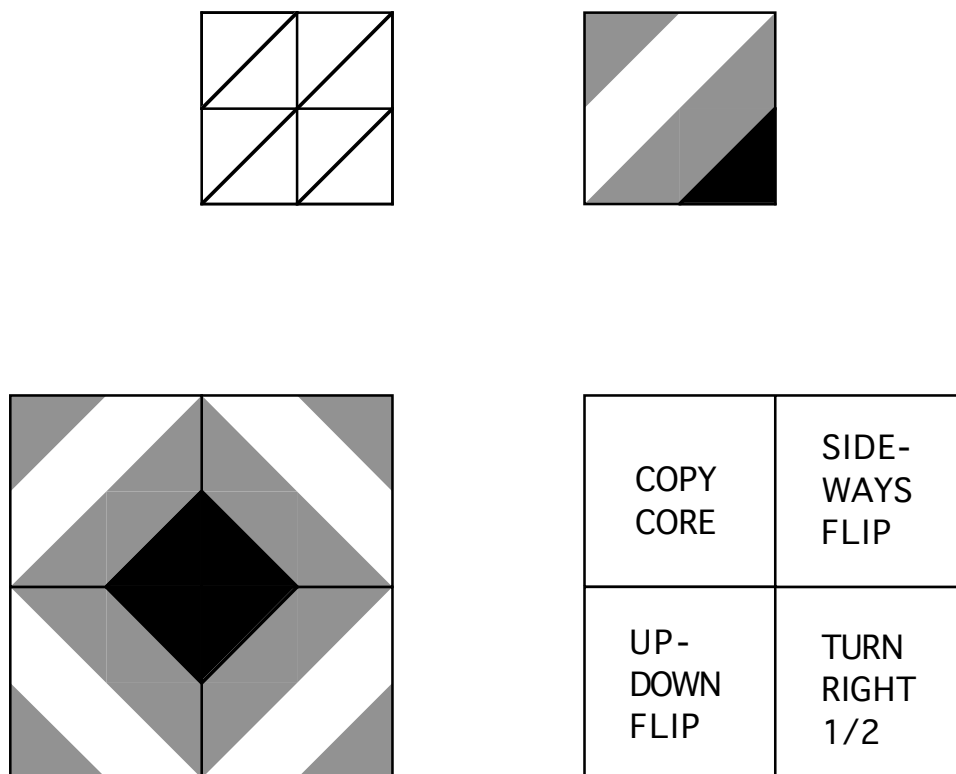
Figure 7. Exploring relations between motion and design.

When we begin the unit, we provide children with Polydrons that they can use to enact the motions literally. They quickly come to realise that the result of a "right" and "left" flip is the same, so they unify both as "sideways" flips. Of course, the very idea of symmetry is one of invariance: Symmetries are those motions that do not result in change, as illustrated by one child's work, depicted in Figure 8 (thanks to James Hamblin and Mazie Jenkins).
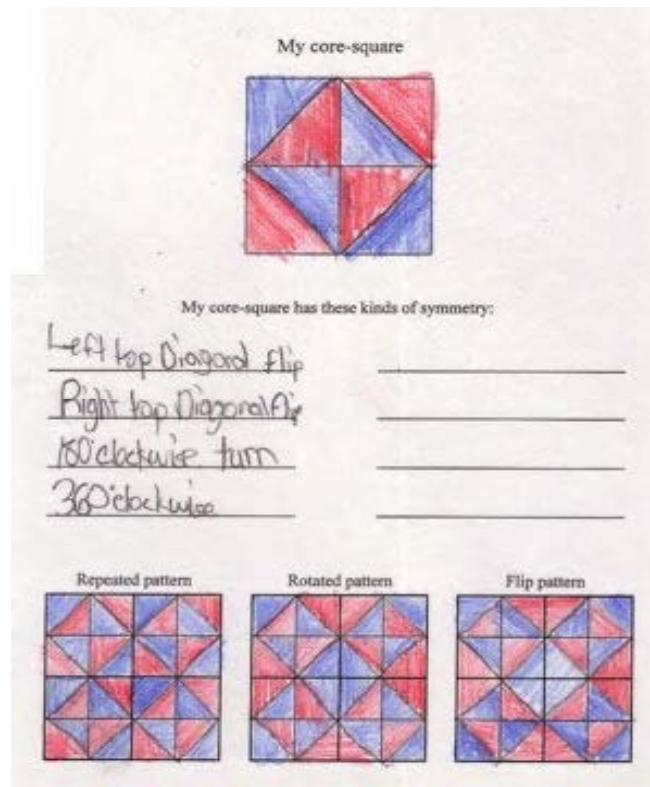
Figure 8. Exploring Invariance

Relational thinking: Thinking about systems

All of the mathematical investigations, that illustrated generalisation and invariance, were oriented toward developing relational thinking. The scope of relational thinking can be either local, or of a more general, systematic character. For example, children may wonder about what happens to the opposite side of a triangle when the angle changes, an example of a limited scope: or they might consider the structure of an entire system of symmetry transformations, an example of a more general scope. The important focus in the classroom is on helping students hold onto their history of investigation and innovation, so that mathematical development is not seen as a set of isolated topics but, rather, as the growth of a system of thinking about particular objects and their relations. We are not used to reminding students of this history or even making its explication a goal of pedagogy. But, without this attention, students often experience even interesting forms of mathematics, as a series of fragmented episodes. This is a major challenge for pedagogy, in all realms of mathematics. But again, I suggest that there are some particular advantages for spatial mathematics. For one, the walls of the classroom can trace a visual record of change. For another, the experience of space can be embodied in different ways, and these forms of experience can serve as anchors for a wider mathematical system. For example, thinking of one figure from a path perspective, as the path traced by bodily motions of move and turn, is readily generalised to thinking about other figures, in the same way. This provides a natural trajectory for the development of a series of mathematical objects and relations.

Bridging the mathematics of space to related endeavours

Modelling

The essence of modelling is having one system of objects and relations stand in for another that is the object of scrutiny. Modelling is a scientific habit of mind that can be supported by the development of the mathematical habits of mind. Modelling, like other mathematical habits of mind, should be introduced gently and with an eye on its potential for development. In one first-second grade classroom, the teacher, Elizabeth Penner, was playing a game of tag with her children called "Mother, May I?". She asked children to configure themselves, literally, to ensure a "fair" game (Penner and Lehrer, 2000). The move to modelling occurred when the teacher asked students if their literal configuration could be modelled, using familiar objects of geometry: points and lines. Children initially thought that the configuration modelled by Figure 9 would suffice. But, much to their consternation, it did not. Eventually, by exploring properties of figures, children arrived at the fairest form of all, depicted in Figure 10. Along the way, children learned important lessons both about modelling situation with shape and form and, also, about properties of these forms that they had never suspected!
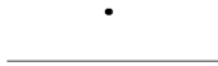
Figure 9. Initial model of fairness.

Figure 10. Many revisions later, the fairest model of all.

Representing data

The use of spatial metaphors and relations is so ubiquitous, in data representation and analysis, that it is largely uncommented upon. We forget that these metaphorical extensions may not be obvious to children. When we forget, children learn to represent by ritual, and so forego the development of representational competence. By representational competence, I mean understanding how a representation goes about its business of representing: what it shows and what it hides about data. In my work with my colleague, Leona Schauble, we seek to make these grounds visible to children and, during the presentation, I propose to illustrate how learning about space bootstraps children's invention and reasoning about data (Lehrer & Schauble, 2002).

Discussion

To briefly recap, I propose that we, as a community, revisit our neglect of the development of spatial mathematics, a children's geometry. Although one can argue that the important business of early mathematics education is number, and thus justify attention to space as a matter of luxury and convenience, this view is not supported by attention to the foundational nature of space. Spatial mathematics ideas are widely employed in related realms of endeavour, such as modelling natural systems or representing data. Apart from these relations, developing spatial mathematics is a natural pathway for developing important forms of mathematical reasoning that serve well across mathematical domains. These habits-of-mind find ready expression and are accessible, even to young children, when developed in contexts of reasoning about space.

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Mathematics is essential in maths education, but what mathematics is then essential?[+]

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As I approach retirement, I find myself marking twenty-one years of further thinking since my previous visit to AAMT[1]. It seems appropriate, therefore, to look back in order to look forward. I will indulge myself in contemplating what it is that we, as a community, and I as an individual, have learned over that time, and where we might expect to look in the future. Participants will certainly be invited to engage in some mathematical thinking, and to ponder and re-articulate, not only why it is important for you to continue to work on mathematics when working with others on issues in teaching and learning mathematics, but what mathematics it is essential to continue to re-experience.

Introduction

I looked back at the paper that accompanied my talk at AAMT in 1986, twenty-one years ago. I have to say, I thought it was one of my better papers. I still agree with everything I wrote, and I have rarely, if ever, been so fluent in expressing it! I choose, therefore, to ask myself the questions of my title, by asking myself, "What have we learned as a community over twenty-one years?"

Of course, no matter how hard I try to be objective, this question is necessarily a projection of the question, "What have I learned over twenty-one years?" onto my biased view of the community as a whole, and which includes "what I wish colleagues had learned"!

In this way I anticipate revealing some of the essential components of effective mathematics education, and so to approach the question of what the essence is. You will not be surprised to learn that I reach similar conclusions to those of my original paper, and I end by reflecting on this observation.

What have we learned as a community?

One thing that I hope is being relearned in each generation, is that you cannot discuss issues in teaching and learning mathematics without being firmly and solidly grounded in mathematics. Put another way, as the acting director of our relatively new National Centre for Excellence in Teaching Mathematics has said, every session, every meeting ought to include some mathematics. I take, therefore, as axiomatic that (working on) mathematics is essential in, to, and for, mathematics education.

So saying, here are some things for you to work on. I have chosen them because I think they highlight, or afford opportunity to be reminded about, many of the things that

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[+] Invited paper
[1] See Mason (1986).

we can say we have learned as a community and that we can agree are both essential to, and an essential part of, mathematics education. The presentation at the conference will use different tasks.

Two mixtures

You have a cask not quite full of wine, and a barrel not quite full of water. You also have a jug. You take a jug of wine from the cask and add it to the barrel, mixing thoroughly; then you take a jug of the mixture and add it to the cask. Is there more wine in the barrel or water in the cask?

Someone has already done this sort of transferring several times. When you now do it with your jug, is there more wine in the barrel or water in the cask?

What can be said about the relative concentrations of wine in the cask and water in the barrel?

Suppose there are two people, one with a jug and one with a small bucket. The jug person transfers from the cask to the barrel and then back again; the bucket person then transfers from the barrel to the cask, and then back again. Would it make a difference if they had acted in the opposite order?

What other variations can you think of?

Comment

I like these tasks because it seems (so far) to be rather difficult to "see directly" what is going on. The first parts depend on the "law of the excluded middle": something is either true or false. In this case, water is either in the barrel or in the cask and what is not wine is water. Two-ness has a very special dyadic quality. The extensions afford an opportunity to use symbols not to represent specific numbers so much as unspecified but identifiable amounts. Intuition does not seem to be a good guide, always, but commutativity emerges. In fact, it can all be accounted for within the language of group theory.

Grid movements

It is well known that you can predict the number of routes between two cells (Start and Finish) on a grid, as shown, when routes are restricted to only *right* and *up*. Can you predict the number of routes between two cells when there must also be a specified number of *down* moves, and a specified number of *left* moves?
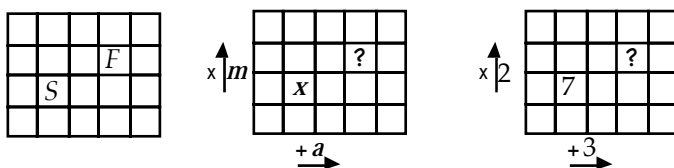
If a number *x* is placed in one cell, and if moving right corresponds to adding *a*, while going up multiplies by *m*, how many different expressions are there for the different results in the final cell? If you specify particular values for *x*, *a* and *m*, how many and how few different values can be achieved in the final cell?

What happens if a specified number of *down* and *left* moves are required (where going left subtracts *a*, and going down divides by *m*)?

Comment

I like this task because it can be tackled by young children using specific numbers (possibly using addition for horizontal and subtraction for vertical!), and it involves them in using brackets to express routes. This means there is both a purpose for paying attention to brackets and to making sure that they mean what you want them to mean,

and a utility (Ainley & Pratt 2002) concerning the use of notation in mathematics. Extending to a specified number of backward movements introduces the possibility of choosing to ignore consecutive occurrences of left and right or up and down movements when considering what counts as "different". Again, the whole can be cast in the language of group theory.

For me, trying to prove the somewhat surprising conjectures about commutativity which I unearthed in the case of the mixtures, and in the grid movements, trying to account for differences between the number of distinct routes, the number of distinct expressions, and the number of distinct values in particular cases, produced lengthy struggle and rethinking. In both cases, trying to find a suitable encompassing theory produced a good deal of struggle to make symbols do what I wanted them to do, and to express generalities and particularities in insightful ways. I am confident that these mirror learners' struggles in classrooms.

> **Scotch fountain**
> The atrium of the teacher's block at Scotch College in Melbourne has a fountain consisting of six nozzles which rotate as they spout.
> What might you expect to see?
> How do you account for the curves in the snapshot apparently traced by water droplets?

Comment

As with any modelling task, the issues that arise are often to do with selecting what is mathematically significant and expressing it in some mathematically tractable form. Although my natural propensity is for modelling structure within mathematics, rather than for modelling material world phenomena, sensitising yourself to opportunities to use mathematical thinking in order to make sense of the world is a vital part of becoming mathematical.

Why engage in mathematics ourselves?

What do you gain from working on mathematics, whether for and by yourself, or with others? What evidence might you put forward that it was worthwhile, perhaps to persuade colleagues or administrators? What evidence would you seek that it improved the experience of learners?

These are non-trivial questions, and I do not have really good answers. I know from my own experience that constantly working on problems, exploring ideas and, particularly, generalising and extending problems, provides me with a focus. It feeds and amplifies a desire to know more, to understand more fully, to appreciate more deeply, to make direct contact with structure and to notice and make sense of phenomena, both in the material world and in the world of mathematical structure. I confess, I am a structuralist at heart, a student of Bourbaki. The various constructivisms provide discourses for trying to express aspects of my experience, but the core of my being is structuralist.

Working on mathematics for, and often by, myself also affords me insight into the use of mathematical powers (what in the 1980s we used to call the "processes of

thinking mathematically") and the pervasiveness of mathematical themes[2]. It enables me to re-experience, freshly and repeatedly, the struggles to get a sense of some situation and to express that sense in mathematically meaningful ways. It also gives me pleasure, and this feeds me energy that can communicate itself to others. I have found it not simply illuminating but vital for me to experience the struggle to express nascent relationships and properties whether conceptual or as part of modelling, to become aware of the use of powers and themes, and to experience again the frisson of having things fall into place or encompassing apparently disparate particulars into a single generality. All this helps to sensitise me to notice opportunities to direct learners' attention to useful things and in useful ways. In the final analysis, it seems to me that what teachers can do is direct learner attention, both directly and indirectly, through structuring experiences.

Since I reject cause-and-effect (especially single cause and single effect) as a useful mechanism for describing education and learning, I neither can, nor wish to try to, point to effects which can be unambiguously and definitively linked to my mathematical thinking as their cause. However, I can point to the energy that I am able to display when working on mathematics with others, which I associate with the enthusiasm and pleasure I get from working on similar or parallel problems, if not actually those problems themselves, and I like to think that I have developed some sensitivity to learners' struggle to appreciate and express mathematical relationships and properties.

Metaphors for mathematics education as a discipline

One thing we have learned, I hope, is that the metaphor for knowledge of laying layer upon layer so as to build up an edifice, is completely inappropriate. For example, we are on the edge of losing the expertise gained from "teaching mathematics investigatively" and "100% coursework"[3]. This includes, not only task construction and teacher–learner interaction in an exploratory mode, but also appropriate assessment. In one instance, in complete ignorance of the recent past, it was asserted at a recent national meeting that "we do not know how to assess coursework the way art students are assessed". In fact "we" do know, or have known, rather more!

I am thrown back on a statement made in my original paper, "each generation needs to re-express the same truths in new vocabulary and new settings," but that most of the issues and concerns go back at least as far as Plato, and probably much earlier. In other words, the essence of mathematics education is unchanged. It is a dynamic of enquiry and challenge in order to be awake to the current situation, in order to be open and fresh to learners as they are, now. In other words, teaching mathematics is both a caring profession and a discipline. It is necessary to refresh both the mathematics and the sensitivity to learners so that we can "be mathematical with and in front of learners".

Contribution to policy

Have we significantly influenced policy generation? In the short term, I think there is plenty of evidence of influence. Government policy documents often pick up on technical terms, such as learning styles, discussion, assessment for learning, constructivism, problem solving, course work, and so on. However, their use in policy documents is usually at best problematic. What we have definitely learned is that once

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[2]   See for example Mason & Johnston-Wilder (2004/2006 or 2004).

[3]   Boaler (1997) studied a comparison with more traditional teaching; Ollerton & Watson (2001) described some of the practices and approaches.

policy documents take up technical terms, their usefulness goes into immediate decline. People naturally adopt meanings that are compatible with their own way of thinking, so the import of technical terms is very quickly watered down or, in some cases, becomes so extreme as to be quickly replaced. I have in mind the temporary phenomenon of advisory teachers in the UK advocating "never telling learners anything, never addressing them in plenary." It was quite quickly counteracted by an insistence on the use of black (now white, smart and interactive) boards and plenary sessions at the beginning and end of most lessons. I also have in mind the insertion of the "three-part-lesson" into primary and then secondary classrooms, which was at first taken up by some as a fresh idea, but by many as a rephrasing of what they were already doing.

I would like to believe that in Australia you have been much more successful at altering vocabulary, without losing its essence. I note, however, that the problem-solving initiative, which began with introducing exploratory problems at VCE level, has been changed to much more directive tasks. Coursework in the UK degenerated so fully that many of us are pleased that it is now to be abandoned. It is reappearing in a fresh form, as ministers warm to the idea of thematic work in both primary and secondary school: what further evidence does one need of cycles in educational values?

In my view, we have a responsibility as a community to construct fresh terms for any terms taken up by policy makers, in order not to lose the substance, the distinctions afforded by those technical terms. Put another way, we do not seem to have learned how to engineer the use of effective fresh vocabulary in a large community. Perhaps this is just as well!

Despite, or perhaps because of, some short-term successes in influencing policy, we do not seem to have learned how to influence long term policy. Perhaps it is not possible, because politics works on short cycles of ministerial careers and elections. The movement towards evidence-based action in the professions is certainly gaining ground, despite the fact that even medicine is already discovering that the particularities of the individual often outweigh the generalities of the evidence-base: trying to mechanise choices does not lead to effective practice. This leads me to another set of observations.

Search for an El Dorado of a simple "fix"

One thing I think we can say the community could learn, even if it has not as a whole yet grasped the implications, is that education in general, and mathematics education in particular, is a complex system. Reductionist attempts to locate a principle cause which, if modified would produce a sizeable effect and "fix" things simply, cannot succeed. I have in my time seen groups of researchers focusing their attention on a sequence of components:

      learner misconceptions
      teacher beliefs
      learner beliefs
      learner motivation (including realistic and authentic mathematical contexts)
      use of apparatus and manipulatives
      teacher questioning
      teaching for understanding
      assessment for learning
      task design (including researched and engineered tasks)
      modes of interaction (especially "discussion")

> situated cognition and distributed cognition[4]
> learning styles
> semiotics
> reification and procepts
> symbolic interaction
> behavioural, cognitive and affect psychology
> psychological constructivism, radical constructivism, social and socio-cultural constructivism, socio-cultural-historical perspective…

In each case, research focusing on one element reads as though people hope that somehow the essential problem can be identified and fixed. As a community, our centre of gravity keeps shifting. In an almost mythical quest for the holy grail, for el Dorado, for Shangri La, we hope that the next perspective will resolve the problem and fix everything. In our search for *the* essential core, *the* underlying cause for why learners fail to appreciate and learn mathematics adequately, we seek something fresh: psychology, hermeneutics, affect and beliefs, sociologies of various kinds, anthropology, semiotics, and so on. We explore and develop the new perspective, promote it to teachers, and often see it become the *zeitgeist* for a time. But it too eventually fails. Dissatisfaction returns and the search begins again.

Such a search is hopeless. As long as our research efforts remain fragmented then, despite refinements in what we can discern, in relationships we can come to recognise, and in properties we can come to perceive as applying more generally, we are doomed to revolve in repeated cycles of change of focus, without contacting an essence of teaching and learning mathematics. Some people may wish to conclude that there is no essence. I choose to try to access essence through recognising and maintaining complexity, while at the same time trying to penetrate beneath the surface. As Italo Calvino observed: "It is only after you come to know the surface of things that you venture to see what is underneath; but the surface of things is inexhaustible," (Calvino, 1983).

It must be clear by now that human beings are complex entities. They have enactive, affective and cognitive aspects which interweave in complex ways, interacting with both attention and will. I note that this is the essence of the ancient metaphor in the Upanishads, for human beings as a chariot or carriage: "The carriage or chariot is the body (enaction); the horses are the affect (motivation, drive, energy); the driver is the intellect (cognition, awareness, attention); the reins are mental imagery; the shafts are feelings; the owner is the will," (Zaehner, 1966, p. 176).

A metaphor is useful only if it informs action or awareness. Much can be derived from this metaphor about the functioning of human beings in general and, more particularly, about learners encountering mathematics. For example, it led me to the Gattegno-based "three onlys": "Only behaviour is trainable; only emotion is harnessable; only awareness is educable," (Mason, 1994; Mason & Johnston-Wilder, 2004/2006), which help me to integrate a range of perspectives, to structure the aspects which make up a mathematical topic, and to make choices while interacting with learners. It certainly makes no claim to do the whole job!

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[4]   You cannot imagine how grateful I am that the pilots of the planes we have taken to get here had more than 'participation in distributed cognition' of how to fly a plane!

The procedural–conceptual divide

Plato inveighs against Greek education for failing to use apparatus the way the Egyptians did, as a means for developing conceptual understanding; Cicero railed against the students of his time who only wanted to be taught procedures, rather than to think for themselves; governmental reports have almost always had something to say about the difference between rote learning of procedures and conceptual understanding. Richard Skemp (1976) used the language of *relational* and *instrumental* understanding (inspired, he said, by an idea of Stieg Mellin-Olsen) to express much the same idea, and the phrase "teaching for understanding" has been used, as if, as David Wheeler pointed out, anyone ever taught for anything else: either you understand in the relational sense of multiple connections and the possibility of re-constructing things for yourself, or in the instrumental sense of "understand what to do" to use a technique. Of course, this then raises the thorny issue of what kind of *knowing* is available: knowing how, knowing what, knowing that, knowing about, or knowing to act suitably in a fresh situation, are different states. Another way of describing the same notion is to distinguish between *syntactic knowledge* of how to behave (manipulate symbols, spout slogans, display practices demonstrated by others) and *semantic comprehension* which is based on awareness of relationships, connections and structure and which, in turn, enable reconstruction and regeneration of requisite facts and techniques when required.

There is nothing new about a procedural-conceptual divide. In a culture which stresses behaviour over being, action over reflection, production over process, instant gratification over Puritan work ethic, cheapness over quality, exploitation of resources over husbandry, it is not surprising that it is an uphill struggle to engage people in taking initiative, in valuing participation in process as well as, and as part of contributing to, a quality product. Fortunately, there are, and always have been, people who respond to quality, to process, to being encouraged to use their own powers rather than having things done for them. It is just that sometimes it seems possible to lose sight of them!

All of this is manifested clearly in the ongoing struggle to engage learners in understanding and appreciating, rather than simply mastering examinable behaviours. In many ways, I think we are much more articulate about this endemic tension, and about the forces and tensions which underpin and feed it. For example, the *didactic transposition,* the *didactic contract*, and the *didactic tension* are useful as reminders of the endemic tensions (Brousseau, 1997; see also Mason & Johnston-Wilder, 2004). At the same time, the cultural ethos of short-term targets set by others and the pervasive influence through the media of short-term satisfactions over long-term aims, is exacerbating the tension for teachers and learners. It seems to me that I am hearing more and more about learners who are accustomed to doing what they are told rather than thinking for themselves, and who not only do not respond with alacrity to being given opportunities to think and to make significant mathematical choices for themselves, but actively even sometimes reject these opportunities. As a community we have enormous distributed intelligence about how, as a caring profession, it is possible to counteract these cultural pressures. Whether it is possible to engineer effective inroads, or whether this is part of the ongoing personal reconstruction and development in which each teacher is engaged as a career long process, I am not sure. My suspicions lie with the latter rather than the former.

What have I learned over twenty-one years?

At various times I have kept notebooks of my reflections on events and other stimulations, such as reading and conversations. Every so often I look back through them, and each time I am astonished to find that the issue which is uppermost in my mind currently actually appears in my notes five, ten and more years previously. It would be tempting to ask whether I have learned anything at all, or whether I am simply recirculating and re-expressing old ideas, failing to recognise that current insights are neither new to me nor to the community at large.

I will confess that I have noticed more and more at international meetings that people raise questions which I recall having addressed in the past and often, to my mind, more effectively then than now. I am minded to take this as evidence that it is time for me to retire and leave the field to younger and more agile minds. I confess to sometimes feeling that we seem not to have learned from the past, and I wonder if we are condemned to cyclic repetitions of the past. As I wrote once: "One thing we do not seem to learn from experience is that we do not often learn from experience alone." Later it was pointed out that this mirrors a statement about history by George Santayana: "Those who cannot remember the past are condemned to repeat it," (Santayana, 1905, p. 284).

But I *have* learned. I feel as though I have learned a great deal. However, in our evidence-based culture, my feelings are not enough. I must address the question of what evidence I can put forward.

I can honestly say that I have access to a much wider technical vocabulary for aspects of teaching and learning. I refer here to a whole range of pedagogical constructs such as situated cognition, ZPD, the onion model of understanding in mathematics and, of course, the structure of attention. These enable me to make finer distinctions than without them, whether in planning, in conducting sessions, or in analysing data and theorising about teaching and learning. Of course, many of these distinctions were elaborated before 1986 but, over time, I believe that I am more effective in interweaving pedagogical constructs and mathematics and as evidence I put forward (Mason et al., 2004; Johnston-Wilder & Mason, 2005; Mason & Johnston-Wilder, 2004/2006).

Final reflections and pre-flections

What emerges from these contemplations, for me, is that mathematics education *can* be a life-transforming, life-enhancing discipline grounded in the experience of thinking mathematically. I say life-transforming because, as I said in 1986, to be engaged as an actively enquiring teacher my perceptions, the details I discern and distinguish in my professional and personal life are constantly under scrutiny and challenge, and so developing. I am kept on my toes, alive to the worlds in which I am operating, through being challenged and stimulated. To be alive as a professional teacher I need to be re-searching. It is not always necessary to search in new domains: sometimes re-searching, revisiting old constructs, old distinctions reveals fresh, if not deeper, insights.

The essence of mathematics education is that each teacher needs to re-articulate, for themselves, the truths of the past, especially those concerning mathematics, mathematical pedagogy, and mathematical didactics, not just once, but in an ongoing process of reconsideration. It is essential as a functioning teacher alive to one's professionalism to make informed, even principled choices with increasing sensitivity.

This means noticing opportunities to act freshly, and fresh actions to initiate, so as to guide and support learners' attention[5].

There are no shortcuts, no standing on the shoulders of giants. Yes, there are shoulders on which to stand, but care must be taken that in perching on one set of shoulders, other perspectives available from other standpoints are not overlooked. Complexity must be retained in order not to become lost in local dead ends of enquiry and policy.

Mathematics education is never going to "solve" the problems which teachers and learners face. But it can provide a discipline within which a there is a supportive and productive environment in which personal and collective development is encouraged, fostered and sustained. One way in which I have tried to contribute to this is by initiating a wiki for mathematical pedagogy (http://ncetm.gov.uk) through which the community (national and international) can refine articulations, focus on precise meanings and details, and support each other in making informed pedagogical and didactical choices in the future.

A plausible, fascinating, and rather informative conjecture is that, just as the things we dislike most in others are often qualities which we dislike about ourselves, so too the things we choose to research or enquire into are what is particularly problematic for ourselves. It is important to bear this in mind then, as I draw these reflections to a close with some conjectures about important topics for the future.

I am convinced that "maintaining complexity" could be a useful watchword for the future. To my mind, co-emergence[6] is a much more fruitful metaphor for how learning takes place than the mechanisms of cause-and-effect. I am also more than ever convinced that where learners are induced and empowered to make use of their own natural powers (of course through and within their socio-cultural-historical milieu), to make significant and pertinent mathematical choices, they are more likely to engage with mathematical thinking than to be content merely with mastering procedures.

I am convinced that school mathematics revolves around the expression and consequent manipulation of generality as a means to make sense of, and to organise our experience of, the many worlds which we inhabit. This, for me, is and remains the essence of mathematics and of what it is essential learners encounter in mathematics lessons. Our failure to awaken this in all learners who get to school is a cultural and social tragedy, as well as being disempowering for the individuals.

I am also convinced that whatever else is going on in a classroom, however learners and teacher are interacting, and mediated by whatever cultural tools, the structure of teacher and learner attention is of vital importance: how is attention attenuated and amplified, focused and dispersed? — but then, given my interest and background, I would think this, wouldn't I?

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[5] This is the aim and purpose of the Discipline of Noticing (Mason 2002) whose articulation begun a at ICME 1984 in Adelaide.
[6] Varela et al. (1991); Davis et al. (1996)

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Addressing the needs of low-achieving mathematics students: Helping students "trust their heads"[+]

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This paper is based on a national intervention and research program. The program has the generic title QuickSmart because it aimed to teach students how to become quick (and accurate) in response speed and smart in strategy use. This intervention seeks to improve automaticity in students' responses, which is operationalised as students' fluency and facility with basic academic facts and procedures in mathematics. This is achieved by reducing working-memory demands on routine tasks, and freeing cognitive resources for higher-order processing, using mathematical procedures and problem solving.

The QuickSmart program supports those students in their middle years of schooling identified as consistently low-achieving. The program runs for approximately thirty weeks with pairs of students involved in three thirty-minute sessions per week. Results of the program indicate that students decrease their average response times significantly, correct inaccurate or inefficient strategies, and develop less error-prone retrieval actions. The results also indicate that by the end of the program these students exhibited strong gains on standardised test scores of higher-order thinking, as well as improvements on State-wide testing measures neither of which were the focus of instruction. Finally, there is evidence that the results are sustained at least 24 months after the intervention.

This paper provides the background, theoretical basis and description, of the program, as well as findings from 2006. Four important aspects of the program that we believe: contribute to its success; have important implications for classroom practice; that are most likely to facilitate improvements in students' learning; and highlight the practical and theoretical significance of having students "trust their head", are also discussed.

Background

Students who experience ongoing failure in school face a myriad of difficulties in achieving long-term employment, and useful and fulfilling occupations. Those who exhibit consistent weaknesses in basic skills, such as the recall of number facts and other basic mathematics skills, are particularly vulnerable.

National test data provide a compelling case for the need to develop programs that improve numeracy outcomes for students, who are performing at or below the National Literacy and Numeracy Benchmarks. There is a specific need for such programs to be effective for Indigenous and rural students, and those with a language background other than English. In addition, national data identify a substantial systemic decline in both the number and percentage of students achieving Numeracy Benchmarks in Year 3,

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[+] Invited paper

Year 5 and Year 7. This trend needs to be attended to as a matter of urgency. It is our contention that, by the time these students reach Year 5, it is particularly difficult to bring about sustainable change within "normal" classroom environments. Consequently, there is a need for educational researchers to design and investigate interventions that support students who experience these difficulties.

QuickSmart (Pegg, Graham and Bellert, 2005) is an example of an evolutionary program of research that is having a strong impact with low-achieving students. The research program associated with *QuickSmart* is one of a few programmatic interventions conducted in Australian schools. The development and monitoring of the program has been supported by a number of different funding sources over the past seven years. Initially in 2001, the Commonwealth Government funded *QuickSmart* for one year, under its innovative project scheme. Subsequently, the collection of follow-up data during 2002, 2003 and 2004 found that these students had maintained their performance improvements 24 months after they completed the intervention program.

Because of the very positive results of the initial *QuickSmart* program, and the data indicating its continued effectiveness, an Australian Research Council Discovery grant by Pegg, Graham, and Royer (2003–2005), "Enhancing basic academic skills of low-achieving students: The role of automaticity in numeracy, reading and comprehension", allowed important aspects of the program to be researched and refined.

In 2006, with support from the federal Department of Education Science and Training, and the Department of Transport and Regional Services, the program comprising two aspects was extended to 12 schools. The first aspect involved improving basic mathematics skill levels in 11 top-end schools in the Northern Territory. A detailed analysis of the results indicated the effectiveness of the program. As a result the NT DEET is conducting a more intensive program in 2007 involving 20 schools. The second aspect concerned a disadvantaged rural school with large numbers of persistently low-achieving students (students below national benchmarks in numeracy and literacy). The program involved 87 students and constituted the largest single student cohort, within a single school, to be involved in the *QuickSmart* program of research. Impressive gains in student performance during 2006 were evident and placed the school among the best in NSW, in terms of value-added results for the Year 7–8 cohort.

Purpose of *QuickSmartT*

The underlying purpose of the *QuickSmart* program is to reverse the trend of ongoing poor academic performance for students who have been struggling at school, and are caught in a cycle of continued failure. These students experience significant and sustained learning difficulties in basic mathematics, and had been resistant to improvement despite attempts to overcome their learning problems. They were unable to draw benefits from other in-class, and withdrawal, instructional activities.

An additional purpose of the program is for classroom teachers, special needs support teachers and teacher aides (referred to below by the generic term "teachers"), to learn how to work with and significantly improve the learning outcomes in basic mathematics skills, of underachieving students, in the middle years of schooling. The program offers professional learning and support for teachers to work in a small class instructional setting, with two students, using a specially constructed teaching program supported by extensive material and computer-based resources.

Theoretical underpinnings of *QuickSmart*

The *QuickSmart* assessment and intervention approach is an innovative instructional method informed by research findings (e.g., Baker, Gersten, and Lee, 2003; McMaster, Fuchs, Fuchs, and Compton, 2005; Royer, Tronsky, and Chan, 1999). Underpinning the program is the establishment of a motivational learning environment, which places an emphasis on fluency, automatic recall of basic skill information, strategy use, and timed and strategic practice. The aim of the program is to improve students' information retrieval times to levels that free working-memory capacity from an excessive focus on mundane or routine tasks. In this way, students become better resourced to undertake higher-order mental processing and to develop age-appropriate basic mathematics (and literacy) skills.

There are theoretical and pragmatic reasons that support the importance of basic information retrieval to both basic mathematics (and literacy) skills. First, it is generally accepted that the cognitive capacity of humans is limited; i.e., working memory has specific constraints on the amount of information that can be processed (Anderson, 1983). As such, there is a strong theoretical basis upon which to expect that improving the processing speed of basic skills frees up capacity, which is, then, available for the cognitive processing of higher-order problem-solving tasks.

Research has already indicated that the ability to recall information quickly is often not subject to conscious control and, subsequently, uses minimal cognitive capacity (Ashcraft, Donely, Halas, and Vakali, 1992; Hanley, 2005; Zbrodoff and Logan, 1996). Another reason why automaticity in basic information retrieval is of prime importance is that it allows for small decreases in "time to accrue" in undertaking sub-tasks associated with a question, again, freeing up working memory. Even small decreases in the time taken to process information in working memory during basic problem-solving situations can be significant. Thus, speed of information retrieval plays an important role in determining the success or otherwise of students undertaking basic mathematics (and literacy) tasks.

The *QuickSmart* Program

In order to contextualise the importance and effectiveness of the *QuickSmart* program, it is necessary to describe the intervention, in some detail. Individually designed intervention programs are developed and implemented as part of *QuickSmart* , in order to strengthen students' problematic skills: e.g., recall of number facts, strategy use, and basic computation. The program is intensive and requires students to work in pairs with an adult instructor, for three 30-minute lessons each week for about 30 weeks.

The *QuickSmart* program:

- is designed to improve students' information retrieval times;
- frees working-memory capacity from an excessive focus on routine tasks;
- fosters automaticity in basic tasks;
- utilises explicit teaching based on understanding, not rote learning, and deliberate practice;
- has time (as well as accuracy) as a dimension of learning;
- integrates assessment tasks into each lesson with a focus on individual improvement;
- maximises student on-task time in a structured but flexible lesson format;

- provides extensive materials including teaching resources, speedsheets, flashcards; and

- incorporates a computer program called the Cognitive Aptitude Assessment System (CAAS).

In addition to specially developed paper and material resources, *QuickSmart* utilises a Cognitive Aptitude Assessment System (CAAS) to support learning, and to assist with obtaining reliable assessments of student performance. This system was developed at the Laboratory for the Assessment and Training of Academic Skills (LATAS) in the University of Massachusetts (e.g., Royer and Tronsky, 1998). The CAAS system is installed on a laptop computer and enables precise measurements of students' accuracy and information retrieval times on numeracy tasks. Importantly, the assessment tasks used are designed and sequenced in order to help identify particular obstacles that may impede student learning (Royer, 1996).

Specifically, when a stimulus is presented to a student who responds into the microphone, the system records vocalisation latency and a scorer pushes one of the two buttons on the computer, to record the accuracy of the response. At the end of a task, the software computes a mean and a standard deviation for response latencies. Also, the software automatically cleans the data by eliminating, as outliers, responses two standard deviations from the mean, such as impossibly fast or unusually slow scores. When the student is finished, the percentage of correct responses, as well as the mean and standard deviation are immediately available and can be recorded and shared with the student. These data are also retained to assist, in part, in the analysis of change in students.

The professional development program accompanying *QuickSmart* is focused on supporting teachers to understand and provide:

- effective instruction that maximises student on-task time, and provides learning scaffolds to ensure students experience improvement and success;

- deliberate practice that is integral to every lesson, allows for success and is focused on providing targeted feedback to improve learning;

- guided and independent timed practice activities;

- strategy instruction and concept development;

- confidence to their students by encouraging a "can do" attitude;

- appropriate teacher and peer modelling; and

- motivational academic activities that are opportunities for modelling and to develop fluency.

As a consequence of the project, and professional development experiences, teachers learn to:

- use time as a dimension of learning and practice;

- incorporate concepts of automaticity (Quick) and accuracy (Smart) regularly in their teaching;

- structure learning activities built around deliberate practice to help encourage success;

- address individual student needs in their planning over an extended period;

- assess and monitor student needs, unobtrusively, in their teaching programs;

- create a highly motivational learning environment for students;

- integrate assessment tasks into each lesson, alongside a non-competitive focus on individual improvement; and
- design and develop activities that improve students' information processing abilities by freeing up working memory.

Also teachers come to experience:

- how automaticity requires conceptual understanding and efficient, effective strategy use; and
- how assessment provides formative information relevant to the progress and design of each individual's program.

QuickSmart results from 2006

In 2006, with Federal Government support, the *QuickSmart* program was expanded and approximately 300 students undertook the program in the Northern Territory (NT) and New South Wales (NSW). In the NT, 203 students were in the program and there were 111 comparison students. In NSW at Orara High School 87 Year 7 students (over 60% of the Year 7 students), who were identified as not meeting national benchmarks, took part in the program. In both settings, many of these students were Indigenous.

The graph below prepared by John Bradbury, Curriculum Officer Numeracy, Teaching, Learning and Standards Division NT (2006) compares pre- and post-scores on a basic skills test for the *QuickSmart* cohort and groups of comparison students.
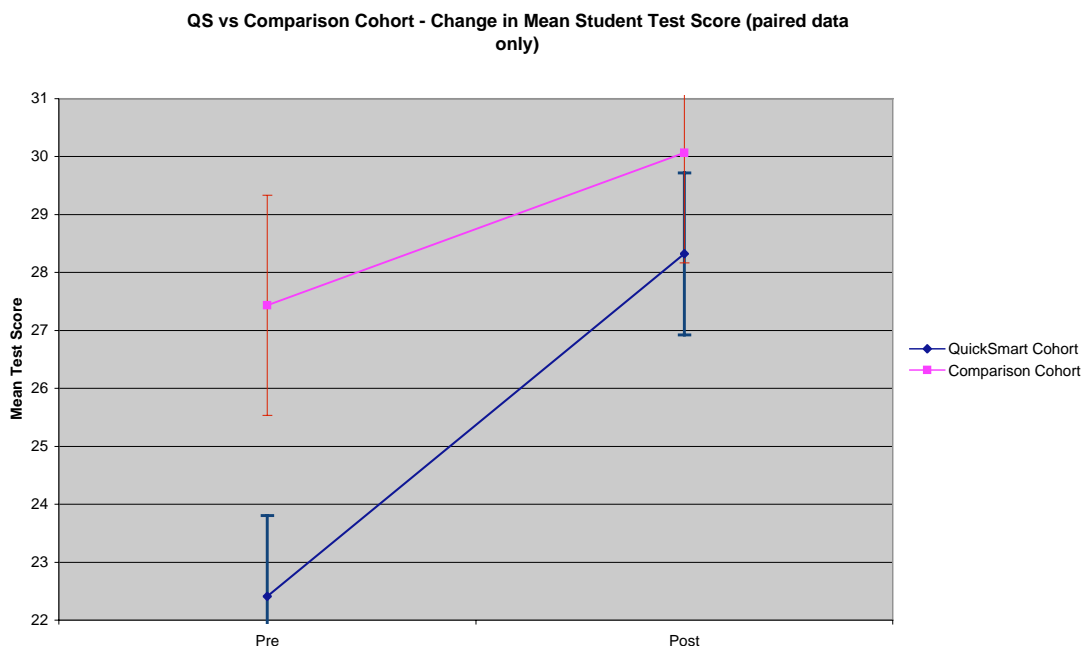
Northern Territory results on 2006 intervention

The graph illustrates the gains made by the *QuickSmart* group of students, as compared to their average-achieving peers. The two cohorts were statistically significantly different at the start of the program from the comparison students, and were not statistically different on the post-test.

In the case of Orara students: they attempted the State-wide Secondary Numeracy Assessment Program (SNAP) in Years 7 and 8. The results were summarised in an article written in a NSW DET by Cotton (p.5, 2006) in *Side-by-Side*. The first three paragraphs are reproduced below and report the gains made at Orara High School:

> It's not often that a school records a meteoric rise in student performance over a single year. So when Orara High School recorded the highest growth in its history for Year 8 literacy and numeracy, the principal, Graham Mosey, summed it up in three words: "We were thrilled!"
>
> Last year almost half of the school's Year 7 cohort was under the national benchmark for literacy and numeracy. But in 2006, all of the students, now in Year 8, performed above the benchmarks — almost doubling the state average growth in their English Language and Literacy Assessment results, and more than doubling the state average growth in writing. Similar results were brought home for the Secondary Numeracy Assessment Program.
>
> "Anecdotally, we'd been told things were really improving, but it was good to get some data that confirmed that was the case," Mr Mosey said.

Both sets of results point to how *QuickSmart* helped "narrow the gap". Analysis has identified impressive statistically significant gains that mirror the qualitative improvements reported by teachers and parents.

Finally, it is worth reporting on the parents' perceptions of the program, in order to "bring to life" the results already presented. Parents were interviewed about how they felt their children reacted to the *QuickSmart* program. In all cases their views were positive. Examples of parents' comments are included below:

> Parent 1: Our daughter thought she learnt heaps. It helped her greatly. We appreciated the opportunity the program offered and we believe the benefits for our child were great.
>
> Parent 2: He told me how well he was doing and how he was improving. His speeds were getting better and so was his accuracy. He enjoyed the work on the laptop. Yes, it was a good experience for my son and he is a lot more confident in his approach and more willing to take risks with his maths.
>
> Parent 3: Joe told us about his lessons. He is very proud of his progress. It is a good program and should continue for a longer period.
>
> Parent 4: My daughter has improved her basic maths knowledge. She no longer uses her fingers. I believe she has learnt a lot. She enjoys maths in the normal classroom now.

These comments indicate that parents perceived improvements in their children's mathematical skills that went far beyond accuracy and retrieval times for number facts. Many of the parents commented on an increase in personal confidence that their children felt, as a consequence of the *QuickSmart* intervention. The realisation by students that they can learn mathematics, and that they can play an active and positive role in the classroom, was routinely commented upon by students and their parents. Towards the end of the *QuickSmart* program, for example, one particular student observed that he could "now think like the brainy kids." It is comments such as this that imply the greatest possible long-term value of the *QuickSmart* program: it brings about changes in self efficacy for students based upon their realisation that they have made (and can feel) genuine improvements in their learning, and understanding of their learning.

Overall, *QuickSmart* has accrued an extensive evidence base, covering several years, showing that there is an alternative to failure for many middle-school students who are not meeting National Benchmarks. The program provides a fourth, and potentially last, phase intervention that will enable students to proceed satisfactorily with their studies, for the remainder of their schooling. Many teachers who have been involved in the program believe that *QuickSmart* is their last realistic chance of being able to help low-achieving students in a sustained and, for students, in a sustainable way.

Four features of learning

This section reports on four features that have emerged from our observations of students and teachers within the *QuickSmart* program. These features — student characteristics, cognitive processing, deliberate practice, and feedback — are particularly relevant to the target group of low-achieving students in the middle years of schooling, but are also relevant to other groups of students who are not reaching their learning potentials. At the basis of these four aspects lies the notion of developing, within students, the ability to "trust their head".

Student characteristics

Students who are persistently low achieving in mathematics exhibit a number of similar characteristics. They utilise inefficient and error-prone approaches to learning and recalling information. Effortful calculation of basic arithmetic facts precludes focus on procedures and problem solving. Often finger strategies dominate simple tasks and this compounds poor speed and accuracy with "the basics". Geary (2004), is but one of a growing group of researchers, who suggests that disruptions in the ability to retrieve basic facts from long-term memory might be considered a defining feature of mathematics learning disability.

These students also have learning gaps or misconceptions that impact on their class or test performance. This translates in performances below national benchmark figures. As a result they report not having a feeling of control over their learning. Compounding feelings of low self efficacy is the fact that, by the time some students are in the middle years of schooling, they have been targeted for support in various forms for many years, without success. These students could be described as "treatment resistant". This is an unfortunate term, but one that focuses attention on the grave plight, and difficult-to-reverse situation, in which these students find themselves.

This point highlights what research has been telling us that low-achievers in mathematics have considerable difficulty in developing automaticity in their number facts. However, if this situation is not addressed, then the achievement "gap" between these students and average achievers gets wider. Really, students need to be proficient or fluent in basic mathematics before the end of primary education (when they are around 11 or 12 years of age). Any real chance of students developing number sense, or forms of mathematical reasoning in secondary school, depends on this occurring.

While poor self efficacy is prevalent, these students can also be described as "classwise". This term (Pegg & Graham, 2007) is analogous to people being described as "streetwise". It highlights how these students have become familiar with the ways of the classroom and how to "survive" within it. Characteristics of this form of student behaviour include the ability to have their lack of learning and understanding become "invisible" to the teacher. For example, students may conceal their lack of basic skills

through various behaviours like copying and denial, so that neither the teacher nor peers are fully aware of their academic difficulties.

What features bring about change? The salient points related to *QuickSmart* instruction were addressed earlier, however, a few need to be emphasised. Students need *time* to acquire the desired skill and understanding level, and *time* to establish new neural pathways. They need to be aware that during changes to cognitive functioning, particularly early in the process, people are extremely vulnerable as they let go of familiar routines and embrace new ones. Motivation is a key factor underpinning the will of the student to try again because they want to improve their performance and because they realise that simply doing what was unsuccessful before is not the best approach. Improvement requires a genuine cognitive reorganisation of the processing underlying the skill needed. One catalyst *QuickSmart* uses to bring about this change is using time as a dimension of learning to build students' awareness of their progress and possibilities for improvement: i.e., through feedback and deliberate (systematic) practice that is targeted at particular goals that are achievable and understandable for the students such as between 35 to 40 flashcards correct in one minute.

Cognitive processing

There are three elements to this discussion of the role of cognitive processing in learning. The first is about the meaning and functioning of working memory; the second concerns the importance of automaticity, and; the third discusses how these ideas are operationalised through the theoretical frame of the SOLO (Structure of the Observed Learning Outcome) model (Biggs and Collis, 1991; Pegg 2003).

Our view on mental activity is guided by Baddeley (1986) and his co-workers who introduced the notion of working memory. Working memory is defined as a processing resource, of limited capacity, involved in the preservation of information while simultaneously processing the same and/or other information (Baddeley and Logie, 1999). This differs from long-term memory in which procedural and declarative information is stored for long periods of time, and short-term memory where small amounts of material are held passively and reproduced in an identical form to which they were encoded. Activation of short-term memory draws upon minimal resources in long-term memory.

Working memory is considered by Baddeley (1986) to have three components. These are: a central executive system that interacts with two subsidiary storage systems: a speech-based phonological loop for storage of verbal information and: a visual-spatial sketchpad that is involved in the generation and manipulation of mental images. The central executive system coordinates these two subsidiary systems, as well as activating information from the long-term memory. Swanson and Siegel (2001) stated that there is also a mental work space that has limited resources and has a combined processing and storage facility that is under the control of the central executive system, and can operate in a distinct fashion from the two subsidiary systems.

We believe that difficulty with working memory capacity underlies many of the problems low-achieving students experience in acquiring mathematical competence, or undertaking more difficult mathematics tasks. Hence, a critical step in supporting these students is to provide them with experiences that enable them to reduce the cognitive load of processing basic skills. Similarly, if we can support students to replace effortful (high cognitive load) strategies with more strategic, and less demanding, approaches then their performances in mathematics will improve. One approach to reducing

cognitive load and, hence, free working memory space is to develop automatic responses in routine tasks.

It is our belief that automaticity in basic mathematics facts and skills is fundamental to a student being mathematically proficient, and able to achieve success in higher mathematics. Hence, an important part of teaching is helping students reduce the cognitive load associated with basic and routine tasks to facilitate deeper mathematical experiences. There are large processing demands associated with inefficient methods and finger counting strategies, etc., as opposed to direct retrieval approaches.

The SOLO model offers us a potential framework to consider when and how teaching might facilitate student development. In particular, SOLO can provide ideas on where direct teaching, explicit teaching, and drill and practice are more appropriate than indirect teaching where problem-solving inquiry, and reflective discussion, might be more useful.

The SOLO model posits that there is a learning cycle comprised of a focus on a single aspect (referred to as unistructural), followed by a focus on several independent aspects (referred to as multistructural) and subsequently a focus on the integration of the individual aspects (referred to as relational). This unistructural, multistructural and relational cycle repeats itself with the acquisition of new ideas and concepts as well as adapting to accommodate the growing abstraction of ideas.

QuickSmart is primarily focused on unistructural elements of learning where students are helped to understand separate individual aspects related to basic mathematical facts and, then, provided with an opportunity to focus directly on these specific aspects through deliberate practice. The purpose of focusing on these unistructural elements is to reduce working memory demands that, in turn, frees working memory resources and facilitates the development of multistructural thinking. At the multistructural level, students have sufficient working memory space to access several aspects separately and to undertake sequential procedures that do not require interconnections among the aspects to be utilised.

Hence, for both the unistructural and multistructural levels, directed learning or explicit teaching is beneficial, and required, to help students come to know the individual elements needed and to practice and consolidate their understandings. Instruction that targets the integration of ideas, and attempts to move students into the relational level, is more about creating an environment for students to make the links themselves through their own motivation and understandings.

Deliberate practice

Practice, in terms of repeating similar procedures or exercises, has value in terms of establishing routines for certain activities and, hence, reducing cognitive load. However, in terms of moving students beyond their current state of performance, practice can actually limit what can be achieved in education. Most practice, even when engaged in over a long period of time, leads to plateaus or ceilings in performance. The amount of practice, past a certain point, does not necessarily lead to ongoing improvement in performance. The reasons for this is that if students are to improve they must either think differently about situations or replace inefficient strategy use. To obtain improvement in performance there needs to be a cognitive reorganisation of the skill, which is accomplished through targeted practice activities. This is achieved by applying deliberate effort (or practice) to improve performance.

We use the term "deliberate practice" drawn from research that has explored and attempted to explain expert performance in a range of areas outside of education

(Ericsson, Krampe, and Tesch-Romer, 1993). For us, deliberate practice within an education context takes four key positions. It:

- is a highly structured activity that has been specifically designed to improve the current level of performance;
- allows for repeated experiences in which the individual can attend to critical aspects of tasks;
- involves specific tasks that are used to overcome weaknesses; and
- enables performance to be monitored carefully to provide feedback.

Students are motivated to exert effort to improve because focused practice improves their performance. Evidence of this improvement is available to all observers, and to the students themselves. In *QuickSmart* deliberate practice takes the form of consistently-encountered, supported and timed tasks that are graduated in terms of difficulty and cognitive demands.

Feedback

Like practice, feedback is a complex feature of teaching and learning that is fundamental to improvements in student achievement. However, there are some features of feedback that make it particularly effective. We draw on the work of Hattie and his colleagues (e.g., Hattie and Timperley, 2007) to explore these ideas.

Feedback needs to be carefully defined and used thoughtfully as an integral part of instruction in order to engender student improvement. Hattie identified four levels of feedback:

1. Feedback about the self unrelated to performance on a task.
2. Feedback on self-regulation so that the student knows how to complete the task with less effort and more success.
3. Feedback aimed at how the task is completed. This includes feedback on strategic levels of understanding and how to process information required to complete the task.
4. Feedback about the task that allows students to acquire more, different, or improved information.

Hattie's argument is that these levels of feedback are least effective at the first level, powerful at the second and third levels in terms of deep processing and task mastery, and most powerful at the fourth level when information is used to improve strategic processing.

With regards to *QuickSmart*, feedback is continuous — even relentless. It is our belief that, without adequate feedback, students will not automatically improve. We provide feedback in the form of praise when both the teacher and student can see that there are genuine improvements in understanding or performance. However, the majority of feedback is focused at a more strategic level. Feedback on activities completed as part of the *QuickSmart* program provides information to students on what they understand or do not understand, why the student is correct or incorrect, what needs to be changed or improved, and what information needs to be focused on or practiced in order to improve.

This form of feedback is linked to formative assessment practices, where the teacher uses assessment information to focus and guide teaching approaches. Formative assessment concerns finding out what the student understands and can do, during the teaching/learning process as students are forming their ideas. It is this information,

when shared with students, that seems to have the greatest possible effect in terms of bringing about real change in student learning.

There are three further important characteristics of feedback. Firstly, feedback needs to provide information to the student on the substance of their performance and, at the same time, be supportive, yet challenging, to students. Secondly, feedback needs to be delivered in such a way that it sets a context that will move students on, from their current performance to the attainment of improved performance. Thirdly, feedback is instrumental in allowing teachers and students to set realistic and attainable goals that are clearly-defined, shared and continually move students towards improved performance.

Within *QuickSmart* the process of effective feedback is facilitated because of the small class instruction mode of delivery that enables and expects the teacher to monitor and react quickly to students' approaches to tasks, their understandings and errors. Small class instruction provides a context for immediate feedback to students, while the consistent lesson structure of *QuickSmart* allows teachers the time and space to follow the performance of students, through repeated trials (what we refer to as deliberate practice) over an extended period of time.

It is possible that one important reason for low-achieving students' poor performance is that, in large class instructional settings, these students have not been able to receive sufficient feedback on their performance to enable them to make the changes necessary to improve their performance in mathematics.

Conclusion

In short, the *QuickSmart* program represents an innovative direction for supporting both basic mathematics (and literacy) skills development. Our monitoring and evaluation of the *QuickSmart* instructional approach since 2001, using quantitative and qualitative indications, have already established that this program significantly improves basic mathematical (and literacy) outcomes for educationally disadvantaged students (e.g., Graham, Bellert, Thomas, and Pegg, in press).

Since its inception in 2001, approximately 800 students have been involved in the *QuickSmart* program. Without doubt, the focus of this work on changing the performances of low-achieving students is an important one in school education. It is also particularly important, in terms of intervention research, that findings are rigorously evaluated because the student population targeted in this work is among the most vulnerable in our education system (Dobson, 2001; Fuchs and Fuchs, 2005). It is obvious that educationally disadvantaged students should only participate in interventions that are accepted as educationally sound. Interventions, based on unsubstantiated ideas, have the potential to take up these students' valuable instructional time and result in little, or no, maintained gains in performance (Strain and Hoyson, 2000).

Central to the research and ideas reported in this paper is the belief that carefully obtained data collected over time are powerful in determining the robustness and utility of educational interventions. In the case of *QuickSmart* our research has provided additional insights concerning the role of working-memory and automaticity in information processing. It also has highlighted the need for further research.

This work is not easy. There are no quick fixes for students who have significant difficulties in mathematics. For example, it takes considerable financial and human resources to run the *QuickSmart* program and it is difficult to obtain sufficient funds to provide a robust intervention to a sample population sufficiently large, so that statistical

procedures can be appropriately employed. The importance of control and comparison groups adds further to the cost and complexity of intervention research. However, such work must be pursued so that an important avenue of help for low-achieving students is not lost, but carefully explored and fully justified.

The benefits for students are immense. Programs such as *QuickSmart* change students' lives in profound ways. They allow students, who are consistently achieving poor results in their classrooms, a chance to become active participants in the "main game" of mathematics. Students who have been involved in *QuickSmart* report that they:

- come to understand and are able to talk about their own learning as the program progresses;
- are able to establish goals and targets for their learning;
- begin to feel they can perform "just like the good kids"; and
- experience genuine improvement and success that encourages them to expend more effort to improve — they are motivated from within.

Most importantly, students who have participated in *QuickSmart* begin to embody a new confidence in what they have learnt, based on genuine observable improvements that are obvious to their peers, parents, teachers and themselves. As students gain confidence and become active contributors to their own mathematics learning, they begin to succeed in ways that surprise themselves and that they can build on their developed foundation to achieve further classroom success. *QuickSmart* students report that they come to "Trust their heads" as effective learners of mathematics.

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Major presentations

Learning difficulties in middle years mathematics: Teaching approaches to support learning and engagement[+]

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Excellent teachers of mathematics seek to support all students in their classrooms (AAMT, 2006a). This aim can be challenged when teachers are confronted with a class of 30 students from diverse backgrounds, with varying levels of understanding, and in some cases, particular learning needs. In addition, student disengagement in mathematics adds another layer of challenge. This paper summarises findings from the literature into learning difficulties in mathematics and presents a range of modifications and accommodations for the middle years classroom. While targeting students with special needs, all students can benefit from the recommended teaching approaches.

The AAMT communiqué *Quality Mathematics in the Middle Years* (AAMT, 2006b) highlights the importance of supporting students at this critical time in their schooling.

> [Students] experience mathematics as a coherent, meaningful and purposeful aspect of their schooling that is connected to their lives as learners and as adolescents developing into adults. Teachers gain great satisfaction from meeting the intellectual and professional challenges they face, and working with colleagues and the community to turn into a reality their belief that all students can learn, do and enjoy mathematics (p. 5).

Most teachers are familiar with the experience of students in their classrooms who have difficulty learning mathematics. These students often require considerable assistance from the teacher and yet they still seem to have problems retaining information and improving their knowledge and understanding. The types of learning difficulties in mathematics are many and varied, all requiring specific modifications and accommodations to support student learning. Individualised instructional materials, withdrawal schemes and other special programs are all attempts to try to overcome the burden placed on teachers. While there may be some advantages in using these support procedures, teachers of mathematics are often best placed to support their students' learning by selecting teaching strategies suitable to their needs (Westwood, 2000).

Appropriate assessment and early identification of students with specific learning needs are critical. In this paper, identification of these students is discussed and accommodations, which relate to good teaching practice are outlined. Some students who are identified as having learning difficulties are actually disengaged for other reasons, which need to be considered and addressed by teachers. The importance of

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addressing language difficulties in the learning of mathematics is highlighted followed by a range of appropriate teaching approaches.

Identification of students with learning difficulties

The focus of this paper is to consider strategies for supporting those students who are having difficulty comprehending and retaining information in mathematics lessons, and for those who are generally disengaged. Students with learning difficulties in mathematics usually have processing problems that could include visual, auditory, motor or memory problems (Steele, 2004). These difficulties may then affect visual processing such as difficulty copying work from the board, auditory processing such as difficulty interpreting oral instructions or general task-related behaviours such as insecurity and lack of confidence. In addition, language difficulties can interfere with reading, vocabulary, listening and making sense of new ideas (Bley & Thornton, 2001). The discussion in this paper does not include those students who have special needs such as the hearing impaired, sight impaired, emotionally disturbed or whose development is severely handicapped.

Students with learning difficulties in mathematics classrooms may be readily identified as they can exhibit one or more of the following behaviours: disruptive or withdrawn; ask many questions to the point of being a nuisance; leave books and equipment at home or lack general organisation; have a very short attention span; readily forget the work from yesterday's lesson; have trouble finding the questions in the textbook, or confuse the meanings of symbols and words in mathematics. Targeted diagnosis is necessary and may be facilitated by the special needs teacher at your school or the school counsellor. Interview protocols or specifically designed tests are also available for this purpose (e.g., Booker, 1994). However, when attempting to diagnose the nature of students' difficulties, Geary (1994) notes the following word of caution.

> In our studies, roughly half of the children who had been identified as having a learning problem in mathematics did not show any form of cognitive deficit… These children probably perform poorly in mathematics because of a lack of experience, poor motivation, or anxiety… (p. 157)

The real issue then becomes one of deciding exactly what can be done in the classroom to help support the disengaged students as well as the students with learning difficulties as they attempt to learn mathematics. Consideration needs to be given to enhancing engagement and motivation as a first priority through the provision of a supportive learning environment and opportunities for success, as well as to exploring the possible modifications and accommodations which address learning difficulties. So what advice is provided for teachers in the literature about student engagement and learning difficulties in the middle years of schooling?

Recommended teaching approaches

Excellent teachers make "a positive difference to the learning outcomes, both cognitive and affective, of the students they teach" (AAMT, 2006a, p. 4). To meet the needs of the full range of students, teaching practices should incorporate a range of approaches and accommodations to promote attitudes and engagement as well as understanding and meaning making of mathematics. These accommodations can involve changes in teaching style which could include variations to the pace of instruction, increased use of

suitable materials and a wider variety of activities, and changes in the curriculum to include problem solving as well as consolidation (Chinn, 2004; Steele, 2004; Westwood, 2000).

Addressing attitudes and engagement

Engagement is an important element of quality teaching and learning which helps to promote working mathematically (Anderson, 2005). Lack of engagement may be evidence of a poor attitude which interferes with learning. What causes such attitudes is an important consideration when planning for quality teaching and learning. Students will often develop a poor attitude if they see mathematics lessons as boring, the work as difficult or irrelevant, or if they continually fail assessment tasks. Certainly students need to become competent in carrying out basic numerical operations but if these procedures are rehearsed daily using the same teaching strategies then it is not surprising that many students will describe mathematics lessons as boring. Review is important and consolidation is necessary to enhance recall and support automaticity but a diet of pure arithmetic is hardly inspiring, particularly if this rehearsal is continually performed alone from sheets or textbooks. A variety of teaching strategies including cooperative learning activities, games, prompts, demonstrations and concrete materials may assist. However, organisation, planning and structure are still important.

Students with learning difficulties need teachers to spend most of their time teaching within a positive, orderly and predictable classroom (Spencer, 1994). Students should know exactly what is required of them and be given clear direction before independent practice. Identifying errors in work samples helps to highlight misconceptions and allows for targeted practise and a focused teaching approach (Anderson, 1996). Regular and consistent feedback is necessary and students need to be aware that they can learn from errors.

For those students who have only ever received poor marks in mathematics, the expectation of continued failure is enough to destroy any interest or enthusiasm to learn and may result in lack of participation and disengagement. This can then lead to anxiety about mathematics with a fear of taking examinations. Praise and positive support are critical so that students believe they can be successful and will be prepared to be challenged and even take some risks. Martin (2003) recommends the use of PBs (personal bests) so that students set goals and aim to improve their individual performance. Keeping a track of performance through charts and graphs (Steele, 2004) or personal logs (Martin, 2003) assists goal setting and enhances motivation and engagement. Building sound relationships between teacher and student helps to enhance a positive self-concept and the feeling that improvements are possible and worthwhile aiming for.

Addressing learning difficulties

A range of modifications in practice can be used to address specific learning difficulties (Steele, 2004). For example, advance organisers help students keep track of lessons. Noting on the side of the whiteboard the steps in a lesson at the beginning helps students follow the progression of the teaching sequence. If new work is presented too quickly with not enough time for students to think and absorb the information or if the prerequisite skills have not been developed then it is also not surprising that mathematics is seen as a difficult subject. This problem is further compounded if each lesson represents a new and unique idea with few connections made to existing knowledge.

Students learn at different rates and many students are exposed to abstract concepts well before they can make sense of these ideas (Westwood, 2000). Bley and Thornton (2001) emphasise the use of concrete experiences for all new concepts. However, if students experience motor problems, the materials may not be easy to handle. Some students may be distracted by the materials and may not be able to make the necessary links to the mathematical concepts being portrayed (Steele, 2004).

If necessary, tasks need to be modified by altering quantity, level of difficulty, presentation, or the time allowed (Spencer, 1994). Tasks become more manageable by breaking them down into subtasks. When asking questions, students with learning difficulties need a longer wait time. Spencer surveyed students who reported that they learn better if they are given one instruction at a time, they have more time, they can see a purpose in the learning, and if the teacher helps them to organise their work. Students need to be taught how to organise themselves as well as how to summarise, revise work, study for a test, and sit for an examination.

Language difficulties in learning mathematics

Language helps students organise knowledge in both written and oral form (MacGregor & Moore, 1991). To that end, students need opportunities to read mathematics, to write about mathematics, and to talk about mathematics using their own language. For some students, language difficulties interfere with their learning as they attempt to make sense of new language demands. For example, students may experience difficulty converting between words to numbers (Miller & Mercer, 1993). In addition, the vocabulary of mathematics can create problems for students (Chinn, 2004; MacGregor & Moore, 1991). Students who have difficulty with reading and comprehension experience problems when mathematics is presented in real contexts, frequently with much information to comprehend and interpret (Newman, 1983).

Many students have difficulties with the technical as well as the symbolic language of mathematics and reading mathematical textbooks is challenging. The vocabulary, symbol system and technical language confronted in textbooks must be discussed with the students (Shield, 1991). Textbooks should be read aloud in class so that students practise saying mathematical terms, they ask questions to clarify understanding, and become familiar with the language used in problems and exercises. Students often report that they did not do set homework because they could not interpret the questions or instructions. Extra time in class needs to be allocated to reading the homework exercise particularly for the benefit of those students who experience reading and comprehension difficulties.

Investigations and projects require clear communication of mathematical thinking. Writing activities support the development of students' use of mathematical language (MacGregor & Moore, 1991). However, higher-order thinking skills are required and this can be extremely difficult for students with learning difficulties. Allowing students to work in mixed-ability groups to complete writing tasks will help to reduce anxiety and give students the opportunity to share and clarify understandings (Ridlon, 2004). Suitable writing tasks include writing prompts, open-ended questions, and journal writing.

Supporting problem solving

Many students perceive that school mathematics is totally irrelevant and does not have any application to their future aspirations. Achieving good grades in external examinations may be the only motivation to try to understand the concepts confronted

in each lesson. Mathematics would be seen as more useful if connections were made with problem-solving situations related to everyday experiences.

Ridlon (2004) explored the impact of a problem-centred mathematics curriculum on 26 low-achieving Grade 6 students over a nine-week period, and compared their learning to a control group who were taught the same concepts using a more traditional textbook approach. She reported significant improvements in both attitudes and achievement for the experimental group citing evidence from the students, their parents, and an independent observer in the classroom who could not believe they were the "low achievers".

Modelling the process of problem solving and providing prompts helps students to ask the right questions and organise their thinking. Another useful strategy is to use acronym mnemonics, which help students to remember the steps in the problem-solving process. One example is SOLVE (Miller & Mercer, 1993) where each letter stands for a separate step:

1. **S**tudy the problem.
2. **O**rganise the facts.
3. **L**ine up a plan.
4. **V**erify your plan with computation.
5. **E**xamine your answer. (p. 80)

This strategy helps those students with memory deficits although all students will benefit from using this organiser as it promotes the use of metacognitive strategies. Awareness of one's own thinking is critical in the problem-solving process as it gives the student control and helps them to organise and plan a solution. Not understanding what the problem is about and misinterpreting the language used in mathematical problems can lead many students to additional difficulties in mathematics (Newman, 1983).

While these recommendations are not new and represent good teaching practice, many teachers still spend a disproportionate amount of time on the practice of computation skills without trying to develop understanding or without giving students the opportunity to think mathematically. Problem solving is accessible to these students provided the teacher chooses the right problems. Open-ended problems are ideal for use in mixed-ability classrooms as they allow all students the opportunity to make an attempt at the problem. Sullivan, Zevenbergen and Mousley (2005) have used open-ended tasks with enabling prompts for students who experience difficulty making a start on the problem.

Conclusion

In this paper, many issues related to teaching students with learning difficulties have been outlined as well as a range of recommended teaching approaches to support learning. Critical to this support is the provision of a classroom environment which supports the learning of all students. Often this will mean addressing poor student attitudes and disengagement, possibly resulting from continued failure. Providing opportunities for success in a positive, non-threatening classroom will do much to help many students begin to want to learn and enjoy mathematics.

Other important modifications may be required to support the learning needs of particular students. The difficulties which these students experience can be a result of ineffective processing of information, poor retrieval skills or language problems

Suggestions have been made to help mathematics teachers in their endeavour to teach these students. These are worth considering as all students may in fact benefit from the variety of new teaching strategies.

Finally, good teaching practice provides opportunities for students with learning difficulties to experience success. Teacher expectations can strongly influence the performance of these students and so teachers are encouraged to reflect on their attitudes towards learning difficulties students. There is evidence that teachers tend to demand less work, give less feedback, ask fewer questions, criticise more often and have less friendly interactions with low achieving students (Brophy & Good, 1987). It is important that all students are supported to improve their knowledge, skills and understanding in mathematics by focusing on effort to enhance students' self-esteem (Martin, 2003).

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From here to there: The path to computational fluency with multi-digit multiplication[+]

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Drawing upon research, theory, classroom and personal experiences, this paper focuses on the development of primary-aged children's computational fluency with multi-digit multiplication. Getting children from 'here' (current strategy use) to 'there' (a more efficient strategy) is often not a straight-forward path. The critical links between number sense and a child's ability to perform mental and written computation with ease are examined.

Many readers will know the story of the famous mathematician Johann Carl Friedrich Gauss (1777–1855). As a young boy he was prone to daydream in class. One day his teacher decided to punish him for not paying attention. He was asked to add all the numbers from 1 to 100. Much to the annoyance of the teacher, young Carl was able to derive the correct answer in seconds. Fortunately for Carl, he knew a short-cut. He realised that adding pairs of numbers (e.g., 1 + 100, 2 + 99, etc.) all equalled the same number: 101. He figured that there were 50 such pairs, so calculated the total to equal 50 × 101 or 5050.

Recently I related this story to a group of primary-school teachers. One teacher immediately asked, "But who taught him that?" This question sparked a discussion about the critical relationship between a person's understanding of mathematics and their computational fluency. The teachers agreed that Carl's in-depth understanding of mathematics enabled him to see patterns and relationships that made the computation more manageable, but that his knowledge of basic facts and the fluency with which he could compute were equally important. The teachers concluded that understanding without fluency can inhibit the problem solving process.

This paper focuses on the development of primary-aged children's computational fluency. It emphasises the critical links between number sense and a child's ability to perform mental and written computation. The case of multi-digit multiplication is used to illustrate these important links.

Computational fluency:
Number sense and the standard algorithm

The idea of teaching mathematics for understanding and for meaningful learning to occur has been advocated for over half a century (Brownell, 1935). However, it was not until the 1980s that the term "number sense" was first used to refer to those who had a deep understanding of numbers. The focus on number sense is manifested in the recent and on-going emphasis in international curriculum and policy documents on mental

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computation (e.g., Australian Education Council, 1991; National Council of Teachers of Mathematics, 2000). Research has shown that those who are good at mental computation possess a well-developed sense of number (McIntosh & Dole, 2000).

The increased emphasis on mental computation and number sense has seen a corresponding de-emphasis in curricula on standard algorithms. An algorithm is a specified multi-step procedure that produces an answer for any given set of problems and is characterised by long-term practice. While still recognised as important, some Australian state syllabus documents have delayed the introduction of standard algorithms for around two years to allow a focus on mental strategies for as long as possible (e.g., Board of Studies, New South Wales [BOSNSW], 2002). The worry with an early emphasis on standard algorithms is that students will shift their focus to executing convenient procedures rather than on understanding the mathematics.

A concern is that educators will view the development of number sense and fluency in written and mental computation as separate bodies of knowledge requiring separate instruction. In fact, computational fluency, whether employing mental or written methods, and number sense are intertwined and should be developed together. The aim of the following sections is to examine how children develop proficiency in their computational methods while instruction remains focused on learning with understanding.

Understanding the development of children's strategies

While a number of research-based "frameworks" provide excellent descriptions of learning pathways by which children's computational strategies develop, they fail to tell us about how children progress to use a more efficient strategy in preference to another less efficient one. It is imperative that teachers understand how children make this shift.

As children become more competent mathematicians, they develop a variety of thinking strategies for solving mathematical problems. Generally, children initially apply basic counting strategies to help them solve simple numerical problems before moving onto using more complex non-counting strategies. While the strategies that develop usually become more sophisticated as children learn more efficient ways of doing mathematics, it is now well acknowledged that at any one time, a child will use a multiplicity of strategies and that often these strategies will not be the most efficient ones a child is capable of performing. Such inefficient strategies persist because while they may be slow, they eventually yield the correct answer (Gould, 2000). When a child is placed under some form of cognitive demand, such as an imposed time limit, mental fatigue or even boredom, they will often revert to a less sophisticated strategy that they know well and can perform with minimal effort. A nine year old explained this to me once while I questioned her about her strategies for addition:

> I know when I just have to add a small number—say five or less—then its fast for me to count by ones. But if its 20 or 30 to add, then I will stop and think of a better way that does not use just ones because I know it will take me too long to count that many. Sometimes I just want to count by ones because it's too hard to think of another way.

I learnt from this little girl that children modify their strategy use according to at least two things: the demands of the mathematical problem and the limitations of their knowledge. Another influence on children's choice of strategy that I have observed during my time in schools is that of textbooks or even teachers themselves. In the

attempt to introduce students to a variety of mental and written methods, instructional material may overemphasise or specify the use of a particular strategy or scaffold (e.g., the empty number line) when students are already working beyond what is specified (see Bobis & Bobis, 2005). The challenge for teachers is to encourage the development of, and consistent use of, more efficient and appropriate strategies for solving mathematical problems without it being "too hard" for children. To do this, it is imperative that teachers not only understand what these strategies are, but how a more efficient strategy becomes a student's *preferred* strategy even when placed in a stressful situation. The diagrammatic representation of Siegler's (2000) overlapping wave theory has helped further my own understanding of how this can be achieved (see Figure 1). I have shared Siegler's theory with practicing and prospective teachers and found it beneficial in explaining how a more efficient strategy can become a child's preferred strategy.

Siegler's (2000) overlapping wave theory is based on three assumptions: (1) children typically use a multiplicity of strategies to solve a single problem; (2) less and more efficient strategies may coexist over prolonged periods of time and not just for short periods of transition; and (3) the relative reliance on existing and more efficient strategies can be changed given *appropriate* experiences. The first two assumptions are represented diagrammatically in Figure 1. The third assumption is addressed later in this paper.
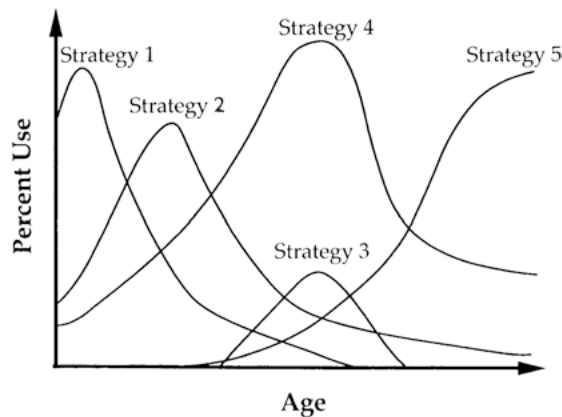
Figure 1. Diagram representing Siegler's overlapping waves model.

It can be seen from Figure 1 that at any one point in time, a student may use a range of strategies. However, the relative frequency with which particular strategies are used over time may vary continuously, with new strategies becoming more prevalent and some more inefficient strategies stopping. By following the path of a single strategy, it can illustrate how some strategies will often be used for a prolonged period of time even after more efficient strategies have been introduced. This can be exemplified by a student who uses counting-on by ones to solve simple addition problems such as 7 + 2 as a 5 year old, and who continues to use the same strategy to solve 47 + 12 as a 10 year old. Siegler suggests that as a child progressively learns more efficient strategies they pass through four dimensions or components of change. These components range from the initial use of the strategy, which in some cases may at first be used at an unconscious level, to a stable, precise and efficient use of the strategy. The four dimensions along which learning occurs include:

1. The acquisition or introduction of a more advanced strategy;

2. An increased reliance or frequency of use of the new strategy within the set of the child's existing strategies;
3. An increased *appropriate* choice of the strategy; and
4. An improved execution of the more advanced strategy that can lead to it becoming increasingly automated.

While this model for strategy development is based on the assumption that children learn by doing, it is important to emphasise that simply drilling the strategies is not enough. Understanding is also crucial. We know that the greater the degree of understanding, the less practice that is required to obtain fluency and to sustain the change in strategy use. Additionally, each new strategy competes for a long time with more familiar strategies, so it may not be used consistently as their preferred strategy for some time and there may be occasions when a child seems to regress in their strategy use. In other words, getting children to move from their current array of preferred strategies (the "here" strategies) to a more efficient strategy (the "there" strategies) is not a straightforward process.

The case of Crystal and multi-digit multiplication

I first met Crystal when her Year 6 teacher asked me to assist with the development of an intervention program for a small group of students in her class. These students were experiencing difficulty with the algorithm for multi-digit multiplication and the teacher was unsure what remediation was needed. This section details the journey to computational fluency of one child from that group.

Frameworks describing developmental pathways of children's thinking strategies for addition, subtraction and single digit multiplication are now quite common (see, Bobis, Clarke, Clarke, Thomas, Young-Loveridge, Wright & Gould, 2005) and some are actually embedded into curricula (e.g., BOSNSW, 2002; Van den Heuvel-Panhuizen, 2001). However, much less is known about multi-digit multiplication. Fuson (2003) reports preliminary research that reveals children use a progression of strategies from (a) direct modelling with concrete materials or semi-abstract drawings, to (b) methods involving repeated addition, such as doubling, to (c) partitioning methods. Partitioning strategies normally include the partitioning of one number or both numbers into tens and ones or partitioning by a number other than 10.

The standard algorithm for multi-digit multiplication most commonly used in NSW primary schools requires a number of steps involving multiplication and addition. It also relies on the answers at each step being properly aligned according to their correct place value. Such alignments can be accomplished without any understanding of a number's true value. In Crystal's case, errors in her multi-digit multiplication were the result of a range of factors. The single-digit multiplication work samples in Figure 2 indicate that Crystal could efficiently solve single-digit computations when multiplying by numbers less than 7. However, she did not know all her multiplication facts from 7 onwards, thus hindering her computational fluency. This was later confirmed in an interview with Crystal. She had memorised most facts to 6×10, but seemed unaware of the commutative property of multiplication. Hence, she was unable to see that 6x8 was the same as 8x6. In addition, the work samples indicate that Crystal was not only making procedural errors when carrying, but that she had little understanding of place value when multiplying by tens. This is a very common error in students' execution of the algorithm for multi-digit multiplication and is generally a result of learning the procedure by rote. To overcome these procedural and conceptual errors, Crystal needed to understand the distributive property of multiplication.

Figure 2. Examples of Crystal's single digit and multi-digit multiplication.

A program of work starting with Crystal's understanding of the commutative property of multiplication was implemented by the classroom teacher. It was decided to strengthen Crystal's knowledge base of single-digit multiplication before moving to the more difficult multi-digit multiplication computations. While this initial instruction spanned a few weeks, it is the understanding of the mathematics underlying multi-digit multiplication that is my focus here. It was during our search for a strategy to help Crystal understand the underlying mathematics that the classroom teacher and I learnt most about Crystal's mathematical abilities and about teaching multi-digit multiplication via a number sense approach.

We soon learnt that if Crystal was going to develop an understanding of the distributive property of multiplication, it needed to be presented in a visual form. Early attempts to explain this property through purely abstract means (e.g., $14 \times 5 = 10 \times 5 + 4 \times 5$) had little success. Visual representations of double-digit numbers became very cumbersome and messy for Crystal, thus making the learning and teaching tedious. It was at this point that we encountered a method involving partitioning of numbers according to their place value and a convenient visual model (Fuson, 2003). We started by introducing Crystal to array's incorporating tens and ones (see Figure 4 for an expanded and abbreviated model of an array). The visual representation supported Crystal's understanding of multiplying all the combinations in two double-digit numbers.
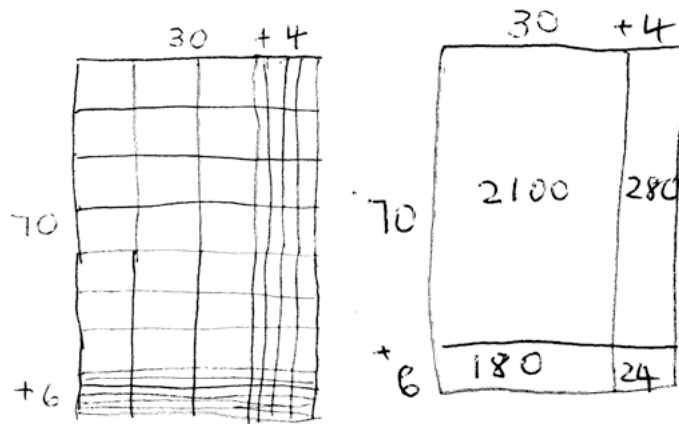
Figure 4. Array structures used to model all combinations in multi-digit multiplication.

The array models scaffolded the introduction of mental strategies involving partitioning, and at the same time provided a convenient representation of the distributive property of multiplication. Within two weeks of instruction, the visual representation of the array was unnecessary and Crystal was able to record her thinking numerically (see Figure 5). As she gained more confidence with this process, Crystal eventually took short-cuts and discarded recordings to the right of the algorithm.

$$
\begin{array}{rcl}
34 & = & 30 + 4 \\
\times 76 & = & 70 + 6 \\
\hline
2100 & = & 70 \times 30 \\
24 & = & 4 \times 6 \\
180 & = & 30 \times 6 \\
280 & = & 70 \times 4 \\
\hline
2584 &
\end{array}
$$

Figure 5. The distributive property is emphasised to assist understanding of the algorithm.

While this sequence of instruction was first introduced to cater for the needs of Crystal and a few other students in the class, the teacher decided to integrate the array model into her regular classroom teaching of multi-digit multiplication. After witnessing the benefits of this process of instruction the teacher interviewed more students from her class to determine their level of understanding of multi-digit multiplication. She was alarmed to find many other students implementing the standard algorithm correctly, but without understanding the underlying mathematics.

Conclusion

High levels of efficiency in computation remain a goal of our mathematics curricula; the process by which it is achieved needs to take account of how students develop a sense of number. The path to computational fluency is not a straight-forward one for most students. However, it is clear that the promotion of number sense is critical to a basic understanding of mathematics and to a child's ability to compute easily.

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Collaborative development of a framework for numeracy: A case study[+]

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A major curriculum reform in Tasmania has provided the impetus for teachers to reconsider the place of numeracy in the curriculum. As one of the Key Elements in an Essential Learnings Curriculum, "Being Numerate" has been the focus of professional learning throughout the state, addressing teaching, learning and assessment. This paper describes a project implemented at one primary school where teachers negotiated with researchers to develop a framework for the teaching of numeracy in the school, drawing on curriculum support materials and teachers' understanding of the school context.

Introduction

Curriculum reform in Tasmanian schools has centred around the implementation of an Essential Learnings framework (Department *of* Education, Tasmania (DoET, 2002; 2003). This framework has provided a catalyst for professional learning programs and the development of centrally produced support materials designed to assist teachers in implementing the new values-based curriculum.

Being Numerate is recognised in this curriculum framework as a cross-curricular understanding, and one that is important for development in all students:

> Being numerate involves having those concepts and skills of mathematics that are required to meet the demands of everyday life. It includes having the capacity to select and use them appropriately in real life settings. Being truly numerate requires the knowledge and disposition to think and act mathematically and the confidence and intuition to apply particular principles to everyday problems. (DoET, 2002, p.21)

This shift in emphasis, together with a set of defined outcomes and standards (DoET, 2003) has required teachers to reconsider the teaching of numeracy, in a climate where they have been met with a number of other challenges. These challenges have included the need to adopt different planning practices — using the features of the Teaching for Understanding (TfU) framework (Blythe, 1998) and where possible to plan cross-curricular and collaborative units. Teachers have also needed to think outside of the traditional Key Learning Areas as organisers and, with content not prescribed, consider what they will teach and importantly, why they are teaching it. This is then linked closely with how the teaching will occur — closely matching purposes with appropriate teaching, learning and assessment activities.

The MARBLE (Mathematics in Australian Reform Based Learning Environments) research project was conceived in the context of this curriculum reform, focussing on

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the Middle Years (5–8). Recognising this as a critical time to support teachers as they work through the challenges outlined above, the project aims to improve student numeracy outcomes through provision of teacher Professional Learning. The importance of paying attention to teaching and learning practices required for the success of curriculum innovation is not new. The work of Fullan (1993) addressed this in the broader context of Educational reform, with Clarke (1994) specifically addressing this need with respect to supporting reform amongst mathematics teachers.

The researchers recognised that there are a number of important factors that contribute to successful Professional Learning. Two key features that have been identified in the seminal work by Hawley and Valli (1999) are the involvement of teachers in the identification of what they need to learn, the process to be used and the facilitation of collaborative problem solving. In terms of mathematics, Schifter (1998) found that professional learning experiences in which teachers were engaged with the content of the mathematics curriculum that they taught, in ways that challenged and deepened their own mathematical understandings, assisted them to make significant changes in their classroom practice. The importance of involving teachers in the professional learning process, and the need for collaboration is further borne out in the review of Ewing et al. (2004).

In order to address specific areas of concern for individual teachers and schools involved in the MARBLE project, in the manner described above, schools were invited to participate in school-based case studies. The focus of these was negotiated between teachers in schools and the researchers. In the present study, teachers expressed a need for a whole school framework for the teaching of numeracy consistent with the new curriculum outcomes and standards. It was agreed that the development of this framework was to take into account both the curriculum documents and the teachers' own knowledge of context and current practice.

Method

The underpinnings to this research lie at the intersection of pragmatic and participatory research paradigms (Creswell, 2003). Steeped in the "real world" of the teacher making everyday decisions about curriculum content, the researchers have adopted a collaborative approach with the teacher participants, involving them in design, collection of data and data analysis. The final phase of the project will involve teachers and researchers producing a document that can be used to outline a scope and sequence for the teaching of numeracy in the school.

Consistent with a case study approach, the project was sited in a single school which conforms to Stake's description of a "bounded system" that is "complex" and "dynamic" (Stake, 1980). The study took place at Valleyfield Primary School, a government K–6 school situated in a rural area in Southern Tasmania.

The project consisted of the following phases:
1. Identification of the school focus through consultation between teachers and the researchers. A key participant in this stage was the coordinating teacher at the school who worked as part of the research team.
2. Preparation, by the researchers, of rubrics describing teaching emphases at each standard for each strand of the numeracy/mathematics curriculum. These rubrics used the strands defined by National Statements on Mathematics for Australian Schools (Curriculum Corporation, 1990) and the curriculum support document "Teaching Emphases" prepared by the School Education Division of the DoET, (DoET, 2005). The latter outlines the teaching and learning focus for each of the

strands at each of the developmental standards. For the primary school, Standards One to Three are appropriate for most students, with each standard being divided into three divisions (lower, middle and upper). An example of the audit rubric is reproduced in Figure 1 .

3. The undertaking of an audit of numeracy teaching and learning in the school against the rubric by all staff at the school. This occurred after the documents were introduced by the researchers at a staff meeting. Teachers were invited to take away all rubrics that were relevant to the students in their classes. In addition to the audit, teachers were invited to add any teaching and learning activities they had found to be particularly successful or relevant and any resources that were useful.

Standard/ Progression	3 Middle
Thinking and acting mathematically	Focusing on reflecting, refining and sharing strategies for problem solving. Using different strategies to comparing strategies to determine efficiency. Continuing to build specific mathematical terminology and recoding methods. Developing confidence with mathematical language and recording methods to communicating their numerical understandings to a wider audience.
Number	Continuing to focus on mental computation with multiplication and division. Focusing on refining and sharing mental computation strategies in addition, subtraction, multiplication and division. Building a deep understanding of common fractions through the use of a wide range of approaches (not over emphasising "pizza" fractions for example) including an explicit focus on partitioning. * Beginning to independently select a strategy to selecting the most efficient mental computation strategies and justifying their choice. Exploring a range of informal written computation methods for addition and subtraction of 2 digit numbers to establishing a *secure written method* for addition and subtraction
Measurement	Developing skills in measuring length, mass, time, angle and capacity Providing opportunities to quantify attributes [length, mass, time, angle, volume (capacity) and temperature] for the purpose of comparing and/or ordering. Using formal measurement units (and tools) with teacher guidance to using the appropriate unit (or tool) for length, mass, time, angle, volume (capacity) and temperature more independently
Patterning and algebra	Making predictions based on the growth of a pattern e.g. 'If one costs fifty cents, then I can find how much it costs to buy 20 by multiplying fifty cents by 20. So, 10 lots of $0.50 is $5 and I can double that Making predictions using simple patterns based on costs to making predictions using a broader set of contexts. Systematically recording patterns in a table to connecting how a table, a set of diagrams and words can represent the same pattern.
Proportional reasoning	
Space	Focusing on how 3D objects are constructed from 2D nets. Identifying shapes to constructing and deconstructing shapes and objects based on specific attributes e.g., "What shapes would I need to make a model lighthouse?" Reading maps to constructing simple maps taking into account orientation.
Data and chance	Exposing students to a range of ways to record data and developing their confidence to construct different graphs and charts. Collecting and representing data in simple bar graphs and charts to using an increasing range of data representations — stacked dot plots and pictographs (including use of technology). Interpreting data in order to quantify differences between items in the data set. Analysing data to locate measures of centre (median and mode) and spread (range) with teacher support. Exploring fairness through experiences with chance devices such as spinners, dice, cards, etc.

Figure 1. Example rubric used for audit.

1. Initial analysis of the audit was used to produce a summary of teaching and learning in the schools and which was subsequently presented to staff for comment.

2. Supplementation of the audit data through individual staff interviews.
3. Analysis of data by Strand and Grade and preparation of a draft document.
4. Presentation of Draft sequence to staff for further refinement and for reference to revised curriculum being produced through the DoE.

Results

The initial audit completed by the teachers was analysed against the "Teaching Emphases" document based on the Being Numerate Outcomes and Standards (DoET, 2004). The results are summarised in Figure 2. This graphic representation indicates the proportion of each of the strands (represented by colours consistent with the audit documents) covered in each of the grades at Valleyfield Primary.

Analysis of the audit indicated that the majority of the curriculum for Standard 1 was covered by the Prep teachers. Grades 1–2 appeared to concentrate on curriculum foci for Standard 2 (Lower), Grades 3–4 on Standard 2 (Upper) and Standard 3 (Lower) and Grades 4, 5 and 6 covering material predominantly from Standard 3. Interestingly, the two teachers with composite Grade 5 and 6 classes indicated they also covered material from Standard 2 (Upper).

The extent to which teachers provided additional information in the audit process varied, however the Prep teachers provided comprehensive information on both teaching and learning activities and resources that they had found useful. Nevertheless, many teachers took this opportunity to share resources, and in some cases noted units or cross-curriculum contexts for numeracy that they had found to be successful:

Where is number in our world?
Attendance
House numbers
Telephone

Measurement: Use with integrated units
Growing plants
Body measurements

Two significant findings from this audit concern what had been indicated as not covered by the teachers. No teacher had indicated coverage of the curriculum designated at the Standard 2 (Middle) level. Additionally, there were substantial areas of the curriculum that did not appear to be covered. Very little Pattern and Algebra was listed as being taught above Standard 2 (Upper). Similarly aspects of Number and Thinking Mathematically were not covered in either grades 4, 5 or 6.

Raising alarm bells, these findings were looked at more closely with a view to uncovering reasons for gaps in the curriculum. It became clear that not all teachers had returned their audit rubrics, and in some shared classes, only one of the teachers had completed the audit. In the latter instance, teachers had under-reported what the class had covered, documenting only material taught during their teaching time. In consultation with the coordinating teacher, it was decided to present the data and ask teachers how best they believed we could address the issue of missing data.
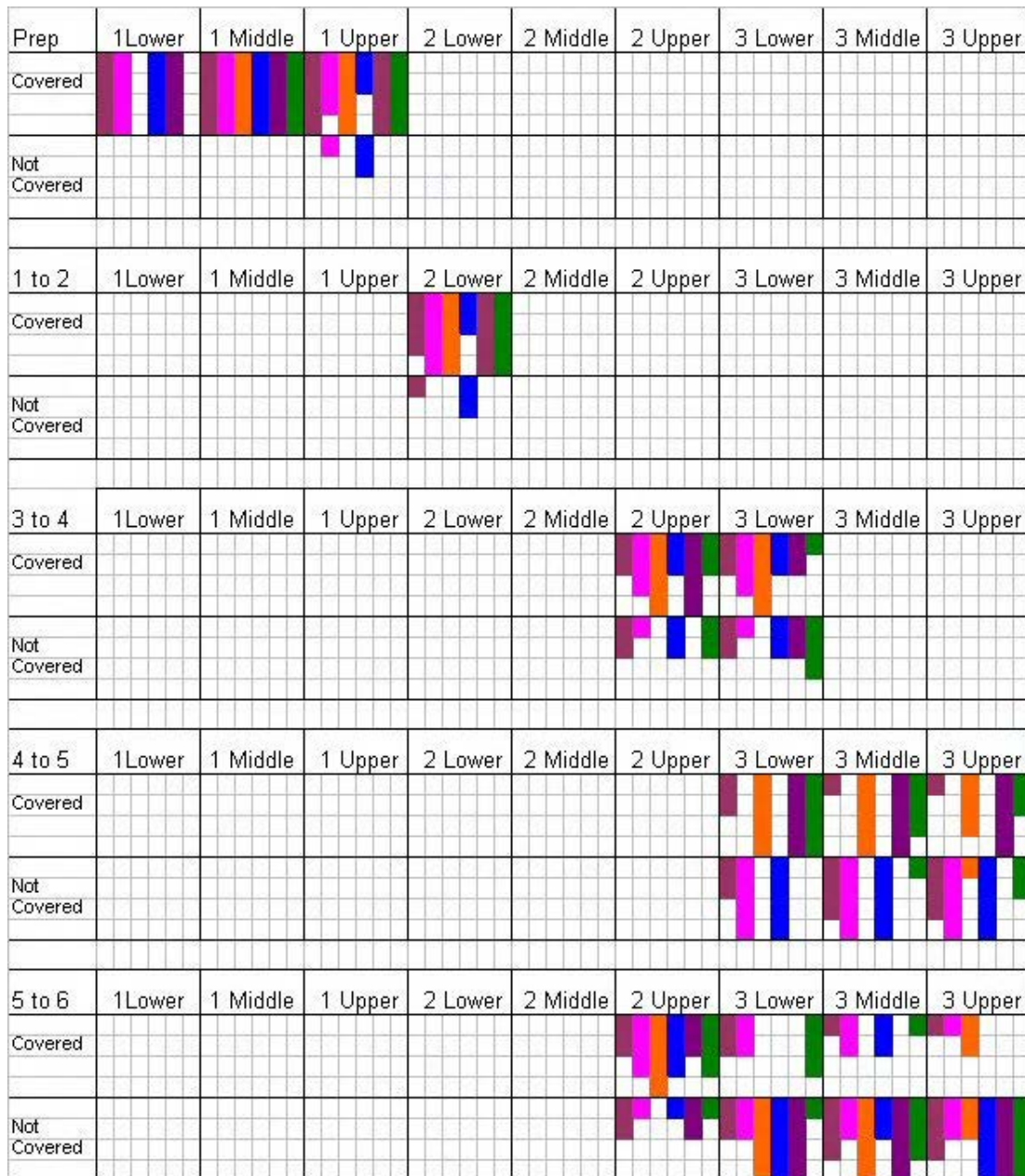
Figure 2. Overview of coverage of each outcome and standard from first reporting cycle.

Presentation of the data in overall form, and Standard by Standard, occurred at a staff meeting and staff were asked to indicate how to proceed. The comprehensive nature of the information provided by the staff of the Prep classes was used as a model, with staff agreeing to pursue the audit via individual teacher interviews with the researcher. Teachers were provided with relief and asked to bring their mathematics plans for the year to the interview.

The teaching, learning and assessment covered by each of these teachers was clearly identified during the interviews and used to amend the audit and identify areas that were poorly represented in the curriculum. This two stage process enabled teachers themselves to recognise areas of the curriculum that were missed, or received scant attention. In particular, Chance and Data, Space and Pattern, and Algebra were singled out by upper primary as receiving less attention, with teachers focussing more on Number and aspects of Measurement.

In addition to informing the audit process, the teacher interviews provided valuable supplementary data to the researchers. This included an insight into the issues faced by teachers in this context and informative comments on the process used during the professional learning program. All teachers interviewed could see the value in a school wide document outlining the numeracy curriculum. In particular, teachers of the higher grades faced with struggling students saw value in knowing what material had been covered previously, and how this had been covered. Teachers new to the school also found it difficult to plan their mathematics program in the absence of any whole school plan.

In terms of the process of data collection, one teacher commented that he had found the process valuable because he "was put on the spot." The opportunity to reflect on his own mathematics plan, and its implementation, was clearly personally valued. Although timing of the interviews (in the last full week of the school year) was slightly problematic, all teachers indicated they found the process straightforward and thought the exercise worthwhile. Teachers also indicated support of the structure of the professional learning program MARBLE, seeing the individual school Case Study as complementing the cluster-wide professional learning days. Indeed, all teachers indicated that they had successfully incorporated ideas and activities from these days into their classrooms.

The amended audit, including teaching and learning ideas and teacher resources, was then used by the researchers to construct a draft framework. The curriculum documents "Teaching Emphases" (DoET, 2005) and the Outcomes and Standards for Being Numerate (DoET, 2004) were used as a basis and interwoven with the teachers' own teaching emphasis for each grade. Further consultation with Tasmanian curriculum documents "K–8 guidelines" (Department of Education and the Arts Tasmania [DEAT], 1992) and the Margate Primary School Whole School Plan (at http://ltag.education.tas.gov.au/focus/beingnumerate/Margateplan.doc) allowed the development of Key Understandings in each of the strands for each grade. The understandings have been presented in conjunction with the relevant teaching emphases for the grade together with possible activities. These activities were primarily drawn from teachers' reports of their own classroom activities, with additional activities sourced from the K–8 Guidelines (DEAT, 1992), where necessary. An example of the format is reproduced in Figure 3.

Year: Five to Six
Key Understanding
Understands how to select and justify effective mathematical strategies and choose the most appropriate strategy for communicating information and solving problems in a variety of situations.
Possible Activities
Mathletics
Problem solving – set formulae
10 mental maths problems first thing each lesson
Heads and Tails –lesson 41 (Maths300)
Farmyard Friends –lesson 47 (Maths300)
Links to Numeracy Standards: Teaching Emphases

3 Lower	3 Middle	3 Upper
• Focusing on building the 'mathematician's tool box' [See Addendum] of strategies to use when attempting a problem. • Explicitly teaching problem solving strategies will be a major teaching focus throughout standard 3. • Teaching of mathematical terminology and recording methods.	• Focusing on reflecting, refining and sharing strategies for problem solving. • Continuing to build specific mathematical terminology and recoding methods. •	• Consolidating students' problem solving strategies. • Refining and extending mathematical terminology and recording methods. • Giving opportunities for student's to challenge each other's thinking. •

Figure 3. Excerpt from framework document: Thinking and Acting Mathematically, Grades 5/6.

Discussion

The numeracy teaching framework has been produced in collaboration with the teachers at the Valleyfield Primary School, in response to a perceived need for a school wide progression document. The progressions, based on the teachers' own reporting of their numeracy practices in their own classrooms, have been divided into strands and linked to the Tasmanian Being Numerate standards (DoET, 2004).

The document has been presented in two ways: by strand and by year group. Each year and progression has a Key Understanding, representing the primary numeracy concept for each year and numeracy strand. Following this are some possible activities that may aid development of this understanding. Presenting the results by year/class group provides a 'one stop' resource for teachers to plan their numeracy units across the year. As these are primarily based on teachers' current numeracy plans, this does not radically change the emphasis of their present teaching and recognises their expertise in the context in which they are working.

The alternative presentation, by strand, provides the opportunity for teachers to see the progression of students' understanding through the primary years of schooling. Firstly, this provides an expectation of knowledge and understandings for teachers at the beginning of the academic year. Secondly, by looking at earlier and later year groups it provides a resource identifying possible activities for those students that may be falling behind the majority of their class, for classes finding a concept difficult to grasp, and for those exceeding standard expectations. In this format, teachers can use the document to ensure their students continually improve and extend their numeracy skills.

Clearly, the linking to particular numeracy standards can be seen as a limitation. Since beginning this project a new Education Minister has mandated a revision of the curriculum delivered in Tasmania government schools, and a refinement of curriculum is currently underway. However, the use of Key Understandings-supported numeracy standards is a model that can be adjusted to incorporate future changes in numeracy outcomes. This document, therefore, should be viewed as a work in progress.

This school-based project, driven by pragmatic needs of teachers, has been enthusiastically received by the teaching staff. The planning has been aligned with the recommendations of Hawley and Valli (1999), involving teachers in identification of issues and in collaborative problem solving. Not withstanding the support from the teachers, this Case Study has also uncovered the importance of the context in influencing school based research. In this case, the broader context of curriculum reform at state level has provided both the impetus for the project and barriers to its progress. Introduction of a new (refined) curriculum at the concluding stages has halted the implementation phase with researchers waiting for an opportunity to align this work with the new documentation.

The context of the school itself is also of interest. External projects, regardless of relevance, impinge on the time of teachers who, in Tasmania at present, are feeling the strain of a major reform process. Using an interview process where teachers were released from class (and teaching relief supplied) proved to be an effective way of involving teachers in the project without undue additional work. As well as providing information with respect to teaching and learning, the interview process itself was able to uncover aspects of context which are useful in planning support for teachers. At Valleyfield Primary, issues such as removal of teachers from class for administrative duties, and the appointment of short term contract and permanent replacement teachers were clearly seen by the teachers themselves as impacting on the program for students.

These particular contextual issues provided even greater support for the objectives of this particular project, as a way of addressing the needs of teachers in planning for quality teaching and learning.

Conclusion

The documents produced to date are seen as important works in progress that will be refined by the teachers and researchers during the current school year. At time of writing, a revised curriculum for Tasmanian schools is being written for implementation in 2008. As a first step, key ideas in the draft document are being reviewed in light of the revised Numeracy curriculum document. Following this process, teachers will be asked to critique and amend the document. The updated version of the framework, together with an outline of this refinement process will be presented in July.

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An investigative approach to teaching secondary school mathematics[+]

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A new syllabus for mathematics in Years 1–10 was launched in Queensland in 2004. The challenge for teachers implementing this syllabus lies not only in using a new structure for curriculum planning, but also in designing learning experiences that take an investigative approach to "working mathematically". This paper describes a professional development project that supported a group of secondary mathematics teachers in implementing the new syllabus. A case study of one pair of teachers shows how they developed their planning, pedagogy and assessment to engage students in meaningful mathematical investigations.

Queensland's new *Mathematics Years 1–10 Syllabus* (Queensland Studies Authority, 2004) has an outcomes focus that gives it a different structure from syllabuses previously developed in this state. Instead of specifying what should be learned in particular years or grades of school, the mathematics syllabus is organised around key learning area outcomes that describe how students think, reason, and work mathematically, and core learning outcomes that describe what students should know and do with what they know in the strands of Number, Patterns and Algebra, Measurement, Chance and Data, and Space.

The challenge for teachers implementing the new syllabus lies not only in using the new structure for curriculum planning, but also in designing learning experiences and assessment tasks that take an *investigative approach* to "working mathematically". This was the focus of a professional development project commissioned by the Queensland Department of Education and the Arts and undertaken by researchers from The University of Queensland from October 2005 to February 2006 in conjunction with teachers from secondary schools in the Mackay District. The main aim of the project was to explore the potential for the new syllabus to support improved classroom practices by documenting innovative aspects of, as well as hindrances to, successful implementation.

This paper describes the professional development program and its impact on a pair of teachers in one school. The purpose of the paper is to give these teachers a voice in reflecting on their own professional growth and to identify issues that are critical to the success of professional development initiatives such as this. The next section takes a research perspective on factors that influence professional learning. This is followed by a brief outline of the professional development program in which the teachers and researchers jointly engaged. A case study of the pair of teachers shows how they developed their planning, pedagogy and assessment to engage students in meaningful mathematical investigations. Implications are then considered for extending similar

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professional development opportunities to secondary mathematics teachers in other schools.

Teacher professional learning

Researchers have identified a range of factors that influence teacher professional learning. Rather than considering each separately, it is helpful to organise these factors into three "zones of influence". The first zone represents *teacher knowledge and beliefs*, and includes teachers' disciplinary knowledge, pedagogical content knowledge (knowledge of how to represent concepts and to use examples and analogies, as described by Shulman, 1986), and beliefs about their discipline and how it is best taught and learned. The second zone represents constraints within the *professional context*. These may include teacher perceptions of student background, ability and motivation, curriculum and assessment requirements, access to resources, organisational structures and cultures, and parental and community attitudes to curriculum and pedagogical change. The third zone represents the *sources of assistance* available to teachers that define which teaching actions are specifically promoted. This assistance is typically provided by colleagues and mentors in a school or by formal professional development activities.
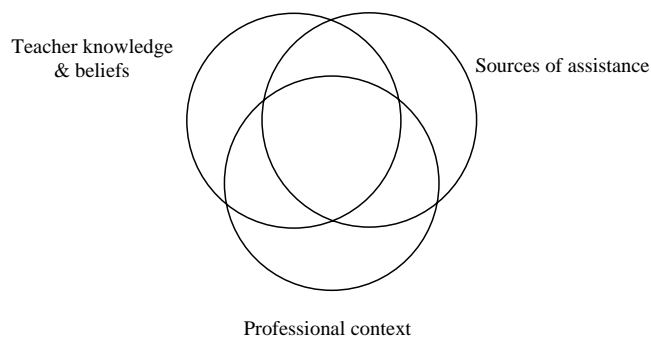
Figure 1. Three "zones of influence" in teacher professional learning.

To understand teacher learning we need to investigate relationships between these three zones (represented by the overlapping circles in Figure 1). Professional learning may be most effective when teachers experience enough challenge to disturb the balance between their existing beliefs and practices, but also enough support to think through the dissonance experienced and either develop a new repertoire of practice or a new way of interpreting their professional context that fits with their new understanding.

The professional development program

Four pairs of teacher from four schools in the Mackay region volunteered to participate in the project. Two schools were in Mackay, one was in a small rural town, while the fourth school was located on the coast approximately 125 km from Mackay. The university-based researchers made three visits to Mackay to work with the whole group of teachers, each time for two consecutive days. During these visits we gathered information on the teachers' *knowledge and beliefs* and their *professional contexts* (see Figure 1) via questionnaires and structured group discussions.

As a *source of assistance* that was deliberately promoting new teaching approaches (see Figure 1), the professional development program recognised the importance of providing teachers with authentic, practice-based learning opportunities that included examples of mathematical investigations, opportunities to experience these investigations as learners themselves before planning their own investigations and trying them out with their students, and opportunities to share their ideas and experiences with colleagues, including the challenges encountered and their insights into the process. An action research approach (Loucks-Horsley, Love, Stiles, Mundry & Hewson, 2003) was adopted to provide for iterative cycles that would allow the teachers to reflect on the initial experience teaching their investigative unit and apply those experiences and learning to a second iteration, and for an on-site visit to each teacher's classroom to gain insight into their classroom and school context.

The case study that follows illustrates how relationships between these three "zones of influence" shaped the professional learning experiences of one pair of teachers.

Sugartown[7] State High School: Engaging learners

Skye and Chris had been teaching for only three years. Skye qualified as a middle years teacher and taught primary school for eighteen months before moving to the secondary school so she could specialise in mathematics teaching. She was motivated to volunteer for this project because she was assigned to teach a new subject, Practical Mathematics, designed for Year 8 students who were not achieving success in regular mathematics classrooms. Chris had completed a degree in secondary mathematics teaching, and previously had almost always taught senior mathematics classes. He volunteered for the project because he was looking for help in planning new programs and devising new forms of assessment.

Knowledge, beliefs, professional context

Skye's and Chris's questionnaire responses showed they held very similar beliefs about the nature of mathematics, and mathematics teaching and learning. For example, they both agreed that there are many ways of interpreting and solving a problem. However, Skye also agreed with more traditional views that in mathematics something is either right or wrong, and solving a problem usually involves using a rule or formula. Both expressed agreement with strategies such as showing students many ways of looking at a question and negotiating meanings through class discussion. Yet they also saw their role as showing students the proper procedures for answering questions. Both agreed with emphasising understanding rather than getting the right answer, encouraging students to build their own mathematical ideas, and using manipulatives and real life examples; but they were uncertain about the benefits of more traditional approaches such as memorisation and practice. These responses suggest that Skye and Chris were interested in moving towards more student-centred, investigative teaching practices, and that they needed to try out these practices with their own classes to find out whether this would lead to improved learning.

Skye and Chris considered their students' mathematics achievement and social background to be the key characteristics influencing their professional learning goals. Many students entering Year 8 were achieving at below the Year 7 numeracy benchmark, and both teachers were very concerned that these students "had no idea about a lot of things" when they started secondary school. Both teachers also referred to

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[7] Names of schools and teachers are pseudonyms.

the students' family and community context. In common with many small rural communities, Sugartown was experiencing social and economic challenges exacerbated by the town's proximity to the thriving city of Mackay. Many students also came from low socio-economic status, single parent families, and the teachers felt that, on the whole, parents did not care much whether their children were learning anything at school. In these circumstances, they struggled to interest students in academic work.

Sugartown State High is a medium sized school with about 600 students. The Head of the Mathematics Department was very supportive of Skye and Chris's desire to be involved in this project, organising their timetables so they had spare periods at the same time that they could use for collaborative planning.

Goals

Skye and Chris agreed that the most frustrating obstacle in their professional context was the students themselves, and their apparent lack of interest in learning. This was evident, for example, in their disruptive and uncooperative behaviour, and their frequently stated belief that they were "dumb" and simply could not do mathematics. These teachers' personal histories and the experience of teaching unmotivated students led them to formulate a goal of *engaging learners*, or, as Skye explained, "for them to learn maths without being terrified of it". Both teachers saw investigations as a way of presenting mathematics differently that would allow them to make mathematics more interesting for students by engaging them in purposeful tasks with real world relevance.

Skye and Chris planned and implemented two units of work with Year 8 Practical Mathematics classes. For the first unit, taught towards the end of 2005, Chris volunteered to give up one of the spare lines in his timetable to work with Skye in team teaching her Year 8 class. Together they planned and taught a unit on Shopping involving operations with whole numbers, common and decimal fractions, and money. The following year, Chris asked to be given his own Year 8 Practical Mathematics class, and both teachers team-taught the two classes. The first unit they taught together, "School Rage", asked students to design and administer a survey to find the school's favourite top 20 songs.

Shopping investigation

In developing this first investigation, Skye and Chris decided that the most important aspects to consider were the students' previous experiences in learning mathematics, and the school's geographical and social contexts. They drew on the experience of families having to travel to Mackay to do their grocery shopping. The task was to investigate which situation is more economical, purchasing groceries in the local supermarket in Sugartown or driving to Mackay to buy groceries. In their final report on the investigation, students had to include a supermarket shopping list of at least 20 items that their family would normally purchase, a price list for these items from the local supermarket in Sugartown and a supermarket in Mackay, a price comparison between the two supermarkets, a calculation of the cost of fuel in driving to Mackay, and a comparison of advantages and disadvantages of shopping at the two chosen supermarkets.

Although Skye and Chris saw evidence that around half the students were much more engaged with the mathematics and more willing to persist with tasks than previously, they were not entirely satisfied with this investigation because they realised that the topic of "grocery shopping" was not of much interest to young adolescents.

This meant that many students did not complete the investigation and did not hand in the final report.

"School Rage" investigation

For their next attempt, Skye and Chris took advantage of local resources in the form of the school based radio station and their students' universal interest in popular music. In the "School Rage" investigation students had to create a Top 20 song list for the school radio station, based on a survey of fellow Sugartown State High School students. To make the task more realistic, Sky and Chris gave students a letter from the "radio station manager" (in reality one of the mathematics teachers) asking them for their assistance in providing input into the design of a new radio program similar to the Rage Top 20. The teachers also told their classes that the group submitting the best quality report would have their Top 20 songs played on the radio station during a designated lunchtime. Thus the task had an authentic purpose and a real audience comprising the entire school community.

Core learning outcomes demonstrated through successful completion of this task were related to designing and carrying out data collections to investigate their own and others' questions, using data record templates, organising data and creating suitable displays, making comparisons about the data, and working with whole numbers, fractions and percentages.

Classroom observation confirmed Skye and Chris's judgment that students were very much engaged in the investigation. For example, although the desks were arranged in rows, students formed informal groups to discuss wording of survey questions and sampling techniques and often moved around the room to speak to other peers or one of the two teachers. The teachers listened to conversations between students, and intervened where necessary to answer questions, repair faulty understanding, or offer suggestions. Skye also brought her laptop computer to the classroom with several relevant websites saved so that students could browse them offline (as there was no live Internet connection available): she suggested they could find out which songs were currently topping the charts based on CD sales or polling of radio station listeners.

Evaluation

Students' comments during lessons showed they recognised the difference between investigations and the more traditional, teacher and textbook centred lessons they were accustomed to. The following exchange, overheard during my visit to the school to observe a "School Rage" lesson, is typical of students' response to the new approach adopted by Chris and Skye:

Student:	Why are we doing this? This isn't maths!
Chris:	What do you mean?
Student:	It's like we're doing SOSE (Study of Society and Environment) — this isn't like a normal maths classroom.
Chris:	Is that a negative thing?
Student:	No, I normally hate maths but I don't mind doing this.

The teachers' own evaluation of the investigative units identified not only the benefits for the students (engagement, confidence, alternative opportunities to demonstrate their learning) but also the challenges the teachers faced and how they overcame them. They were now spending more of their time in class responding to the unanticipated ways in which students tackled investigations, often by asking questions

to scaffold students' thinking, such as "What does it mean if you include the same person twice in your survey?", "What if this person votes for two different songs?", "What will you do if you end up with a list of 60 songs?" However, as Skye pointed out, the unexpected is to be welcomed as this is often a sign that students are thinking in more sophisticated ways than the teacher thought possible.

Skye and Chris identified several reasons why they had been successful in implementing an investigative approach. They frequently emphasised the importance of taking into account the students' prior experience and interests, and the local context of the school and community. Access to mathematics teaching resources was also critical, especially examples of investigations designed by other people. So too was access to human resources in the form of a supportive school administration team, a network of like minded mathematics teachers across the schools participating in the project, and their teaching partner. Planning and teaching as a team, rather than individuals, was a significant benefit for both teachers because they recognised that this reduced their workload, expanded their repertoire of teaching strategies, and provided opportunities for mutual observation and feedback.

Support for development of new teaching approaches
was consistent with teacher knowledge & beliefs

Teacher knowledge & beliefs:
- Well qualified in mathematics and mathematics education
- Mathematical beliefs are student-centred and non-rule based

Sources of assistance:
- Project offered immersion in mathematical investigations, support for curriculum implementation in school, collaborative professional partnerships

Professional context:
- Low achieving students, lacking confidence, streamed into Practical Mathematics in Year 8
- Limited access to material resources
- Yrs 1-10 mathematics syllabus supports investigative approach to teaching and assessment
- HOD arranged timetables to allow teachers to plan together and team teach
- Little parental support for children's learning

Productive tensions between beliefs, student characteristics, and syllabus led to formulation of goals

Investigative approach was feasible in context because of "nothing to lose" approach to student learning & HOD's organisational support for team teaching

Figure 2. Relationships between professional growth factors for Skye and Chris.

Skye and Chris's professional learning experience is summarised by the relationships between their knowledge and beliefs, professional context, and sources of assistance shown in Figure 2.

Implications and critical issues

Each of our case studies documented a different configuration of teacher knowledge and beliefs, professional contexts, and sources of assistance that came together to shape opportunities for professional learning that involved an investigative approach to "working mathematically". Although the four pairs of teachers worked in different professional contexts that offered both support for, and hindrances to, innovation, they were able to draw on their knowledge and beliefs and the sources of assistance available to them to plan and implement teaching approaches consistent with the intent of the new syllabus.

There are several critical issues that must be taken into account when considering how to extend similar professional development opportunities to secondary mathematics teachers in other schools (Loucks-Horsley et al., 2003). One issue concerns the need for *building a professional culture* characterised by a strong vision of learning and collegial interactions between teachers. A second issue involves *developing leadership* in teachers who have the capacity to improve the quality of teaching and learning in their schools. Often the most powerful leadership exercised by teachers is simply in modelling new practices for colleagues to demonstrate that they actually work with students. *Building capacity for sustainability* is necessary to ensure that any changes achieved within the life of a professional development project are sustained after it ends. Similarly, *scaling up* is a vital concern for education systems as teachers and school districts implement new teaching and learning approaches. Finally, *gaining public support* for mathematics education is necessary for building consensus around curriculum and pedagogical reform, thus leading to a more informed public understanding of effective methods for teaching mathematics and of the role of mathematics in preparing young people for productive work, leisure, and citizenship.

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Adventure and adolescence: Learner-generated examples in secondary mathematics[+]

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The Tasmanian document "Essential Learnings" offers opportunities to change radically the teaching and learning of mathematics. In my talk I draw on the learning theories of Vygotsky, and on the psychology of adolescence, to give examples of tasks which afford the generation of abstract mathematical concepts and simultaneously enhance adolescents' feelings of self worth and autonomy. I shall show how even the most esoteric mathematics can be taught in ways that include all students, and that adhere to the principle of essential learning, without "dumbing down" the subject.

In this paper I will show how adventure in mathematical learning relates to the adolescent project of negotiating adulthood. Further, I will relate this to some of the "essential learnings" identified by Tasmanian educational policy-makers by illustrating how autonomous thinking might be embedded in the teaching and learning of mathematics.

This kind of comment is usually taken to mean that mathematical tasks need to be "relevant." Then the word "relevant" is often taken to mean that adolescent interests, (such as music, skateboarding and pocket money), should provide contexts for mathematical tasks — otherwise they will not be interested. The consequence of this line of thought can be that mathematical calculations are dressed up in artificial contexts, yet adolescent students are not fooled by this. A more interesting consequence is that students can be asked to solve "real life" problems in the classroom in the hope that through this process they will learn some mathematics. Thus they calculate batting averages, compare journeys to school, divide *n* dollars between *m* people, but these experiences do not necessarily lead to further knowledge of means, graphs, gradients, or ratio.

We know, thanks to the work of Freudenthal, that we can explain this problem not in terms of poor teaching or lack of learning effort but in terms of the difficulty (Lave would say the impossibility (Lave & Wenger, 1991)) of lifting the ways we think in one context and applying them in other contexts (Freudenthal, 1973, p. 130). When we compare pocket money we are in the world of money and adolescent notions of fairness, not in the abstract world of ratio; when we compare journeys to school we are imagining ourselves and our mates arriving at the school gate, not constructing images of time-distance relationships (Coleman & Hendry, 1990, p. 48). Shifts from proximal, *ad hoc*, methods of solution to abstract concepts are hard to make and need deliberate support — indeed this is what is at the heart of Vygotsky's insistence that talk is a necessary aspect of learning "scientific" concepts (1978, p. 131), otherwise one gets

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stuck with intuitive and everyday notions such as "multiplication makes things bigger" or "the bigger the perimeter the bigger the area" (Fischbein, 1987). This shift towards seeing abstract patterns and structures within a complex world is typical of adolescent development (Coleman & Hendry, 1990, p. 47) but the verbal and kinaesthetic socialised responses to sensory stimuli (including mathematics questions) which have satisfied teachers at elementary level have to be put to one side.

Realistic tasks — tasks that mimic the ways we think in complex, messy, everyday situations — can provide contexts in which mathematics can be learnt, so long as the ideas encountered while doing the tasks can then be related to overarching mathematical ideas and universal structures through paying attention to patterns, properties and relationships. This is what (Treffers, 1987) has called "vertical mathematising." Furthermore, the processes of working with such tasks appeal to adolescents because they provide room for adolescent concerns about identity, belonging, being heard, being in charge, being supported, feeling powerful, understanding the world, and being able to argue in ways which make adults listen (Boaler, 1997).

Such tasks can indeed provide contexts for inquiry, as defined in the Tasmanian curriculum document, which includes "identifying and clarifying issues, and gathering, organising, interpreting and transforming information. It encompasses the processes of creatively, imaginatively and inquisitively thinking about possibilities; analysing, synthesising and evaluating proposed solutions; and explaining and justifying decisions. The skills of inquiry can be used to clarify meaning, draw appropriate comparisons and make considered decisions."

Historically, mathematics has been inspired by observable phenomena, and mathematicians develop new ideas by exploring and inquiring. It is also possible to conjecture relationships from experience with examples, and thus getting to know about general behaviour. But mathematics is not essentially an empirical subject at school level. Its strength and power are in its abstractions, its reasoning, and its hypothesising about objects which only exist in the mathematical imagination. Many secondary school concepts are beyond observable manifestations, and beyond intuition. Indeed, those which cause most difficulty for learners and teachers are those which require rejection of intuitive sense and reconstruction of new concept-based images and understandings. Examples of these problematic topics include probability, proportional relationships, non-linear sequences, symbolic representations and the wretched adding of fractions. In mathematics, inquiry alone cannot fully justify results and relationships, nor can decisions be validated by inquiry alone.

I could argue that, for the adolescent, this can be the beginning of the end of mathematical engagement. If I cannot understand the subject by seeing what it does and where it is and how it works, but instead have to believe some higher abstract authority that I do not understand, then it holds nothing for me. But this misses the point. The authority of mathematics does not reside in teachers or textbook writers but in the ways in which minds work with mathematics itself (Freudenthal, 1973, p. 147; Vergnaud, 1997). For this reason, mathematics, like some of the creative arts, can be an arena in which the adolescent mind can have some control, can validate its own thinking, and can appeal to a constructed, personal, authority. In mathematics, I can always support my thinking by looking at my work in a different way and adjusting it if necessary or seeking help on my own terms. Thus, in mathematics, there is always the possibility that learners can be absolutely sure they are right, and have grounds to argue with. By "sure" I do not necessarily mean the use of mathematical proof — although if a learner understands a proof, this is one way to be sure. Instead I mean that they can back their

arguments with demonstration, generalities, counter-examples and other tools of intellectual expertise.

Jenny Houssart, in her research with pre-adolescents, spotted some students who were rapidly becoming disaffected in the classroom, with a hostile atmosphere between them and the teacher. One feature of this deteriorating relationship was that they were sometimes right about mathematics when the teacher was wrong, but their comments were ignored, sidelined, or even punished by the teacher (2004).

By contrast, students whose ideas and thinking are valued explicitly in lessons are more likely to feel and behave as if they belong. Thus some adolescent boys arrange themselves to be directly in the teacher's line of vision and make sure she knows that they know the answers, or that they understand what is going on. As the Tasmanian curriculum document says: "People who have a sense of competence in their ability to think and learn… will be eager to pursue questions that really matter."

Those whose thinking never quite matches what the teacher expects, but who never have the space, support and time to explore why, can become disaffected at worst, and at best come to rely on algorithms. While all mathematics students and mathematicians rely on algorithmic knowledge sometimes, to have that as the only option places learners totally at the mercy of the authority of the teacher, textbooks, websites and examiners for affirmation. Since a large part of the adolescent project is the development of autonomous identity, albeit in relation to other groups, something has to break this tension — and that can be a loss of self-esteem, rejection of the subject, or disruptive behaviour (Coleman & Hendry, 1990, pp. 70, 155). I would argue that presenting mathematics in imaginary contexts does not necessarily touch their need for self-actualisation; nor do contrived references to what might be useful in future employment. What is required instead are ways to engage the person as they are here and now in the human activity of doing mathematics using their own thinking. This might include contexts of genuine interest, and information about how human beings developed the subject by asking pertinent and curious questions, but it could also include intriguing and puzzling situations, unanswerable questions, ways to use what they already know to generate new big ideas.

My aim is to develop ways in which *all* students, not just those whose arm-waving attracts the teacher's positive attention, can be engaged in mathematics for its own sake and thus begin to see that mathematical thinking is a part of who they are, and might form a part of who they become. In the development of adolescent identity, I suspect that using football as a context for mathematics affirms the identity of football fans; using the school disco affirms the identity of organisers and music freaks. The ability of the stronger mathematicians to take over such contexts and claim ownership can exclude and deskill and therefore further dishearten some of the students whom the context was supposed to help. In addition, Cooper and Dunne's finding that context can confuse students from disadvantaged groups because it supplies a layer of necessary discernment about what is, and is not relevant (2000). Instead, there is a mathematical component of identity — the human capacity to reason spatially, numerically and logically, which can be nurtured by participation in mathematics, by having one's thinking valued, and by having some autonomous control over the locus of mathematical understanding in lessons.

I shall now give some examples of tasks which generate and nurture this aspect of identity, and which fit very well into Tasmania's description of "essential learnings". Each task is of a type that can be applied to many mathematical contexts. I am not claiming these task-types are new, but representing them as tools for engaging

adolescents into personal engagement with mathematics, and ceding authority and validation to their relationship with mathematics.

Learner-generated examples

Students in a lesson were familiar with multiplying numbers and binomials by a grid method:

X	20	7
50	1000	350
9	180	63

X	z	+3
2z	$2z^2$	6z
−1	−z	−3

They had been introduced to numbers of the form $a \pm \sqrt{b}$. The teacher then asked them to choose pairs of values for a and b, and to use the grid method to multiply such numbers to try to get rational answers. Students worked together and began to explore. At the very least they practised multiplying irrationals of this form. Gradually, students chose to limit their explorations to focus on numbers like 2 and 3 and, by doing so, some realised that they did not need explicit numbers but something more structural which would "get rid of" the roots through multiplication. Although during the lesson none found a way to do this, many carried on with their explorations over the next few days in their own time.

Tasks in which students gain technical practice while choosing their own examples, with the purpose of finding a particular property or relationship, can be adapted to most mathematical topics. During the work there are many non-trivial features of their thinking that are valuable and can be praised: exemplifying, controlling variables, conjecturing, limiting the range of variation used, observing and testing special cases, designing spreadsheets to carry out the task, seeing implications of some results, generalising and so on.

Equivalence

Shifting the focus of lessons from finding answers to generating equivalence alters the locus of power in lessons. Here are two examples:

> Given a point on a number line drawn on grid paper, how many ways of representing it as a fraction can you find?

This reverses questions which ask people to reduce fractions to their simplest form. It also gives each student a set of fractions they can use for future activity, augmented by each other's findings — a kit of examples for demonstrating and validating future calculations.

> Given $y = 5$, how many equations can you write in which y is secretly 5, but this fact is hidden, such as in $17 = 3y + 2$?

This task generates a set of questions which everyone can then try to answer; the class has created its own practice exercise.

Another and another…

Ask students to give you examples of something they know fairly well, then keep asking for more and more until they are pushing up against the limits of what they know.

> Give me a number between zero and a half; and another; and another… Now give me one which is between zero and the smallest number you have given me; and another; and another…

Each student works on a personally generated patch, or in a place agreed by a pair or group. Teachers ensure there are available tools to aid the generation; in this case some kind of "zooming-in" software, or mental imagery, would help.

This approach recognises their existing knowledge, and where they draw distinctions; it then offers them opportunity to add more things to their personal example spaces, either because they have to make new examples in response to your prompts, or because they hear each other's ideas (Watson & Mason, 2006). Self esteem comes at first from the number of new examples generated, then from being able to describe them as a generality, and finally from being able to split them into distinct classes.

Putting exercise in its place

If getting procedural answers to exercises in textbooks is the focus of students' mathematical work (whether that was what the teacher intended or not) then shifts can be made to use this as merely the generation of raw material for future reflection. Many adolescents have their mathematical identities tied up with feeling good when they finish such work quickly, neatly and more or less correctly; others reject such work by delaying starting it, working slowly, losing their books and so on. Restructuring their expectations is, however, easy to do if new kinds of goal are explicated which expect reflective engagement, rather than finishing, so that new mathematical identities can develop more in tune with the self-focus of early adolescence (Dweck 1999).

Examples of different ways to use exercises are:

> Do as many of these as you need to learn three new things; make up examples to show these three new things.

> At the end of this exercise you have to show the person next to you, with an example, what you learnt.

> Before you start, predict the hardest and easiest questions and say why; when you finish, see if your prediction was correct; make up harder ones and easier ones.

> When you were doing question N, did you have to think more about: method, negative signs, correct arithmetical facts, or what? Can you make up examples which show that you understand the method without getting tied up with negative signs and arithmetic?

Since the Tasmanian "essential learnings" have been written to cover all subjects, many of the special features of mathematical thinking are only hinted at. "Active reflection" it says "enables connections to be made between different types of subject matter, and this enhances the likelihood of knowledge being transferable to new

situations." This is true, of course, but the active reflection I describe here does more: it enables connections to be made *within* particular concepts and methods, so that learners become better at developing critical relational knowledge. It is this recognition of methods at a structural, rather than operational, level that makes adaptable and transformable understanding more likely.

Rules versus tools

"Learning is more effective, interesting and relevant when learners consciously choose and use particular methodologies, devise their own strategies to deal with challenges," says the Tasmanian document.

Student-centred approaches often depend on choice of method, and this, of course, celebrates autonomy. However, mathematics is characterised by, among many other things, variation in the efficiency and relevance of methods. For example, "putting a zero on the end when multiplying by ten" is fine so long as you are working with whole numbers — and mathematicians do not abandon that way of seeing it. Rather than it being a *rule* it becomes a *tool* to be used when appropriate. Adolescents often cannot see why they should abandon methods which have served them well in the past (repeated addition for multiplication; guessing and checking "missing numbers"; and so on) to adopt complex algorithms or algebraic manipulations. One way to work on this is to give a range of inputs and to ask students to decide which of their methods works best in which situation, and why. This leads to identifying methods which work in the greatest range of cases, and the hardest cases. "Supermethods" need to be rehearsed so that they are ready to use when necessary, and have the status of tools, rather than rules.

In all of the above types of task, students create input which affects the direction of the lesson and enhances the direction of their own learning. I see this as an adventure, since they are each starting out from the safe ground of their own knowledge-so-far and moving elsewhere within a mathematical community. Classrooms in which these kinds of task are the norm provide recognition and value for the adolescent, a sense of place within a community, and a way to get to new places which can be glimpsed, but can only be experienced with help. To use the "zone" metaphor, these tasks suggest that mathematical development, relevance, experience and conceptual understanding are all proximal zones, and that moves to more complex places can be scaffolded in communities by the way teachers set mathematical tasks.

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Linking the big ideas in middle school mathematics[+]

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This paper considers three mathematical settings from the middle years of schooling and some links that may help to motivate both teachers and students. Nearly everyone agrees that proportional thinking is the most fundamental building block in the middle school mathematics curriculum. The transition from additive to multiplicative reasoning requires much time and many varied experiences. Three types of experience are suggested: links between percent and data handling, proportional links between geometry and algebra, and links between rates and quantitative literacy.

Introduction

Van de Walle (2007) is among many writers for preservice teachers to stress the importance of proportional reasoning, devoting an entire chapter of his latest edition to it (Chapter 19): "The development of proportional reasoning is one of the most important goals of the 5–8 curriculum… Proportional reasoning is indeed the cornerstone of a wide variety of topics in the middle and high school curriculum" (p. 353). He goes on to list the topics of fractions, algebra, similarity, data graphs, and probability, each of which is considered as part of a separate chapter.

Two questions arise from acceptance of Van de Walle's views and abundance of associated material and suggestions. How is enough time to be found in preservice teacher education programs for primary, middle school, and secondary preservice teachers to understand and absorb the topics, as well as learn to teach them effectively? How can inservice teachers be brought up-to-date on the importance of focussing on proportional reasoning and the links across the mathematics curriculum that will reinforce understanding?

The goal of this paper is to address in a small way the second question. It is likely that many inservice teachers had experiences similar to the author in learning about proportional reasoning: tables relating the cost of various numbers of balloons (multiples of 3) if 3 balloons cost 5¢. This progressed to the famous "cross-multiply" algorithm that allowed for finding any one of the four values of a, b, c, or d, if the other three were known:

$$\frac{a}{b} = \frac{c}{d} \text{ implies } a \times d = b \times c.$$

Large arrows connecting a and d, and b and c, helped one to remember the rule. An off shoot of this was the adaptation for percent problems that solved the three types of missing value problems:

[+] Invited paper

$$\frac{250}{650} = \frac{c}{100}, \quad \frac{a}{375} = \frac{35}{100}, \quad \text{and} \quad \frac{700}{b} = \frac{87.5}{100}.$$

The only difficulty with using these formulas was deciding where the box went and then applying the cross-multiply algorithm correctly.

There is nothing wrong with the above mathematics (except that inflation has affected the price of balloons) but what Van de Walle (2007) and others (e.g., Lanius & Williams, 2003; Thompson & Bush, 2003) suggest is a more developmental approach that is based on problems from many contexts (less artificial than might have been used previously), on specifically contrasting additive and multiplicative situations, and on linking arithmetic to visual and geometric models. Van de Walle is definitely a recommended foundation for any middle school professional development program. Both reinforcement and extension are the goals for middle school mathematics teachers. Avoiding a strictly algorithmic approach from the start is a necessity if students are going to develop the intuitions required for the other topics of mathematics curriculum noted by Van de Walle. Although not disparaging the development of useful algorithms, even for cross multiplying, these must be built upon a strong foundation of multiplicative thinking. Reinforcements and applications from other topics is one way of assisting in this task.

This paper considers three topics encountered in research, teaching, or professional development that may add to what is currently in the literature in the area of proportional reasoning. They may be a little surprising in that they do not directly relate to the form, $a/b = c/d$. First, graphical representations of percent are presented, reinforcing the meaning of percent in a part-whole sense rather than in the formula sense noted above. Second, the multiplicative reasoning that extends a linear proportional relationship is explored in the creation of an algebraic equation to describe a geometry problem, focussing on the ultimate meaning of the various components of the equation. Third, ratios, "halves of proportions," are considered in terms of the ability to summarise and compare information as required for quantitative literacy.

Percent and hats

Although it surely must be the case that a survey of the news section of the daily newspaper would show that reference to percents occurs more frequently than reference to either fractions or decimals, percents seem to be the poor relation of the three in most school mathematics curricula. Considering that conceptually percents are an exact parallel of fractions, it would appear feasible to learn about them at the same time, especially in relation to the well-known equivalents, $\frac{1}{4} = 25\%$, $\frac{1}{2} = 50\%$, $\frac{3}{4} = 75\%$, and the part-whole aspect in applications. Emphasis on "the whole" and "100%" is a common feature for the two topics. Besides talking about discounts, or perhaps today the GST in Australia, examples of the usefulness of percents may be difficult to find.

A new statistical software package for the middle school, *Tinkerplots* (Konold & Miller, 2005), offers the potential to reinforce basic percent understanding at the same time as serving a statistical end. The package is based on the developmental premise that students should be allowed to a large extent to create their own representations (Konold, in press). Data can be entered on cards that appear in "concrete" form but are also stored in tables. Initial graphs of data appear with two unlabelled axes and data values randomly distributed in the two-dimensional space between the axes. Tools are

provided to allow students to create graphs to tell stories that interest them; these are shown in Figure 1.

Figure 1. Graphing tools in Tinkerplots.

Using the tools students can drag-and-drop attributes onto the axes as they desire and they can stack values in bins or along a scaled axis. Two aspects of the tools provided by the software link effortlessly to the concept of percent. For a particular stacked dot plot as seen in Figure 2(a), the values constitute the whole visible data set (100%). This representation is typical of what mathematics teachers would expect to see in a graph. *Tinkerplots*, however, provides the opportunity to group the data in "bins;" movement between representations is achieved by dragging a data icon either to the right or the left. The Count tool in the menu allows for the number of values in each bin or the percent of the data set (or both) to be displayed. This is shown in Figure 2(b) for seven bins. In working with grade 7 students and *Tinkerplots*, it has been observed that many prefer the representation in Figure 2(b) to the more conventional graph in Figure 2(a). They like seeing the groupings, which although similar to histograms, are a form they have not yet met, and they use the language of percent to discuss the relative frequency of values in the bins. The students naturally see the largest percent in terms of the whole 100% of the data set. Comparing the contents of the bins is also more intuitive using percents rather than numbers.
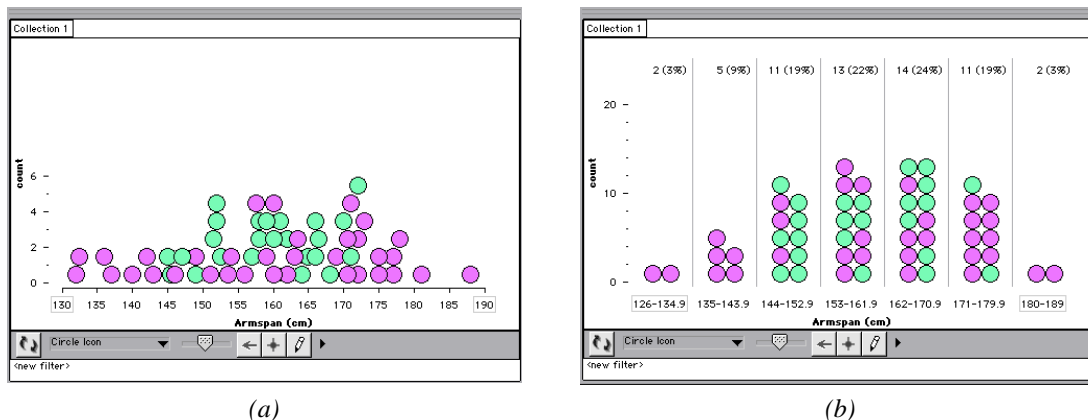
Figure 2. Data displays in Tinkerplots.

When the data are viewed in the form shown in Figure 2(a), the Count (%) tool shows "100%," emphasising the whole data set. The second tool that links to understanding of percent is the Hat Plot. The Hat Plot is a precursor to the Box Plot with which most mathematics teachers are familiar. The basic Hat Plot produces a hat over the top of the data set as shown in Figure 3(a). It has 50% of the data under the crown of the hat and 25% under each brim. A difference in the lengths of the left and right brims provides an opportunity to discuss how the data are represented along the scale on the horizontal axis and how cramped or spread out the data appear to be. The middle 50% or half of the data can be checked by counting points. The Hat Plot always appears with the data below it, never alone, reinforcing what it represents. Students

have been observed to be quite comfortable with this representation and able to explain the representation in terms of the placement of the data on the scale. They also are able to compare two representations by comparing the shapes of their respective Hat Plots, as shown in Figure 3(b).
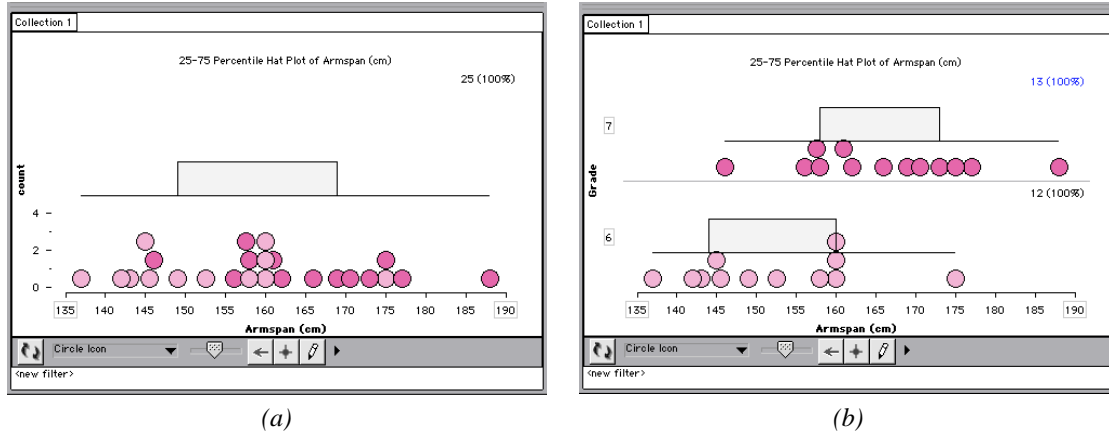
Figure 3. Hat Plots for one (a) or two (b) data sets.

It should be noted that *Tinkerplots* has the capability to merge back and forth from a Hat Plot to a Box Plot. This can greatly assist in understanding the slightly more complex nature of the Box Plot. Again the Box Plot does not appear without the data being present. To most teachers the two representations shown in Figure 4 are nearly identical, with the Hat Plot on the left (a) and the Box Plot on the right (b). For students, however, the presence of the median in the Box Plot and the implications of the smaller part of the box containing more densely packed data values, is notoriously difficult for students to accommodate. The problem has led some statistics educators to recommend that the Box Plot not be introduced at the middle school level (Bakker, Biehler & Konold, 2005). The difficulty of density is not present with the Hat Plot and it offers two visual bonuses for middle school students: reinforcement for the meaning of percent and a way of describing clumps and spreads in a data set.
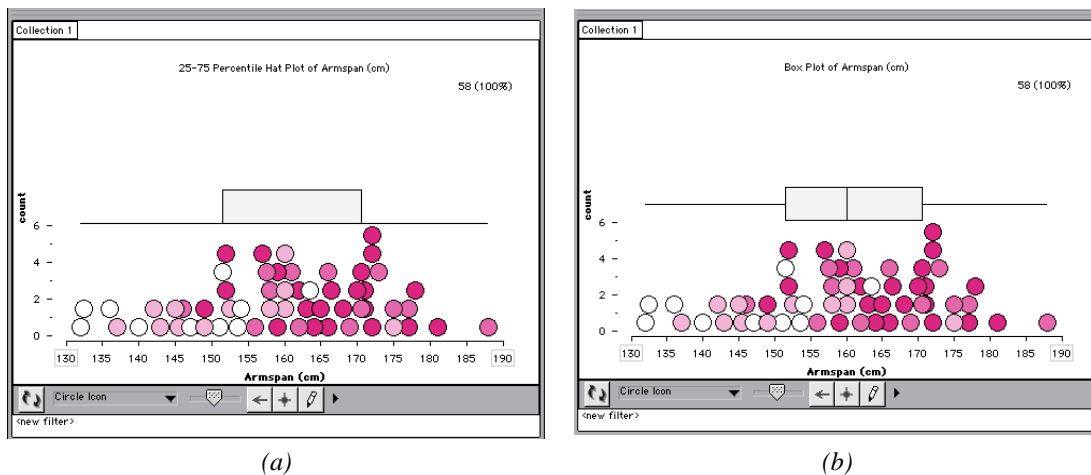
Figure 4. Hat and Box Plots for the same data set.

Cubes, constants and variables

The following comment is from a preservice primary teacher: "I'm really nervous about teaching algebra. You know all those equations? I never know which letters are supposed to be constants and which are variables and why. Everything seems to vary, take $y = mx + c$." This dilemma is reminiscent of a comment made years ago by a colleague impressing listeners about how students must view the mathematics teacher who says, "Let x be a number," when x is obviously not a number. It would be like an English teacher announcing, "Let 4 be a letter." What would the class say to that?

An equation such as $y = 2x + 3$ expresses both a multiplicative and an additive relationship, whereas $y = 2x$ is itself a multiplicative proportional statement. The equation $y = 3x$ represents a different proportion, as does $y = 5x$ or $c = 2\pi r$. This last c however is a variable, whereas the c in the above paragraph represents a constant, but one that may be changed (or varied). For students who try to regurgitate and manipulate the expressions with no conceptual understanding, it is no wonder they suffer frustration and often fail.

Strangely, or perhaps not, given the connections among mathematical concepts, a geometry puzzle may offer some clues as to the potential meaning for variables and as to when multiplicative thinking can be useful. The problem presented in Figure 5 is one that has been used in research to explore students' ability to visualise and manipulate geometric objects to solve a problem (e.g., Campbell, Watson & Collis, 1995). In the research setting students are asked to describe verbally what they do in their minds to solve the problem without any other aids such as pencil and paper. This is also a useful classroom exercise perhaps with students describing their thinking to a partner. The problem, however, has even greater potential for exploring patterns and relationships.

Problem to visualise

A cube that is 3 cm by 3 cm by 3 cm was dipped in a bucket of red paint so that all of the outside was covered with paint. After the paint dried, the cube was cut into 27 smaller cubes, each measuring 1 cm on each edge. Some of the smaller cubes had paint on 3 faces, some on 2 faces, some on only 1 face, and some had no paint on them at all. Find out how many of each kind of smaller cube there were.

Figure 5. The painted cube problem.

In justifying solutions to the problem various facts about numbers and cubes are likely to be employed: $3 \times 3 \times 3 = 3^3 = 27$; a cube has 8 corners; a cube has 12 edges; and a cube has 6 faces. Each corner mini-cube has paint on three sides, each "edge" mini-cube that is not a corner has paint on two sides, each "face" mini-cube that is not an edge or a corner mini-cube has paint on one side, and there may be cubes in the middle of the cube with no paint at all. To find this out a check that all 27 cubes are accounted for can be made: 8 corner mini-cubes + 12 edge mini-cubes + 6 face mini-cubes = 26 mini-cubes, so there must be one left in the centre with no paint. Many people do not have to carry out the subtraction but "see" the central mini-cube with no paint.

Now the problem becomes more interesting: imagine a larger painted cube 4 cm by 4 cm by 4 cm, again cut into smaller mini-cubes with 1 cm on each side. The same

question applies for how many cubes of each type there are. This time there are $4 \times 4 \times 4 = 64$ cubes to account for and different strategies are seen as people work out how many of each type of mini-cube there are. Recalling the numbers of corners, edges, and faces can be useful, as can visualising, drawing, or actually constructing a cube from blocks. Figure 6 shows a $3 \times 3 \times 3$ and a $4 \times 4 \times 4$ cube with different colours representing corners, edges, and faces; the unseen centre is also a different colour.

Figure 6. Constructions of "painted" cubes colour-coded for the number of mini-cube sides painted.

The expression enumerating the numbers of mini-cubes with various sides painted can take several forms. One form is shown in Table 1.

Table 1

Number of painted sides	Part of the cube	Number of these
3	Corner	8
2	Edge (not corner)	$12 \times 2 = 24$
1	Face (not edge or corner)	$6 \times 4 = 24$
0	Centre	$64 - 8 - 24 - 24 = 8$

In working out that there are 24 mini-cubes with two sides painted and 24 mini-cubes with one side painted many people miss the difference in how the two values are related to the number of edges (12) and faces (6). In fact there is an interesting pattern developing here. It may become clear if a $5 \times 5 \times 5$ cube and a $6 \times 6 \times 6$ cube are considered. This is done in Table 2, where the summary is presented in a different form.

Table 2

Size of large cube	Total Number mini-cubes	Number corners (3 sides painted)	Number Edges (2 sides painted)	Number Faces (1 side painted)	Centre (no paint)
$3 \times 3 \times 3$	27	8	12	6	1
$4 \times 4 \times 4$	64	8	12×2	6×4	8
$5 \times 5 \times 5$	125	8	12×3	6×9	27
$6 \times 6 \times 6$	216	8	12×4	6×16	64

For every cube the number of mini-cubes with three sides painted is the same, or "constant." For the other three types of mini-cube, however, there is a multiplicative relationship involved. The number of mini-cubes with two sides painted is proportional to the "length of the large cube – 2." The number of mini-cubes with one side painted is proportional to the "square of the (length of the large cube – 2)," whereas the number of mini-cubes in the centre with no paint is equal to "the cube of the (length of the large cube – 2)." Having expressed this relationship in words several times it would seem appropriate to use a symbol, perhaps   or "n," for the length of the large cube. Then the large cube is made up of $n \times n \times n$ mini-cubes, of which 8 have three sides painted, $12(n - 2)$ have two sides painted, $6(n - 2)^2$ have one side painted, and $(n - 2)^3$ have no sides painted. Putting all of this information together to account for all of the mini-cubes in the large cube reveals:

$$n^3 = 8 + 12(n - 2) + 6(n - 2)^2 + (n - 2)^3.$$

This relatively complex equation is made up of an additive component (8) and three multiplicative components. One factor in the product each time is a constant: the "number of edges" (12), "the number of faces" (6), and "the number of centres" (1).

It is because the length of the large cube, n, can vary, that the proportions involving $(n - 2)$ are introduced. Hence the "variables" that occur in the equation are there because the large cube can increase in size creating more mini-cubes. "How many more" mini-cubes are there each time is related to the constants 12, 6, and 1. The fact that the number of corners, edges, faces, and centres for the large cube do not change is observable from the nature of the cube. The constants and variables in the equation in Figure 7, hence have concrete meaning in the physical construction of the large cube. In one context, algebra, constants, variables, multiplicative properties, additive properties, and geometry are linked.
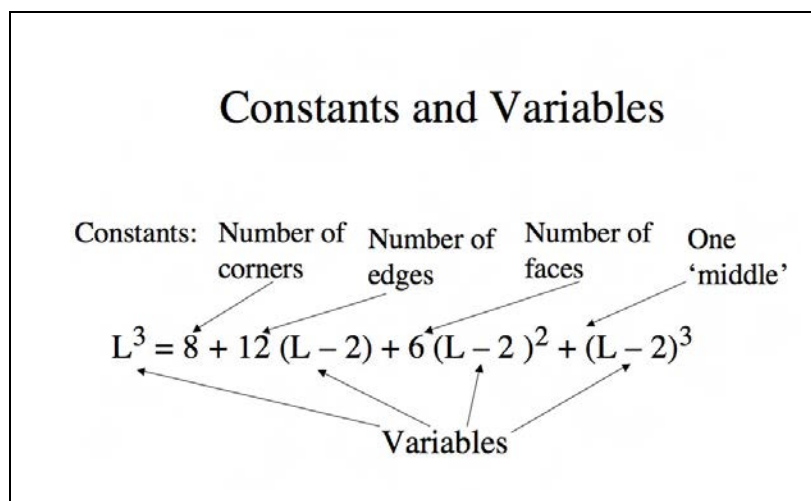

Figure 7. The constants and variables in the "cube equation".

The other interesting observation in the equation in Figure 7 is that 12, the number of edges, is multiplied by the *length $n - 2$*; 6, the number of faces, is multiplied by the *area* $(n - 2)^2$; the centre (of which there is 1) has the volume $(n - 2)^3$. The edge is a length,

the face is an area, and the centre is a volume, so these algebraic representations with the exponents (1, 2, and 3) fit the geometry of the large cube.

An interesting special case to consider is when $n = 2$. Students who understand what has been happening in this development should be able to explain coherently about the 8 corner mini-cubes accounting for the entire cube (cf. Reys, 1988).

Observations: some primary and middle school preservice teachers completing this activity claimed they had never before appreciated what constants and variables were in algebra.

Rates and risk

Rates are examples of ratios that are useful in a huge number of contexts. A rate is a ratio that compares two measurements in different units, for example kilometres per hour (km/hr), price per kilogram ($/kg), and passengers per busload. This representation is different from the part-whole ratio of a fraction where like units are being compared, e.g., girl students to all students or the number of points in a chance event to the number of points in an entire sample space (probability as a fraction ratio). It is also possible to have part-part ratios, for example Liberals to Labor, or chances for and against winning (odds in probability). All three types of ratios, as well as "famous ratios," such as π, need attention. This section focuses on the rate in the context of measuring risk.

Among the goals of the middle school curriculum is to prepare students to use the content they learn in the world around them. In relation to mathematics this is often termed quantitative literacy and proponents such as Lynn Arthur Steen (2001) advocate that it is an essential contributor to the success of a democracy.

> Quantitatively literate citizens need to know more than formulas and equations. They need a predisposition to look at the world through mathematical eyes, to see the benefits (and risks) of thinking quantitatively about commonplace issues, and to approach complex problems with confidence in the value of careful reasoning. (p. 2)

The examples presented in this final section are intended to motivate teachers and students to think quantitatively with the concept of ratio in authentic contexts. Similar contexts from newspapers have been used by Watson (2004) to exemplify the importance of understanding terminology, appreciating it in context, and becoming critical thinkers where necessary to question or interpret claims.

One aspect of quantitative literacy that is prominent in many aspects of life is that of risk. Risks are well known in all phases of life, from walking across a busy street to drowning in the bath tub. The school curriculum acknowledges the need to address students' decision-making ability in situations of risk (AEC, 1994). Of concern for example are substance use, sexual practices, road behaviour, suicide, and dietary habits. Outside of these there are also concerns about risks from pollution, bird flu, global warming, playground equipment, terrorism, and lightning strikes. How is it possible to compare and contrast these risks? Why do some of them bother people and others do not?

Sandman (1993) made an interesting observation about the public's perception of risk. He claimed

Perceived Risk = f (Hazard, Outrage).

The hazard component is made up of the mathematical risk usually expressed in the form of a rate, such as 45 000 Americans die per year in motor-vehicle accidents, whereas about 82 die per year in commercial airplane accidents. Outrage, however, is a psychological perception that can be manipulated by other factors than the statistics. The media may at times contribute to such perceptions by the desire to make headlines and sell stories. Over half of the deaths from disease in the US each year are caused by heart disease and cancer (1.2 million deaths), whereas exotic diseases, like avian flu and mad cow disease, prevalent in the media, have so far killed none.

It would appear that middle school mathematics can contribute directly to the assessment of the Hazard component of the Perceived Risk equation through an appreciation of rates. The issue of Outrage is a cross-curricular one to which critical literacy can contribute. The two combined should be the goal of critical quantitative literacy. Kluger (2006) recently wrote a well-titled report for *Time* magazine: "Why we worry about the things we shouldn't… and ignore the things we should." Including many examples of relative risk, Kluger considered factors that influence people's assessment of risk, such as evolutionary development, dread of death, immediacy of threat, and comfort associated with control. In questioning how improvements can be made in people's perceptions of risk, he detailed (i) difficulties in people's intuitions in interpreting percents, (ii) deliberate stating of numerical values rather than percents by those who want to increase perception of hazard, and (iii) the need for people "to learn more about the real odds" (p. 45). In considering risks, "If you can just get people to compare… then you're in a situation where you can get them to make reasoned choices" (p. 45). These points are related to the critical quantitative literacy skills required across the curriculum based on rates as ratios.

What are the kinds of rates used to quantify risk? Consider a few examples. A recent snippet in *The Weekend Australian* ("Nature wild about elephants," 2006) brought attention to attacks by elephants that had killed people, for example, "In the Indian state of Jharkhand, near the western border of Bangladesh, 300 people were killed by elephants between 2000 and 2004" (p. 32). This translates to "75 deaths by elephant/year," quite impressive for Australians. This sounds more hazardous for example than fatal shark and crocodile attacks (e.g., Cribb & O'Brien, 1995). Between 1980 and 1990 for example there were 8 accidental deaths caused by crocodiles and 11 caused by sharks in Australia. When a person is killed by a shark or crocodile (or sting ray) the event is likely to attract headlines. The statistics imply that this is because of the low occurrence of shark attacks rather than their high frequency: low Hazard but high Outrage. During the same period (1980–1990), when 1.1 people/year were killed by sharks and 0.8 people/year killed by crocodiles, 3277 people/year were killed in car smashes in Australia. Yet very often a single fatality gets little more than a cursory mention in the local media: high Hazard but low Outrage.

How can the risks reported be compared? Certainly the death rate for elephants is not relevant in Australia. What about sharks or cars in India? From this point the critical aspects of quantitative literacy come to the fore. It is the integration of knowledge about what rates as ratios represent with the context of the information to be compared that forms the foundation for making critical decisions about risk. What are the significant questions that need to be asked in gathering information and then in making personal judgments about risk? These should be canvassed in the classroom and students need to be involved in contributing suggestions not just presented with a list of questions. Some of the suggestions should be the following.

- Over what period of time were data collected? How long is reasonable?
- Have conditions changed over that period of time?
- What parts of the world are the data relevant for? Is geography important?
- For what population of people are the data relevant?
- If based on a sample, what is its size and how was it collected?
- What comparisons should be made? Which can legitimately be made and which cannot?

These questions often will lead to a discussion of conditional risk.

Data on risk and hazard appear in many forms. Sometimes the method of representation is chosen in absolute numerical terms and sometimes it is relative in terms of multiples. This can be misleading, whether done inadvertently or on purpose. Consider the variety of ways in which hazard can be reported.

> Males from 17–22 are 21.9 percent more likely to have multi-vehicle crashes than females of the same age. ("Females come out best," 2006)
>
> Smokers in their 30s and 40s have five times as many heart attacks as non smokers a study of more than 10,000 UK heart attack survivors reveals. ("Heart attack risks," 1995)
>
> Driving 10 km/h over the speed increased the risk of a casualty crash fourfold. (Molloy, 1998)
>
> Australian Bureau of Statistics figures on bee/wasp sting mortality show the average incidence of death is 0.084 per one million population each year. (Carter, 2007)
>
> Tasmania's rate of deaths arising from transport-related injuries in 2003/04 was 13 per 100,000 people, or 50 percent higher than the national average. (Martain, 2007)

In the last two examples the absolute hazard is quantified. For the final one, if this were stated as a probability of death for an individual in this period, it would be incredibly small, 0.00013. This then presents a problem in attracting attention of readers. "Fifty percent" more than the national average, however, sounds more impressive. This makes the national average 8.7 deaths per 100 000 people; and incidentally this is a nice little proportion problem for students to work out from the information given (cf., Tabart, Skalicky & Watson, 2005). These extracts should lead to questions about how the data were gathered, from where were they gathered, and what would the sets of numbers look like? These are all legitimate quantitative literacy questions for class discussion and exploration.

Collating information on relative risk should take into account the above questions and make comparisons of risk as transparent as possible. Consider for example information from the International Shark Attack File (2005) indicating that between 1580 and 2005, there were 468 confirmed shark attack deaths worldwide. This is an average of 1.1 deaths/year, the same as Australia over the 10 years 1980–1990. How can these be compared? For many of the 425 years there were no figures collected by the aboriginal occupants of Australia; but even if the worldwide data were for the shorter period of time, Australia being a relatively small subset of the world, would pose a much greater threat of shark attack. Another source (Burgess, 2006) suggests a current rate of 5–15 deaths from shark attacks per year worldwide. Comparing Australia with the rest of the world, perhaps deaths per 10 million people would be reasonable, since Australia's population is about 20 million. Given the approximate world population of 6.5 billion, the comparison still makes Australia "above the world

average," with 0.55 deaths/10-million-people/year, compared with 0.0077–0.023 deaths/10-million-people/year worldwide. There are quite a few context based reasons for this difference and students can be encouraged to write reports to show their critical literacy, as well as quantitative, skills. One can imagine a headline like: "Shark attack deaths 25–70 times more likely in Australia." What would this do for the tourist industry trying to attract Scandinavians to Australia's beaches?!

Unfortunately it appears to be difficult to find information on the worldwide death rate by elephant attack. Following up on *The Australian* elephant story leads to the *New York Times Magazine* (Siebert, 2006), which contains some further data on the behaviour of elephants. "In the past 12 years, elephants have killed 605 people in Assam, a state of north-eastern India, 239 of them since 2001; 265 elephants have died in that same period, the majority of them as a result of retaliation by angry villagers, who have used everything from poison-tipped arrows to laced food to exact their revenge." Although data were not provided, the report also notes conflict between humans and elephants across Africa from Zambia to Tanzania and from Uganda to Sierra Leone. What about the rest of the world? Among the anecdotes in the article are the stories of Mary, a circus elephant in Tennessee who killed a hotel janitor and Topsy, a circus elephant in Coney Island who killed three trainers. Both were eventually killed by authorities. In fact a recent children's book in Australia tells the story of Queenie, an elephant at Melbourne Zoo who suffered a similar fate after killing her keeper (Fenton, 2006). These "sample of size one" Outrage stories make excellent media headlines but contribute little to the appreciation of death rates from elephant attacks worldwide. Deaths by shark attack are familiar to Australian students, whereas deaths by elephant attack are unlikely to be. Project work comparing the two could produce some excellent examples of quantitative literacy.

In the social sciences, risk may reflect more qualitative than quantitative understandings, especially in the advocacy of strongly held causes. Students need to move from the mathematics classroom armed with an understanding of rates and their use in contexts so that they can question every assertion of outrage and demand accurate assessments of hazard.

Conclusion

Many other illustrations of proportional reasoning exist in text books and the National Council of Teachers of Mathematics recognised its importance in a focus issue of *Mathematics Teaching in the Middle School* in 2003. Besides setting the scene informally (e.g., Lanius & Williams, 2003; Thompson & Bush, 2003), formal links to $a/b = c/d$ were developed (Chapin & Anderson, 2003), as well as ratios linked to polygons (Dwyer, Causey-Lee & Irby, 2003) and a focus on context (Sharp & Adams, 2003). Watson and Shaughnessy (2004) followed up with a consideration of probability sampling and comparing two data sets in linking proportional reasoning to the chance and data curriculum and Austin, Thompson and Beckmann (2006) offered a tantalising exposé on locusts also employing proportional reasoning. The topic of risk and quantitative literacy was further explored by Watson (1998) using a newspaper article claiming that smoking a pack of cigarettes a day for more than 50 years increased the risk of premature wrinkling by 4.7 times. There is no lack of resources for approaching proportional reasoning in intriguing and challenging ways. All of these examples and links contribute to the goal of Quality Mathematics in the Middle Years (AAMT, 2006).

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Bush tucker: Nourishing early mathematics learning in a rural school through a variety of forms of curriculum integration[+]

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Early mathematics learning can be motivating, engaging, and rich in mathematical meanings and skills, particularly if children's interests, and local cultural and environmental contexts are used to focus curriculum planning that caters for diverse levels of achievement and diverse learning needs. This paper examines mathematics teaching and learning in the early years that is developed from principles for multi-age environments, the integration of technology, different learning styles, and integration across curriculum learning areas. It uses examples of curriculum planning and student learning outcomes from a K–3 class in a remote school in Western Australia.

Background

In small remote schools in Australia it is common for students to be in classes of mixed year levels, requiring teachers to plan curricula that meet the learning needs of children from a range of ages, language experiences, family and cultural backgrounds, and developmental and academic achievement levels. One way to cater effectively for this high degree of student diversity within one classroom is to adopt a "multi-age" educational approach in which "graded [Year] distinctions are minimalised and where teaching and learning make use of the range of knowledge inherent in the group" (Rathbone, 1993, p. iv). Children are not compared one to another on their achievements, as is often the case in single age group classrooms, but instead the focus is upon their overall progress. Teachers who adopt a multi-age philosophy must plan for a classroom environment, curricula and related teaching strategies that ensure the learning activities provide opportunities for: peer learning and collaboration, in-depth discussions and sharing of ideas, taking responsibility for one's own and others' learning, and cognitive conflict and challenging of ideas. These principles are consistent with early childhood education philosophy in their focus upon inquiry and exploration, social constructivist learning, and a degree of self-generation of learning directions (Biggs & Potter, 1999).

The class

The K–3 classroom upon which this paper reports was based on a multi-age, early childhood education philosophy. The school had an enrolment of about 80 in Kindergarten up to Year 9, and was located in a small community of about 300 people in what was once a vibrant mining area several hundred kilometres northeast of Perth.

[+] Invited paper

Key aspects of early childhood education emphasised by the teacher included: using hands-on learning activities, encouraging children to be inquisitive and questioning of their experiences with the world around them, and finding out things for themselves rather than being given facts by the teacher. However, to ensure all children made progress in their learning, particularly within literacy and numeracy, there was also explicit, direct instruction by the teacher. The teacher also had to plan for catering for much diversity in achievement levels and rates of progress in learning because the children had numerous social challenges that acted as inhibitors to learning. The 90% indigenous population in the class included children who had not attended school regularly, or who had little language development before beginning school, or who had little social and emotional skill development to support classroom learning.

Planning for an "integrated" curriculum

The outcomes focused, integrated curriculum approach adopted by the teacher was consistent with both early childhood and multi-age educational philosophies, and thereby necessitated curriculum content and assessment practices that were relevant and contextual within the environment of that community and the children's interests. Thus, through a reflective practice approach to long and short term programming, along with ongoing formative assessment of children's learning, the teacher developed and implemented an "integrated" learning program that included mathematics as a focus. How this program was designed, and samples of some of the learning activities and related student learning outcomes are outlined next.

Initial planning via a "theme"

Each term the class picks a "theme" that is a topic of interest to them. For example, China, Fantasy, the Circus, The Man from Snowy River, and Aliens are recent themes chosen by the children. The teacher then does an initial "mind map" of ideas for how the theme might be used to address specific learning outcomes across the eight learning areas of the Western Australian *Curriculum Framework* (Curriculum Council, 1998) and the six early childhood learning domains. Formal assessment and reporting requirements of the state Department of Education are also consulted in this initial process (e.g. Schedule A), so that the emphases for required learning area content and processes are met. To plan within a specific learning area, for example mathematics, a matrix outlining content strands and related targeted outcomes is then constructed so that possible theme-oriented learning activities can be included.

   Figure 1 shows some initial ideas for addressing specific mathematics learning outcomes related to the Aliens theme. Specifically, the plan outlines how shapes will be explored through the use of big fabric stretchy body bags. The children don the bags as their alien skin and then create shapes such as triangles, rectangles, squares, pentagons, and other polygons. The activity can then be extended to integrate with the Arts Learning Area by using dance and movement to explore how the body can be used to create shapes. When put to music this activity can then be used for counting beats and bars of music, or for simple fraction concepts related to music such as whole, half, and quarter.

Initial planning for curriculum integration

Thus, "integration" is achieved in that more than one strand within mathematics is addressed (e.g., space, number, and measurement), and at least one other learning area

is addressed (e.g., Arts). Catering for the diverse learning needs and styles of the children is addressed through the use of kinaesthetic activities alongside the more formal use of technical mathematical language. At the same time, as children move about they talk about what they see and what they are doing, once more emphasising language, along with visually and physically based learning.
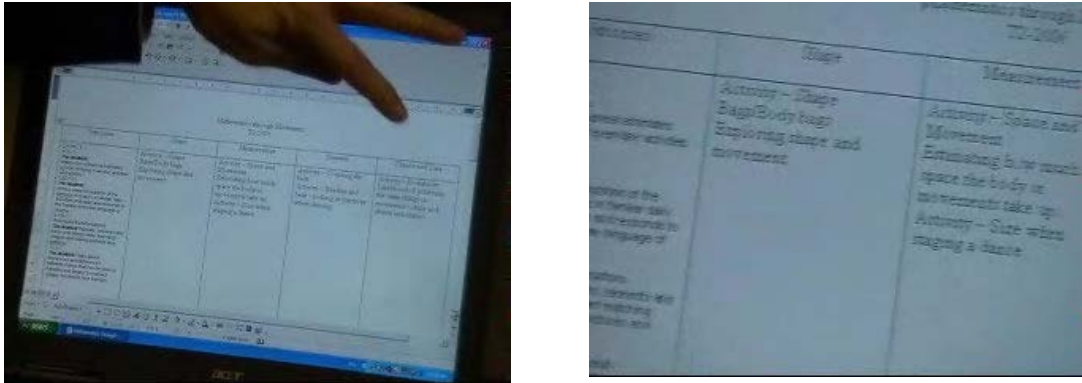
Figure 1. An initial matrix for a mathematics program linked to the Aliens theme.

Implementing an "integrated" curriculum

The examples outlined next include learning activities planned for in advance of a school term, as well as those that emerged from more short-term planning as the children's interests directed them to pursue finding out more about particular things of interest to them.

The Man from Snowy River — Ponchos

During the term when the theme chosen by the students was The Man from Snowy River, one thing the students found out was that migrant workers from many countries participated in building the Snowy River Dam. A country the children took particular interest in was Mexico. From this interest the teacher created the project of having the children construct and decorate their own ponchos (see Figure 2).

Figure 2. An example of one child's Mexican poncho.

The poncho activity involved a range of mathematics skills and processes, including:

- deciding how big to make the initial square of fabric, and then measuring the square;
- deciding how to cut a suitable hole for the head (e.g., folding the shape in half to create a triangle;
- deciding how big to cut a semi-circular shape in the "middle," and then actually measuring and constructing the circular hole;
- designing a pattern to go around the edge (with a requirement that they must use a reflection ("flip") to create the line pattern, and it must also have a colour pattern);
- designing a pattern to go on the front and the back of the poncho (with a requirement that it must using rotation of a shape and must have a colour pattern).

At the end of the poncho project the children learned the Mexican Hat Dance, and then wore their ponchos to perform the dance as a group. Thus, the project was then followed on with a link to physical education and cultural learning outcomes.

Medieval murals

Some of the things the children learned about during the term when the theme was Fantasy were knights and dragons, and this then led to learning about castles in medieval times: how they were built, what shapes were involved, and how they were furnished and decorated. They found out that wall hangings were often used in castles, so the teacher set a mathematics project for the children to create their own wall hangings by making use of 2-dimensional shapes. The Years 1and 2 children chose to make a knight (Figure 3). They designed their knight on paper before constructing the wall hanging with fabric. The mathematics involved included:

- identifying what 2-dimensional shapes can be used and positioned to build an image of a knight;
- deciding upon the needed sizes of the shapes, overall and relative to one another, and then measuring and cutting out the shapes; and
- using 2-dimensional shapes to build a border pattern.

Figure 3. Year1/2 wall hanging of a knight.

Teaching "number" in a multi-age classroom

Not all learning outcomes targeted for a term can be effectively addressed through the theme for that term; that is, the curriculum also must include explicit teaching, particularly within the core areas of early literacy and numeracy. For example, counting is a key concept and skill in mathematics learning, and within a multi-age classroom there will be a diversity of achievement levels for which to cater. One way the teacher structured the explicit teaching and learning of mathematics concepts and skills was by integration with the literacy curriculum through the use of storybooks. A suitable book would be chosen to focus upon a concept or skill, for example counting. The book is then used as a whole class reading activity before children then break away into individual or small group learning activities designed for their current achievement levels. For example, books such as *The Very Hungry Caterpillar* (Carle, 1969) or *Ten Terrible Dinosaurs* (Stickland, 1997), which are basic counting books, can act as an engaging stimulus for the children to think about numbers and the related words and symbols. Subsequent activities can then focus on a wide range of concepts and skills as needed, including:

- counting forward starting at 1, or starting at a different number;
- counting backward starting at 10, or starting at a different number;
- counting forward or backward starting at numbers bigger than 10 (for more advanced children);
- counting on, for addition of two numbers
- recognising and/or constructing the words and symbols for numbers.

The subsequent activities might make use of games (e.g. Snakes and Ladders), the interactive whiteboard (e.g. for tracing or drawing symbols for oneself), or a variety of hands-on or visual materials (e.g. counters, base 10 blocks, plasticine, wall charts).

Community and contextual integration — Chance and data

The theme for each term was not the only pathway by which the teacher planned for learning activities that emerged from the children's interests or world around them. Events within the local community or school also often provided relevant, meaningful contexts by which to plan curricula. For example, to integrate with the *Wastewise* environmental program, the children did an audit of the rubbish in and around the school perimeter. They collected the rubbish and then sorted it into categories (e.g. plastics, paper, metal, etc.). Next they created tables and graphs of the results, and finally they made predictions about the chances on a future collection of finding relative amounts of particular kinds of rubbish (e.g., Would it be likely you would find more plastic than paper? Why do you think that?). Thus, using the relevant local context and happenings, children were involved in Chance and Data concepts and skills related to:

- collecting, organising, summarising, representing, and interpreting data; and
- using everyday language to make statements related to chance.

Summary

There are many ways by which early mathematics learning can be relevant and effective, particularly if children's interests are used as a focus for developing curricula. In particular, the use of themes can allow a teacher to attend to children's interests while also developing meaningful relevant mathematics learning activities. These activities

can often be planned to integrate across more than one content strand within mathematics, and they generally lend themselves, due to their real-world contexts, to integration with other learning areas such as English, science, or the arts. At the same time, the use of, investigations, open-ended as well as more directed activities, hands-on materials, visual and/or kinaesthetic experiences, and discussion and talking about learning caters for diverse learning styles. The range of developmental and achievement levels found in most classrooms, and especially in a multi-age classroom, are also able to be well catered for by the use of these forms of curriculum integration.

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Seminars

Using concept maps and vee diagrams to interpret "area" syllabus outcomes and problems[*]

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Data presented is from a case study, which investigated a primary student teacher's developing expertise concept mapping syllabus outcomes and constructing vee diagrams of problems. Findings suggest the student teacher developed enhanced skills to critically analyse syllabus outcomes and a related problem, competently justify multiple methods using principles and effectively communicate mathematical ideas, and as a result, developed a deeper, conceptual understanding of the developmental and sequential nature of the sub-strand across different stages. Implications for teaching primary mathematics are provided.

Student teachers need to develop deep knowledge and understanding of the concepts they are expected to teach their future students (AAMT, 2006). The New South Wales Board of Studies (NSWBOS, 2002) encourages teachers to develop students' conceptual understanding through investigation and implementation of working mathematically strategies. This paper proposes that the metacognitive tools, hierarchical concept maps (cmaps) and vee diagrams (v-diagrams), and the innovative processes of concept mapping and constructing vee diagrams can influence the (a) development of students' conceptual understanding and (b) dynamics of working mathematically within a social setting.

Ausubel's cognitive theory of meaningful learning underpins the process of constructing cmaps/v-diagrams. Meaningful learning takes place by linking new knowledge to old knowledge via progressive differentiation (reorganisation of existing knowledge under more general ideas) and/or integrative reconciliation (merging many ideas into one or two when apparent contradictory ideas are reconciled) (Novak & Canãs, 2006; Novak & Gowin, 1984). Through social interactions while critiquing cmaps/v-diagrams, teacher and students engage in critical thinking, reasoning and communicating mathematically (Afamasaga-Fuata'i. 2005, 2006).

Cmaps are hierarchical networks of interconnecting *nodes*, representing concepts of a knowledge domain, with *linking words* describing the interrelationships of connected nodes to form *valid propositions*. For example, given

$$\text{Scatterplot} \xrightarrow{\text{may be used to estimate}} \text{correlation}$$

the proposition is, "Scatterplot may be used to estimate correlation". V-diagrams, in contrast, are heuristics that illustrate the epistemological, conceptual and methodological frameworks of a phenomenon (Novak & Canãs, 2006; Novak & Gowin,

1984). Gowin's epistemological vee was later modified for solving mathematics problems (Afamasaga-Fuata'i, 2005) (examples are discussed later).

Research examined the usefulness of cmaps/v-diagrams in the sciences (Novak & Canãs, 2006; Brown, 2000; Mintzes, Wandersee & Novak, 2000) and mathematics (Afamsaga-Fuata'i, 2005, 2004; Williams, 1998), and recently in teacher education with primary (Schmittau, 2004) and secondary student teachers (Afamasaga-Fuata'i, 2006, Brahier, 2005) and with teachers (Liyanage & Thomas, 2002). Three main uses are relevant to this paper. These are as (a) *learning tools* for individual learners to make sense of a knowledge domain, (b) *analytical tools* to analyse the structure of knowledge in a topic; and (c) *pedagogical tools* to guide the planning of learning activities.

The paper's focus question is: In what ways do hierarchical concept maps and vee diagrams facilitate the preparation of primary student teachers for teaching mathematics more conceptually? Presented here is the case study of a BEd (Primary) student teacher (Susan), who constructed cmaps/v-diagrams over a semester, in her third year elective unit.

Methodology

Susan was introduced to constructing cmaps/v-diagrams using simple topics/problems. The unit's main project was the construction of a comprehensive topic cmap in three phases. The topic was to be selected from primary mathematics. Susan was also required to construct v-diagrams of problems, which demonstrate the applications of the mapped concepts. The first phase required that Susan compile an initial list of concepts to construct an initial topic cmap and v-diagrams of two related problems. These were presented and critiqued in class before further revision and expansion. The second phase involved the presentation of a more structurally complex, expanded cmap and v-diagrams of more problems. These were socially critiqued and returned for further revision and expansion. The third phase was the final submission of a more comprehensive, hierarchical topic cmap and more v-diagrams of related problems, which extend upon previous work and incorporating comments from previous critiques, and including a reflection journal.

Cmaps/v-diagrams were collected from the familiarisation phase and the main project including a reflection journal. This paper presents Susan's work to illustrate the application of cmaps/v-diagrams as analytical and pedagogical tools.

Results

Sample cmaps are presented to display Susan's conceptual analyses of the *Area* syllabus outcomes from *Early Stage One* through to *Stage 3* (NSWBOS, 2002) followed by an example v-diagram. The latter demonstrates her understanding of area concepts and their applications in problem solving. The final data are excerpts from the reflection journal.

Concept maps of syllabus outcomes

The cmaps are presented in the order of the staged outcomes to illustrate their use as analytical and pedagogical tools.

Early Stage One MES1.2 syllabus outcome

Susan's conceptual analysis (Figure 1) showed main concepts such as *area* and *closed shapes*, and strategies such as *covering surfaces* and *direct comparison.* Illustrative examples of concepts/strategies are provided.
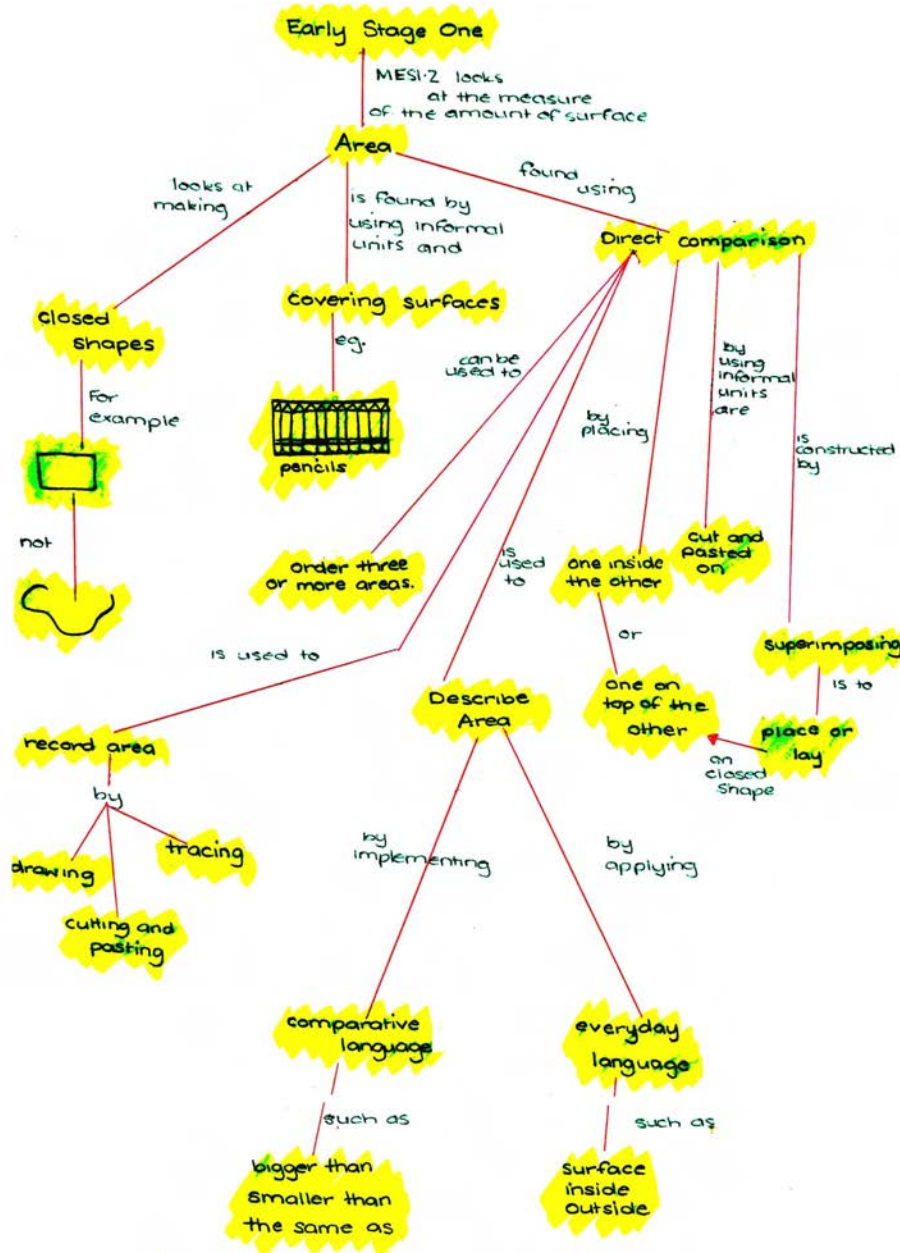
Figure 1. *Area - Early Stage One concept map.*

Progressive differentiating links from *direct comparison* show different strategies appropriate for this stage (*order three or more areas, one inside the other, cut and pasted on, superimposing*). Also displayed are links to *record area* and *describe area.* Whilst differentiating links from the former node display different ways of recording, those from the latter differentiate between *comparative language* and *everyday language* with further links showing examples. Some displayed propositions include "*Early Stage One* MES1.2 looks at the measure of the amount of surface *Area*" and "*Area* is found by using informal units and *covering surfaces.*"

Stage One MS1.2 syllabus outcome

Susan's conceptual analysis (Figure 2) identified main concepts (*surface, area, constant*) and strategies (*measured; estimated, compared and ordered; compared*).

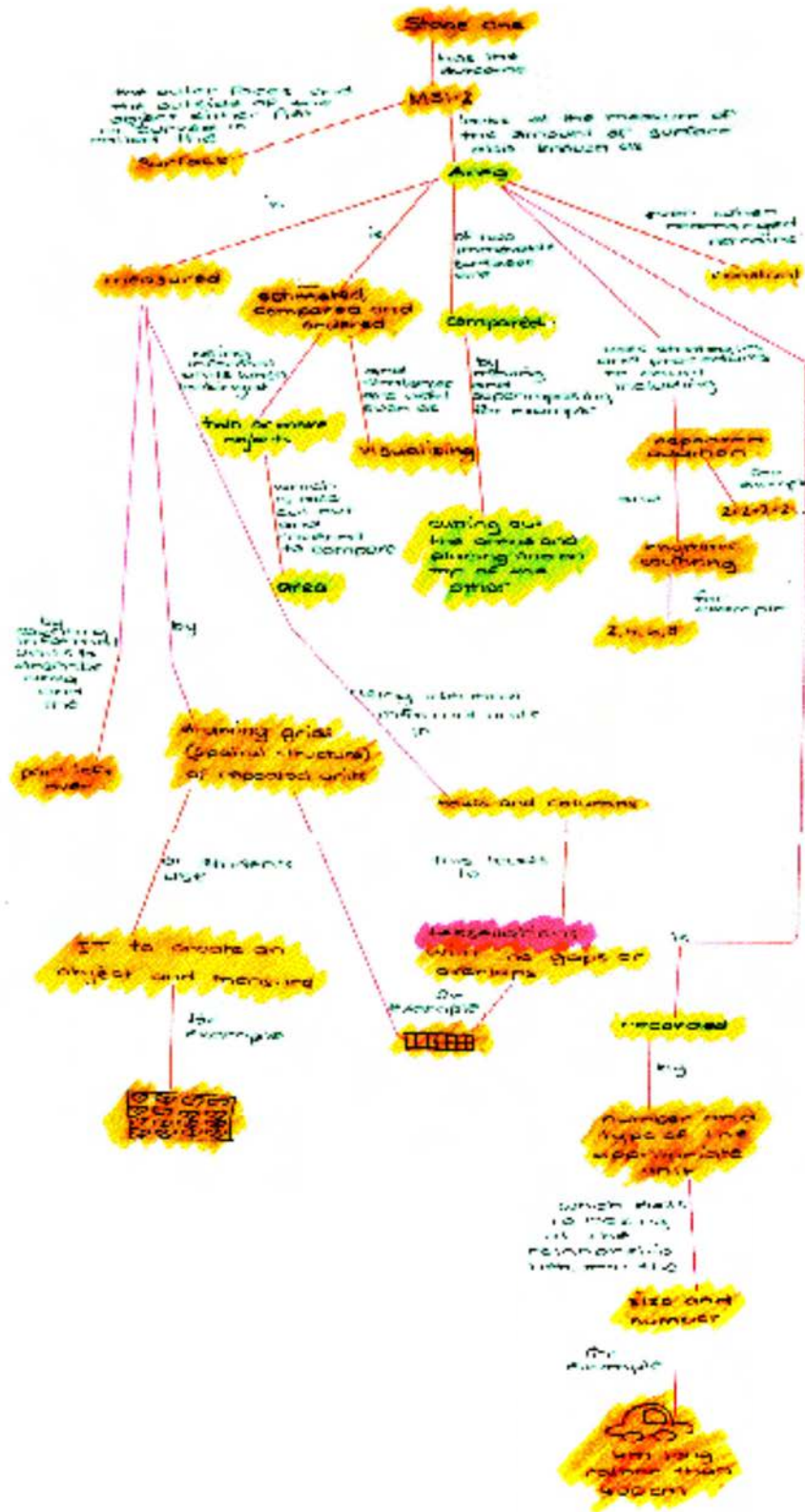*Figure 2. Area - Stage One concept map.*

Progressive differentiating links from *measured* display different strategies, which if read horizontally (left to right), indicate an increasingly sophisticated trend from informal counting of whole *parts (and) left-over* through *drawing grids (spatial structure of repeated units),* to using identical informal units in *rows and columns*; the latter is linked to *tessellations*. Integrative reconciliation links from *tessellations* and *drawing grids* merge at the diagram of a 2-by-7 grid. To the left is a contrasting grid of repeated units with gaps.

Consolidating strategies encountered at Early Stage One is described by the second and third sub-branches (left to right) of the *area* branch. However, new strategies are evident by the next two *area* sub-branches with *repeated addition* and *number and type of the appropriate unit*. The former links to *rhythmic counting* while the latter connects to *size and number*. Each sub-branch terminates with an example.

A comparison of Figures 1 and 2 highlights the progressive development of main concepts and strategies from the concrete and informal to the increasingly more formal *rhythmic counting* in row-and-column arrays. The cmap ends with the selection of units appropriate to the object's size. Nodes are colour coded to indicate connections to prior knowledge.

Stage Two MS2.2 syllabus outcome

Susan's conceptual analysis (Figure 3) showed that formal units of measuring area are introduced at this stage, namely, *squared centimetre* and *squared metre* (symbolised as cm^2 and m^2 respectively). An extended proposition is, "Squared centimetre is used to formally estimate, measure, compare and record a variety of areas by applying (a) variety of strategies such as using a 10 cm × 10 cm tile grid to determine areas that are same as $100\ cm^2$, greater than, (or) less than."

A crosslink between two nodes forms the proposition, "Squared centimetre is one ten-thousandth of a squared metre." Another proposition is "Squared metre which need not be an exact square, for example 2 m × 0.5 m." A crosslink to the next cmap is visible at the top right of Figure 3 (discussed next). Yellow coding indicates consolidation of prior knowledge. A comparison of Figures 1 to 3 indicates increasingly sophisticated means of measuring area and formalisation of units.

Stage Three MS3.2 Syllabus Outcome

Susan's conceptual analysis (Figure 4) identified a new unit (squared kilometre) and area formulas are introduced at this stage. Some displayed propositions include "MS3.2 discovers the need for the squared kilometre and looks at where it is used for example scales and everyday situations which students then interpret" and "MS3.2 is concerned with the surface area of rectangular prisms found by centimetre grids or units which leads to the formula how many rows × number in each row". The rightmost propositions are "MS3.2 discovers the need for a convenient unit to use, the hectare which is the same as $10\ 000\ m^2$" and "hectare which is abbreviated to ha". To the left of hectare are variations of constant rectangular areas with different perimeters, and extensions of areas to surface areas of rectangular prisms. Visible at the top left is an incoming crosslink from squared metre (Figure 3) to squared kilometre (Figure 4) explicitly connecting two staged outcomes. Purple coding a node also reinforces a connection to previous knowledge.

In summary, the four hierarchical cmaps highlight developmental and progressively sophisticated means of measuring and recording area. Colour coding indicates connections to prior knowledge distinguishing them from new knowledge within stage.
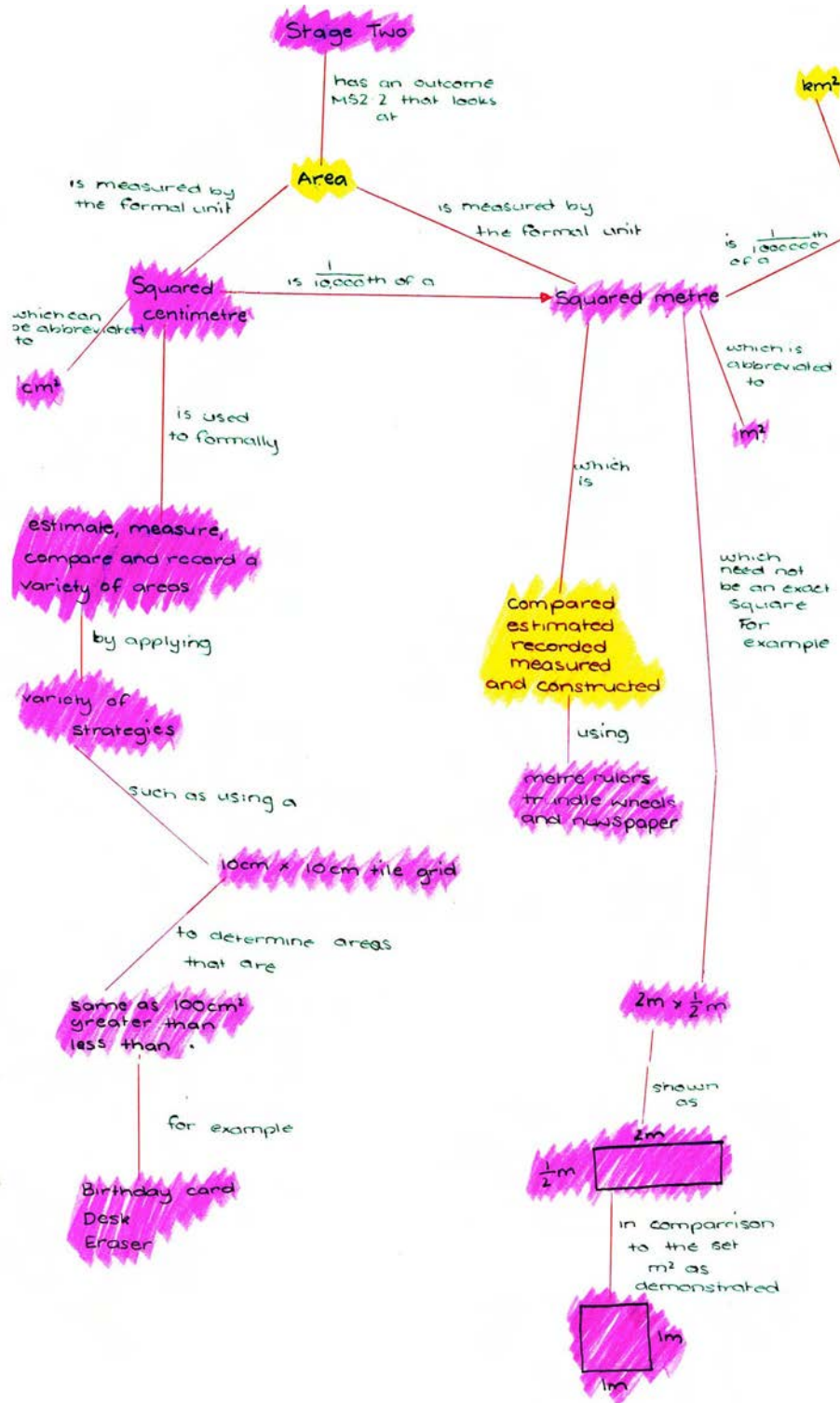
Figure 3. Area - Stage two concept map.

Vee diagram of a problem

Figure 5 shows a v-diagram with a conceptual *Thinking Side* on the left and a methodological *Doing Side* on the right. The vee is contextualised in the *Problem* to be solved with the focus question under *What is the question I need to answer?* The conceptual analysis of a problem and the relevant mathematical principles required to solve it are displayed as responses to *What do I know already?* (mathematical

principles) and *What are the main ideas?* (main concepts). A statement of mathematical beliefs is a response to *Why I like mathematics?*
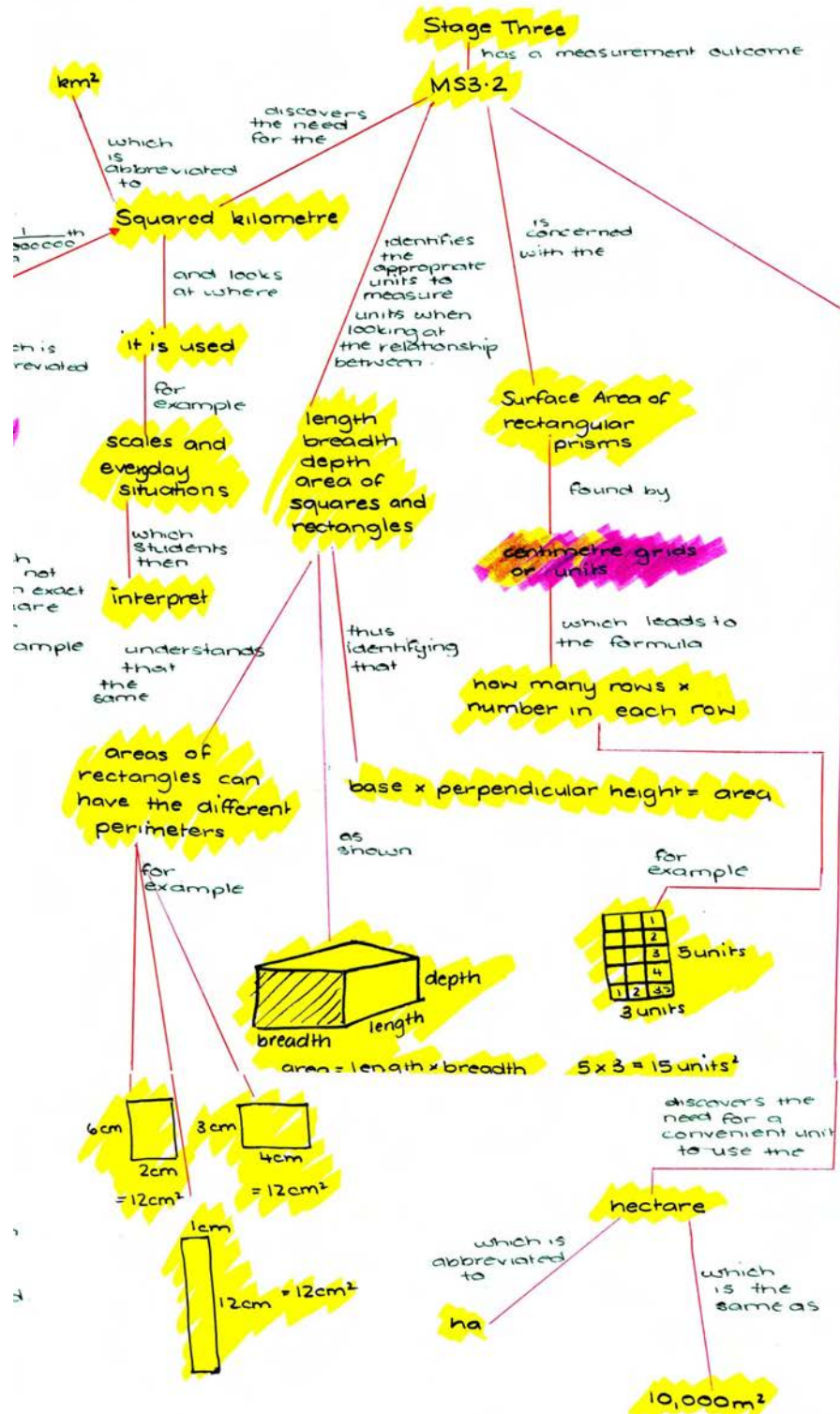
Figure 4. Area - Stage three concept map.

On the *Doing Side*, the given information is under *What is the information given?* and the methods under *How do I find my answers?* Critical reflections are as responses to *What are the most useful things I learnt?* Once the mathematics problem is identified,

completing the v-diagram's sections is done in any order with the provision that the information coherently describes *both* the methods and principles underlying the methods.

The *Thinking Side* (Figure 5) lists eight principles and five main concepts, which Susan concluded, underpin the two methods. Her statement of mathematical beliefs is deeply meaningful given the displayed evidence on the cmaps/v-diagram. Both the conceptual and methodological information (Figure 5) connect to those in Figures 1 to 4.
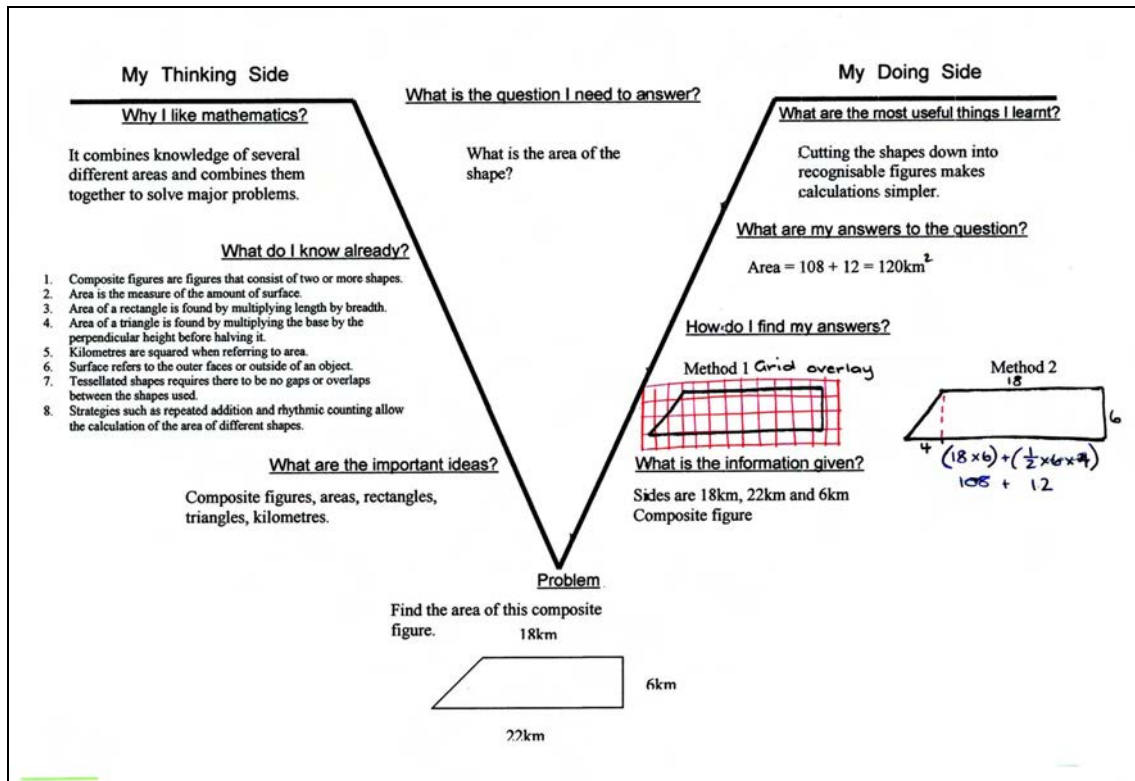
Figure 5. Vee diagram of a problem.

Reflection journal

Susan took the unit to develop a better understanding of the mathematics syllabus and of teaching its concepts to primary students. Rather than the Year 12 style she "was used to, of formulas are everything," she wanted to approach primary mathematics from a primary perspective.

Initially, Susan viewed mathematics problems as simply questions to be answered, and topics as containing information to be taught. However, upon completing the first phase, it became clear that "there was more to a problem than a formula and an answer"; that solving a problem was more than just "an answer finder". Instead, "[it] consisted of a wide variety of factors that contribute to the understanding and subsequent answer" such as prior knowledge one possesses, which influence the selection of methods. Also, constructing cmaps first facilitated the generation of multiple methods and identification of relevant principles.

Susan initially found v-diagrams difficult particularly the *Thinking Side*: "I did not know how I constructed the answer on the right side… thus, did not know what principles I had to list nor the important ideas… I struggled with it as, as a student I had only been taught the formulas never what was behind them." With this self-realisation,

Susan chose to challenge herself in subsequent activities, "Before finding the answer… I would look only at the question and think about what I need to know about it before I actually solved the problem."

Susan admitted, "I always had difficulty in explaining what I wanted them to do… it frustrated me that they did not understand when I explained it the first time." By the end of the semester, her "communication skills verbally [had] been assisted greatly by [her] written communication in both cmaps and v-diagrams." She claimed "I now have the basic [information] written before me and because it was me that had to construct the written version I was able to explain what I did verbally better than I had done before."

Susan's cmap/v-diagram experiences made her realise that a topic has a number of main and relevant concepts and recommended strategies that must be introduced, consolidated and extended through a suitable sequence of learning activities to ensure development of deep understanding. Summarising, she said, "[the Area sub-strand] has many connections that linked across a broad range of subjects and through the construction of cmaps and v-diagrams a deep understanding of the topic was achieved."

Over the semester, Susan developed critical (i) proficiency completing the Thinking Side of v-diagrams "as quickly and as effectively as the [*Doing Side*]" and (ii) analytical skills: "I am now able to see where a problem is going before the actual completion… there are problems I can work backwards [from solution to principles] to see where I am going." She also found cmaps useful guides for completing v-diagrams: "I could see the links and the next step [more clearly] in the solving of problems in relation to the sub-strand."

Discussion

Findings suggest Susan became competent and confident in her critical abilities to analyse syllabus outcomes and problems using cmaps/v-diagrams. She analysed the syllabus outcomes for key concepts, strategies and illustrative examples before placing the results in a conceptual, developmental order within each stage from left to right on cmaps. Making connections between stages was achieved by colour coding nodes (and crosslink) to differentiate between prior and new knowledge within a stage. Using a v-diagram, she systematically analysed a problem to make explicit both the conceptual and methodological information involved in generating plausible solutions.

Communicating effectively with her audience was enhanced through cmaps/v-diagrams. Because she individually constructed them, she is in a better position to explain and justify her ideas publicly, "I had to critically think of the reasons of why each map or diagram is constructed in the way that it is." By the third phase, Susan realised that she could "see the connections that infiltrated the topic" more clearly, consequently gaining a better understanding of how to sequence activities, "I now understand what needs to be taught first and where I need to go from there" by following the visual connections.

Finally, Susan concluded that constructing cmaps/v-diagram had begun "a new chapter in (her) understanding and teaching of mathematics." She felt confident and her understanding of the sub-strand had deepened particularly in "how each and every one of [the concepts and strategies] builds upon the prior knowledge of the last". Findings from this case study contribute knowledge to the development of primary teachers' deep understanding of mathematics and the pedagogical use of cmaps/v-diagrams.

Implications

Constructing cmaps engenders a deep understanding of how concepts and recommended strategies are developmentally progressed and consolidated across the stages. Constructing v-diagrams enhances critical synthesis of the relevant mathematical principles and procedures in generating solutions to problems. These findings imply that cmaps/v-diagrams are potentially useful tools for primary students to use in mathematics learning and problem solving. This is an area worthy of further investigation.

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Concept maps and vee diagrams as tools to understand better the "division" concept in primary mathematics[*]

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The paper presents a case study, which investigated a primary student teacher's developing proficiency with constructing concept maps and vee diagrams as tools to guide the analyses of syllabus outcomes and to facilitate the design of activities that develop deep understanding of mathematics concepts. Findings suggest the student teacher developed a deeper understanding of the developmental nature of the division concept as outlined in syllabus outcomes, which empowered her to confidently provide mathematical justifications for strategies to solve problems.

Introduction

The Australian Association of Mathematics Teacher's *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2006) identified the need for knowledge of conceptual understanding, and the ability to plan learning sequences to develop students' understanding, as essential to achieve excellence in teaching mathematics. Thus, student teachers need to develop deep understanding of concepts they are expected to teach their future students. The underlying philosophy and theoretical principles of the New South Wales Board of Studies' *K–10 Mathematics Syllabus* (NSWBOS, 2002) also encourage the development of students' conceptual understanding through an appropriate sequencing of learning activities and implementation of working and communicating mathematically strategies. This paper argues for the application of the metacognitive tools, hierarchical concept maps (cmaps) and vee diagrams (v-diagrams), and the innovative processes of concept mapping and vee diagramming as viable strategies (a) for developing students' conceptual understanding and (b) for facilitating the design of learning activities.

Literature review of concept mapping and vee diagrams

Ausubel's theory of meaningful learning underpins concept mapping particularly its principle that learners' cognitive structures are hierarchically organised with more general, superordinate concepts subsuming less general and more specific concepts. Linking new concepts to existing cognitive structures may occur via *progressive differentiation* (reorganisation of existing knowledge under more general ideas) and/or *integrative reconciliation* (synthesising many ideas into one or two when apparent contradictory ideas are reconciled) (Ausubel, 2000; Novak & Cañas, 2006). By constructing cmaps/v-diagrams, students illustrate publicly their interpretation and understanding of topics/problems. Hierarchical cmaps were first introduced by Novak

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[*] Paper accepted by peer review

as a research tool to illustrate the hierarchical interconnections between main concepts (nodes) in a knowledge domain with descriptions of the interrelationships (linking words) on the connecting lines. The basic semantic unit (*proposition*) describes a meaningful relationship as shown by the triad

$$\text{valid node} \xrightarrow{\text{valid linking words}} \text{valid node}$$
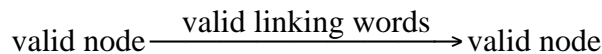
(Novak & Canãs, 2006; Novak & Gowin, 1984). V-diagrams, in contrast, were introduced by Gowin as an epistemological tool, in the shape of a vee that is contextualised in the phenomenon to be analysed. The vee's left side depicts the philosophy and theoretical framework, which drive the analysis to answer the focus question. On the vee's right side are the records, methods of transforming the records to answer the focus question and value claims. The epistemological vee was later modified (Afamasaga-Fuata'i, 1998, 2005) to one that is focussed on guiding the thinking and reasoning involved in solving a mathematics problem (an example is discussed later).

Numerous studies examined the use of cmaps and/or v-diagrams as assessment tools of students' conceptual understanding over time in the sciences (Novak & Canãs, 2004; Brown, 2000; Mintzes, Wandersee & Novak, 2000) and mathematics (Afamasaga-Fuata'i, 2004; Schmittau, 2004; Liyanage & Thomas, 2002; Williams, 1998). For example, investigations of the usefulness of cmaps/v-diagrams to illustrate university students' evolving understanding of mathematics topics found students' mapped knowledge structure became increasingly complex and integrated as a consequence of multiple iterations of the processes of presentation → critique → revisions → presentation over the semester (Afamasaga-Fuata'i, 2004). Two primary student teachers receiving the same instruction on the historical and conceptual development of "multiplication", each constructed vastly different cmaps, "one internalised the concept in its systemic interconnections, while the other continued to see it through formalistic lens" (Schmittau, 2004, p. 576). Others also demonstrated the value of cmaps as pedagogical planning tools to provide an overview of a topic (Brahier, 2005) or to analyse mathematics lessons (Liyanage & Thomas, 2002). As a consequence of in-class concept mapping activities, presentations and critiques and independent practice, a secondary student teacher constructed topic cmaps, which explicitly illustrated both the *conceptual development of derivatives* and an *implicit sequence of lessons* (Afamasaga-Fuata'i, 2006). Research also demonstrated the usefulness of v-diagrams to scaffold students' thinking and reasoning and to illustrate their understanding of the interconnections between theory and application in mathematics problem solving (Afamasaga-Fuata'i, 2005), scientific inquiry (Mintzes, Wandersee & Novak, 2000) and epistemological analysis (Novak & Gowin, 1984, Chang, 1994). In summary, the literature shows three uses of cmaps/v-diagrams that are relevant to teacher education. Firstly, cmaps/v-diagrams as *learning tools* to illustrate students' evolving knowledge and understanding of a content domain. Secondly, cmaps/v-diagrams as *analytical tools* to scaffold the content analysis of topics or problems. Thirdly, cmaps/v-diagrams as *pedagogical tools* to organise and sequence teaching and learning activities using the results from the content analyses of syllabus outcomes.

- **Learning tools**. Individuals construct cmaps/v-diagrams to prompt the development of *deeper understanding* of the conceptual structure of mathematics topics and the *reinforcement of connections* between mathematical principles and their applications in problem solving. The constructive activities can result in a single cmap/v-diagram, which is a visual snapshot of a student's knowledge,

understanding and skills at that time. Alternatively, the activities may be repeated over a period of time, with social critiques and revisions in-between presentations. The resulting parade of progressive cmaps/v-diagrams, records the developmental trend of students' increasingly sophisticated knowledge, understanding and skills. Having school students individually or collaboratively construct cmaps/v-diagrams is a viable "*assessment for learning*" and "*assessment of learning*" strategy.

- **Analytical tools**. Student teachers construct cmaps/v-diagrams to visually display their interpretations and content analyses of syllabus outcomes. The displayed language, principles, concepts, methods and interconnections should be developmentally appropriate to be commensurate with the staged outcomes of the relevant syllabus.

- **Pedagogical tools**. Student teachers construct cmaps/v-diagrams to plan teaching and learning sequences, by distinguishing between prior, new and future knowledge to facilitate developmental teaching and learning approaches. Teacher-constructed cmaps/v-diagrams (of topics, problems or activities) can be used as advance organisers to scaffold instruction and assessment.

The focus question for this paper is: In what ways do hierarchical concept maps and vee diagrams facilitate the development of primary student teachers' deep understanding of mathematics? Figure 1 shows a hierarchical cmap which describes the paper's main ideas. The data presented is from a case study of a BEd (Primary) student teacher (Susan) who was concept mapping and vee diagramming over a semester, in her third year mathematics education unit.

Methodology

The case study started with a familiarisation phase in which Susan was introduced to the metacognitive strategies of concept mapping and vee diagramming using simple topics such as fractions and operations with fractions. The main project for the unit required Susan to construct a comprehensive, hierarchical cmap of a mathematics topic to be selected from the primary mathematics syllabus, and v-diagrams of related problems, which demonstrate the applications of the mapped concepts. There were three phases to the project. The first phase required Susan to compile an initial list of concepts, based on a content analysis of the relevant syllabus outcomes, to construct an initial topic cmap and v-diagrams of two problems. These were presented and critiqued in class before further revision and expansion. The second phase involved the presentation of a more structurally complex, expanded cmap and v-diagrams of more problems. These were socially critiqued and returned for further revision and expansion. The third phase was the final submission of a more comprehensive, hierarchical topic cmap and more v-diagrams of related problems and activities, which extend previous work and incorporate comments from previous critiques, and a reflection journal of her cmap and v-diagram experiences. Data collected included cmaps/v-diagrams from the familiarisation phase and three phases of the main project including a reflection journal. This paper presents samples of Susan's submitted work to illustrate the application of cmaps and v-diagrams as analytical and pedagogical tools for the topic "division".

Findings

Samples of Susan's individually constructed cmaps/v-diagrams are shown to indicate her interpretations of the relevant syllabus outcomes and her conceptual understanding of division problems.
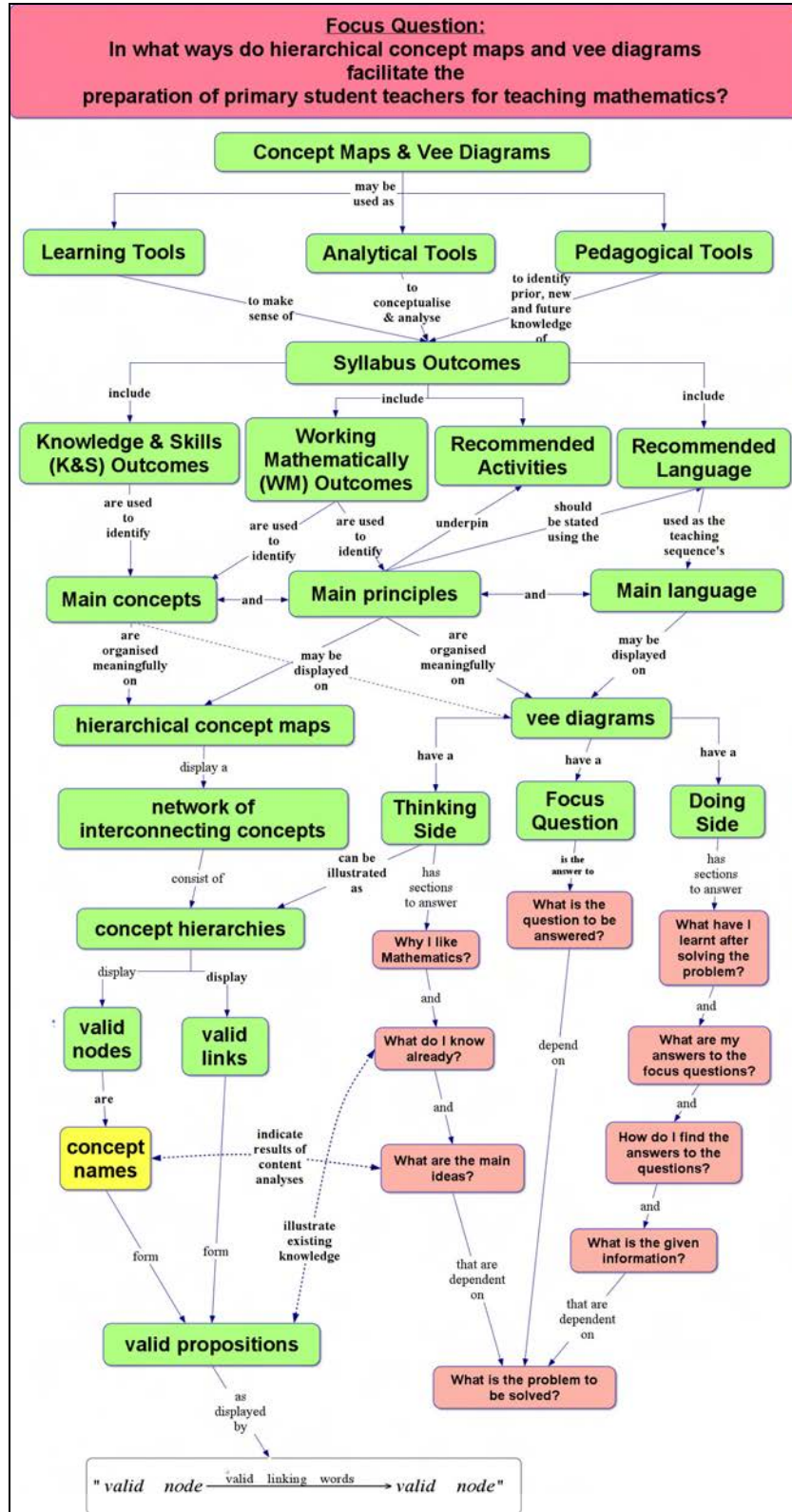
Figure 1. A hierarchical concept map of the paper.

Shown in Figure 2 is Susan's v-diagram of a problem "What is 4263 divided by 9?". The v-diagram's left side is called the *Thinking Side* while the right side is the *Doing Side*. The latter displays the given information, methods, and answer to the focus question including useful things learnt or potential directions for future extension while the former illustrates the relevant conceptual information (mathematical principles and main concepts) and including statements of one's mathematical beliefs.
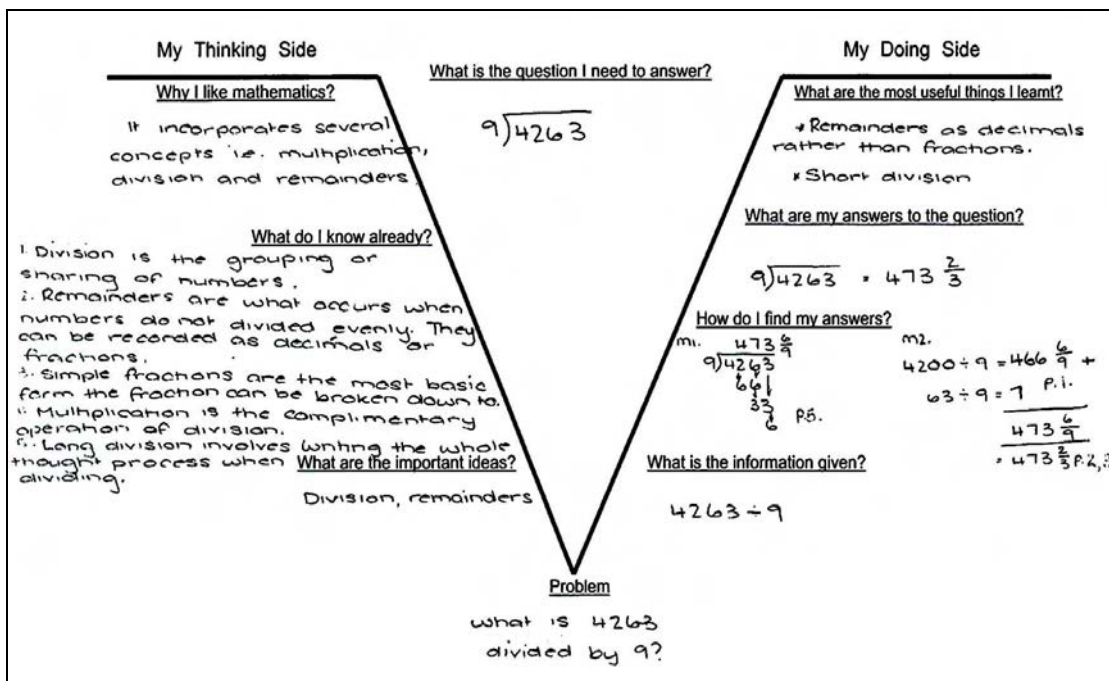
Figure 2. Susan's vee diagram of a division problem.

Displayed in Figure 2 are the results of Susan's content analysis of the relevant syllabus outcomes on the *Thinking Side* in terms of the mathematical principles for division of 4 digit numbers (underneath *What do I know already?*). These principles underpin the two methods of solutions displayed on the *Doing Side* (underneath *How do I find my answers*?). The main and relevant concepts (i.e., *What are the important ideas?*) are listed as "division (and) remainders". At the tip of the vee is the main learning activity, namely, the given problem statement. Moving onto the right side of the vee, the given information is "4263 ÷ 9" (underneath *What is the information given?*) and the answer "$9\overline{)4263} = 473\frac{2}{3}$" (underneath *What are my answers to the question?*). At the top right corner are Susan's perceptions of *What are the most useful things I learnt?*, or, alternatively, *Where do I go from here?* or *What is the future learning?* as a result of solving the problem. These are "remainders as decimals rather than fractions" and "short division", as potential topics for future learning activities. Directly on the opposite side of the v-diagram are Susan's mathematical beliefs (*Why I like mathematics?*) as contextualised in this particular problem, namely, "It incorporates several concepts i.e., multiplication, division and remainders." In summary, the v-diagram displays, on the left, the relevant mathematical principles and concepts, expressed in the appropriate language for primary students at Stage 3 (10–11 year-old students) (NSWBOS, 2002) with the two methods of solving the problem on the right. Susan's mathematical beliefs are on the left with suggestions for future learning on the right.

A cmap of the problem (Figure 3) was constructed using the theoretical information displayed on the *Thinking Side*, namely, the entries for *What do I know already?* and *What are the important ideas?*
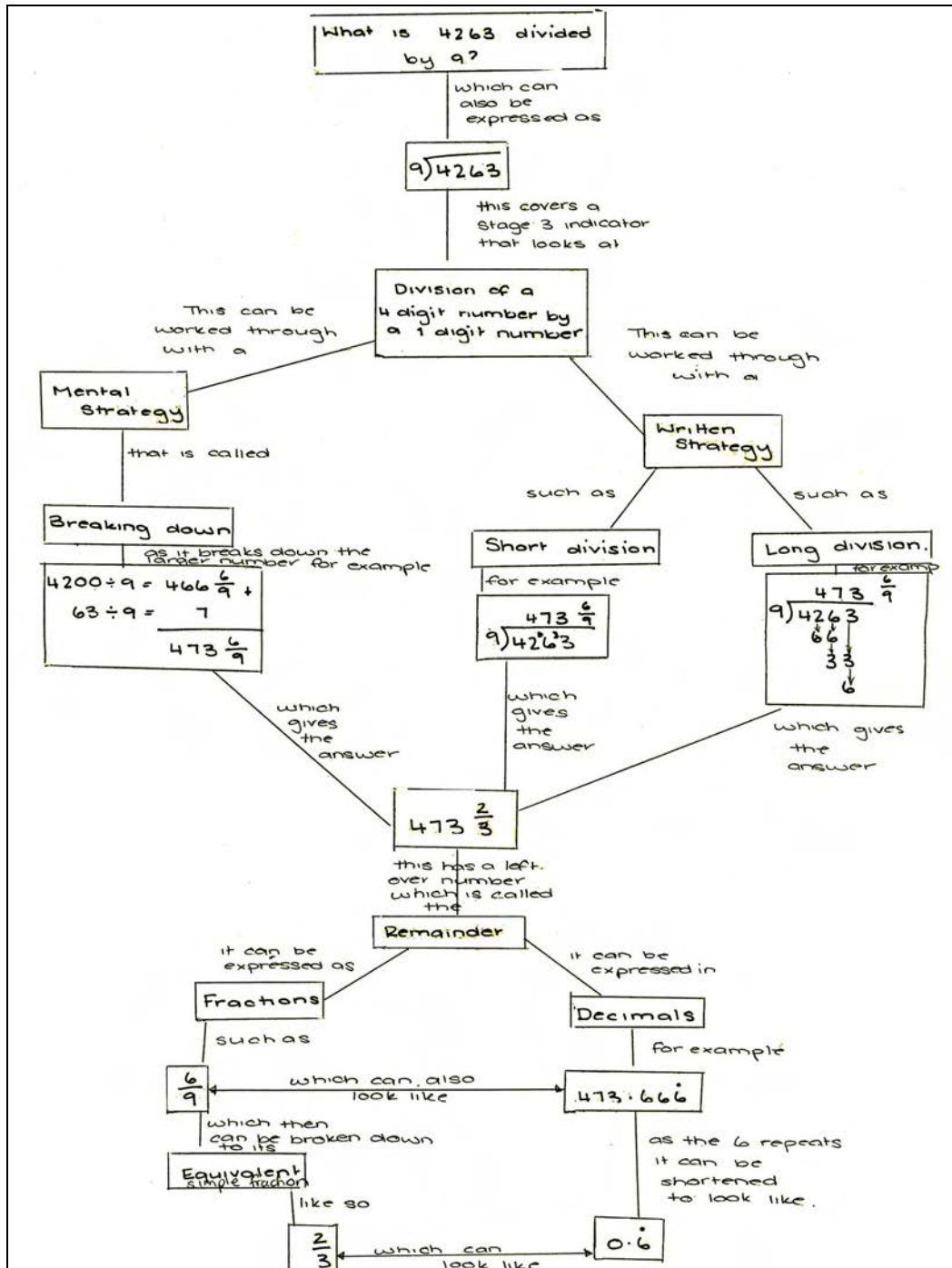
Figure 3. Susan's concept map of the division problem.

Constructing a cmap challenged Susan to meaningfully organise the nodes (concepts) to demonstrate her deep understanding of their interconnectedness and of their applications in solving the problem. The cmap's focus question was "What is 4263 divided by 9?" The cmap illustrates the process of finding an answer to the problem and including the relevant *syllabus indicator* (division of a 4 digit number by a 1 digit

number), *main concepts* (mental strategy, written strategy, breaking down, short division, long division, and remainder), *relevant concepts* (fractions, decimals, equivalent/simple fraction), and various *illustrative examples*; e.g.,

$$9 \overline{)42\,^{6}6\,^{3}3} = 473\tfrac{6}{9} \text{ and } \tfrac{6}{9}.$$

The top two-thirds of the cmap illustrates the process and linked concepts (from the relevant syllabus outcomes) with rich descriptions of interconnections before merging at node "$473\tfrac{2}{3}$", which thereafter shows the relevant concepts "fractions" and "decimals" applied to the "remainder". The latter was flagged in the "future learning" section (top right of the *Doing Side*) of the v-diagram, as a potential extension to the current division problem. The conceptual meaning of remainder and its different representations are further encapsulated by the second principle (underneath *What do I know already?*) on the v-diagram. In summary, the cmap illustrates the connections between different types of strategies and methods, illustrative examples of strategies and concepts, and types of representations of the remainder. Further to showing the two methods on the v-diagram, the cmap also includes "future learning", namely, a third method (short division) and decimal representations of the remainder and final answer.

Evidently, the combined usage of a v-diagram and cmap to analyse a problem (guided by the relevant syllabus outcomes), was an effective combination to cogently communicate the richness of meanings that exist between mathematics principles, main concepts and methods. Susan's v-diagram and cmap of the simple division problem represent the "expert" view of division in primary mathematics and the teacher's conceptual view of the division curriculum to be taught.

Discussion

The presented data is only a sample of Susan's work over the semester. However, it visually illustrated the richness of information that can be captured by the combined usage of cmaps and v-diagrams in analysing syllabus outcomes and mathematical problem solving. Through her statements of mathematical principles using the appropriate mathematical language (v-diagram), Susan captured the conceptual and developmental essence of division as recommended in the syllabus outcomes. The cmap on the other hand, visually illustrated the two strategies to be promoted in primary mathematics, as well as making explicit visual connections between division, remainder and relevant prior knowledge. Future extensions of the current problem are displayed as decimal representation of the remainder and the short division method. The rich linking words describing the nature of the interrelationships between nodes resulted in valid propositions. The latter converted the hierarchical cmap into a network of meaningful, interconnecting propositions that coherently describe an answer to the cmap's focus question.

Susan's cmap/v-diagram both demonstrated that a deep understanding of the division concept can be developed and reinforced. The v-diagram structure provided not only the space to express one's mathematical beliefs and critical reflections, but also projections for future learning as evident from Susan's rationale for an activity sequence. Overall, constructing cmaps/v-diagrams evidently encouraged Susan to move beyond a procedural view to a more conceptually based justification of methods and a purposeful and clearer understanding of sequencing prior, new and future learning activities to minimise student confusion. Findings from this paper contributes knowledge to the

literature on (a) the pedagogical use of concept mapping in mathematics as a means of developing deep understanding of mathematics concepts and (b) the innovative use of v-diagrams to display prior and new knowledge and future learning directions.

Implications

The visual display of the theoretical and procedural information of a problem, on a cmap and a v-diagram, effectively encapsulated the interconnection between the Knowledge and Skills and Working Mathematically Syllabus Outcomes. This suggests the potential educational value of regularly exposing primary students to the strategies of concept mapping and vee diagramming to enhance and develop both their conceptual and procedural understanding of mathematics. Doing so would necessarily enable working and communicating mathematically amongst students in the classroom. Having students construct their own cmaps and v-diagrams before and after a topic, as part of their normal mathematics classroom practices, is an area worthy of further investigation.

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Sketching graphs from verbal, rather than numerical, information[*]

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Contemporary statistics education research emphasises the development of intuitions and sense-making and the cultivation of students' ability to communicate orally, in writing, numerically, and using graphs. Tasks with data-free scenarios provide students with an opportunity to look beyond procedures and the mechanical aspects of data-processing. This article describes a series of three tasks in which students sketched graphs directly from verbal, rather than numerical, information. The tasks also allowed students to develop a deeper understanding of the purpose of creating a graph. The performance of the students was superior to that reported by other researchers.

Introduction

A criticism of traditional statistics education is the emphasis on calculation and procedural competence at the expense of the development of intuitions and sense-making. Influential statistics education researchers (Mokros & Russell, 1995; Garfield & Ben-Zvi, 2004) have argued that traditional teaching actively interfered with students' natural intuitive sense and obscured "the big ideas of statistics".

A key recommendation of current statistics education research is that teachers should facilitate the development of a classroom culture where students communicate their understanding through class discussion, written work, numerically, and the focus of this article: graphs.

To encourage the development of intuitive notions and to shift teaching practices away from procedures, Garfield (2003) recommended the inclusion of data-free scenarios when teaching statistics. Bakker and Gravemeijer (2004) recommended students should be given the opportunity to construct their own representations of information. Mevarech and Kramarsky (1997), in a study of Year 8 students, reported that only one quarter of students could create what they considered a robust graphical representation directly from verbal information. Rossman and Chance (cited in Garfield, 2003) reversed the traditional sequence of students' statistical analysis, and provided students with tasks matching verbal descriptions to plots of data. Friel et al. (2001), in a study of critical factors in comprehension of graphs, recommended tasks where students were encouraged to "read the data, read between the data, and finally, to read beyond the data". The use of graphical representations may also provide an alternative learning pathway, particularly for visual learners.

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[*] Paper accepted by peer review

Method and results

The three tasks were part of a two-week research study of the statistical concept of *distribution.* The study was conducted with a Year 9 class of 29 students in a Hobart metropolitan co-education high school.

The first of the three tasks was highly "scaffolded" and conducted in a conventional classroom, where students were supported by teacher instruction, classroom discussion, and interaction with their peers. The statistics education software Fathom, a product of Key Curriculum Press, was used as a teaching aid. In tasks two and three students worked independently under traditional examination conditions.

A principal objective of the research study was the introduction of continuous distributions as a foundation for the study of normal distributions, in more senior years. All three tasks were selected to provide a familiar context for students. The researcher emphasised to students that the graphs they sketched should "tell a story". All tasks allowed for a wide-range of responses — all students submitted graphs — but more capable students were able to demonstrate their understanding by producing a more sophisticated response.

Task 1: Cross country run

The Cross-country Run task was based upon a task developed by Bakker and Gravemeijer (2004).

Scaffolding and cues for the task was provided by an introductory exercise examining the "Murders in Chicago" dataset, supplied with the education statistics software Fathom. A data projector displayed a Fathom-generated dot-plot as an image on a whiteboard. The relatively large database of 1670 sample points allowed the construction of a smoothed continuous curve. The task was constructed in three stages. Firstly, a whole-class discussion of the dot-plot representation of the "Murders in Chicago" data set identified the key features of the distribution. Secondly, the students were asked to copy the distribution from the whiteboard. Copying a 1670 sample point distribution is clearly impracticable, and the class responded accordingly. The researcher sketched an outline of the dot-plot to demonstrate that a smoothed curve can accurately convey the essential features of a distribution. Finally, the "Cross-country run" worksheet was distributed to the students. Students were asked to sketch the three distributions demonstrating how the performance of a class of students, as shown by the time taken to complete a 5 km run, had improved through training. Additional scaffolding was provided by aligning the "set of three" axes vertically to allow easier comparison of the three graphs. The researcher again emphasised that the graphs should "tell a story."

A Year 9 group is training for a school 5 km cross-country race. The race is to be held in 6 weeks time. The students have decided to train for an hour, 3 days every week leading up to the race. The training program includes a 5 km run every week.
To monitor their improvement, they want to make three graphs: one before the training program starts, one halfway through the training program and the third graph at their final training session three days before the cross-country race.
Draw three graphs that "tell" the following story:
Graph 1: Before training started, some students were slow and some students were already very fast. The fastest ran the 5 km in 22 minutes. Most of the students were on the slow side.

Graph 2: Halfway through the training program, the majority of the students ran faster. The fastest and the slowest students had only made a small improvement in their time.

Graph 3: On the students' final training session the spread, of times was much smaller than at the beginning. Most students had improved their time by about 5 minutes during the 6 weeks. The fastest students had only improved their time by 2 minutes. There were still a few slow students but most had a time that was closer to the fastest runner than at the beginning of the training program.

Draw your graphs on this page. Use whatever scale, notes, type of graph, etc., you think helps best "tell" the story.

A high proportion (83%) of students provided a robust response to the task demonstrating that the task was understood, but the task was highly scaffolded. Examples of a strong response are provided in Figures 1 and 2. Five (17%) of students provided an idiosyncratic, or weak, response showing that the task was not understood. An example of an idiosyncratic response is provided in Figure 3.

Student G1408E provided a response which showed the fastest running time, and that the fastest time had decreased by two minutes. The majority of students are shown as taking "half as long again" as the fastest students, and the shift in the peak shows the majority of students has improved by five minutes. A scale has not been included on the *y*-axis as the number of students is not known. A subtle point made by the student (intentional or otherwise) is that the area under the curve has stayed constant. This is a point worthy for subsequent class discussion as it a key concept underlying normal distribution.
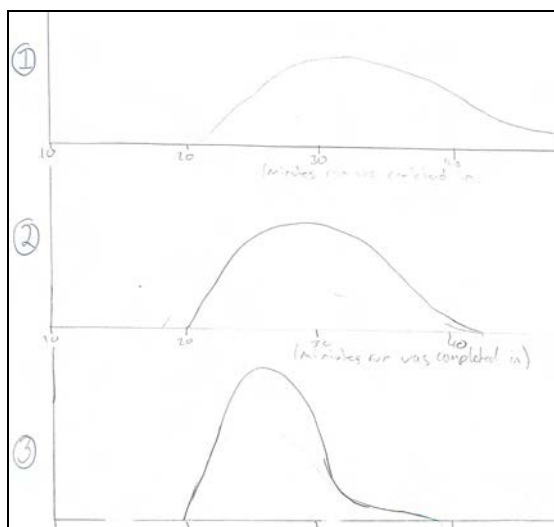
Figure 1. Student G1408E Cross-country run.

Student D2806C provided a similar response with a detailed and accurate time scale, and a *y*-axis without a scale. The student annotated the graph to show how the time of the majority (in this instance, the mode) has decreased by five minutes. The shape of the curve is unconventionally discontinuous and could be interpreted as a response transitional between bar charts and continuous distributions. The student's graph suggested the majority of students was represented by the mode — appearing as the peak of the graph — rather than *the area under the graph*. A point worthy of class discussion is that the graph shows that the slowest time is clearly defined, but that

information had not been provided in the original question. This could be used to prompt a class discussion on how best to use a graph to show this information.
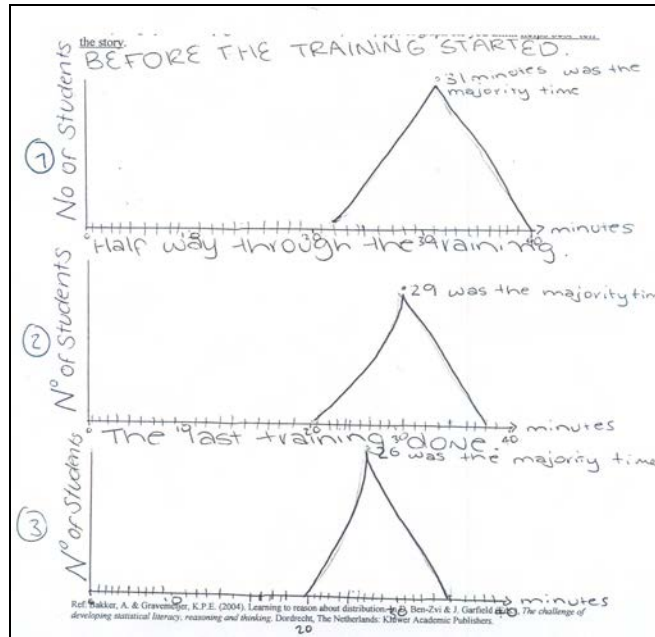
Figure 2. Student D2806C Cross-country run.

Student Y2206D, shown in Figure 3, did not show a strong understanding of the task, or the concepts involved. The student used a bar chart and a continuous distribution. The student appears to consider the data set as a collection of individual students rather than the more sophisticated thinking of seeing it as an aggregate.
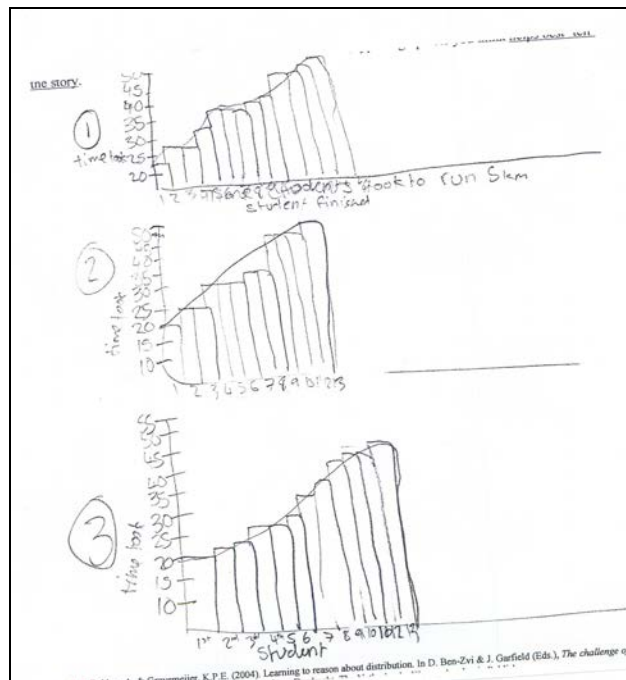
Figure 3. Student Y2206D Cross-country run.

Task 2: Mid-year science examination

This task was conducted in a traditional examination environment, as one question, on an hour-long assessment paper. The phrasing of the question "most students received between 75–85%…" was designed to encourage students to consider "most" students, not as the mode, but as the students lying within a range of scores.

> Sketch a distribution describing the following situation. Your science teacher is providing general feed-back to the class on their performance in the mid-year exam. She said all students have passed. Most students received between 75–85%. A few students who have missed lessons because of illness or overseas travel received about 60%. The top mark was 96%. Clearly mark or show all the information.

More than 70% of students provided a strong response showing that the task was understood. Only one student provided an idiosyncratic response. Although not specifically requested, almost all students used a continuous distribution.

Student G1408E provided a response showing that most students received a mark between 75–85% and the highest and lowest marks awarded. A feature, common to many other students' responses, was the reversal of the scale, with the scale decreasing from left to right. This is not incorrect, just unconventional.
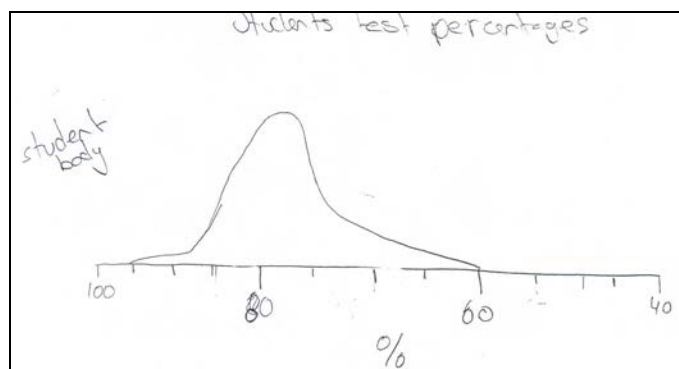
Figure 4. Student G1408E Science examination.

Student L2103S provided a highly idiosyncratic response that shows features normally portrayed in a cumulative frequency chart. The axis orientation is unconventional with the "number of students" given as the independent (x-axis) variable. The axis has been labelled but a scale has, quite correctly, not been included
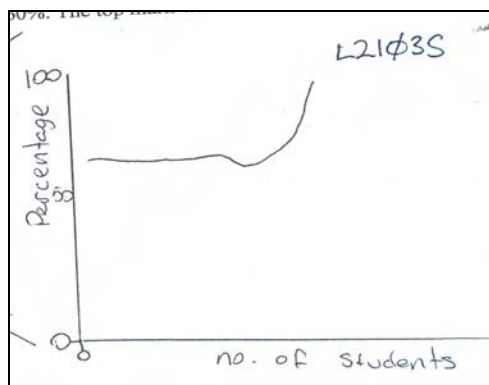
Figure 5. Student L2103S Science examination.

Task 3: Metro bus-service

This task was also conducted under traditional examination conditions, and included in the same examination paper as Task 2.

> Sketch a distribution describing the following situation. As the principal private secretary to the Minister of Transport, you have been asked to sketch a distribution of the performance of the Metro bus service. Your minister is keenly aware that the voting public expect the buses to run on time.
> By definition, any bus arriving more than 4 minutes behind schedule is late.
> Almost all buses arrive within 4 minutes of the scheduled time. About 5 percent arrive late and a few buses in non-peak times arrive early. Occasionally buses do not arrive at all. Clearly mark or show all the information.

This is a more complex task designed to extend students to the limit of their knowledge and skill. Students provided a rich range of responses including continuous distributions (45%), bar charts (36%), dot plots and tally sheets, and idiosyncratic responses that appear to be based upon cumulative frequency graphs. The use of bar charts was an unintended outcome, as the research study had focussed on continuous distributions. The strong response of students using bar charts showed that many students had made a considered decision as how best to represent the information.

A significant and sophisticated feature of one student's response (Figure 6) was the use of negative numbers to describe buses that had arrived early. A point worthy of class discussion is that the distribution shows buses are as likely to arrive early as late, but this is entirely consistent with the statement "almost all buses arrive within four minutes of the scheduled time."
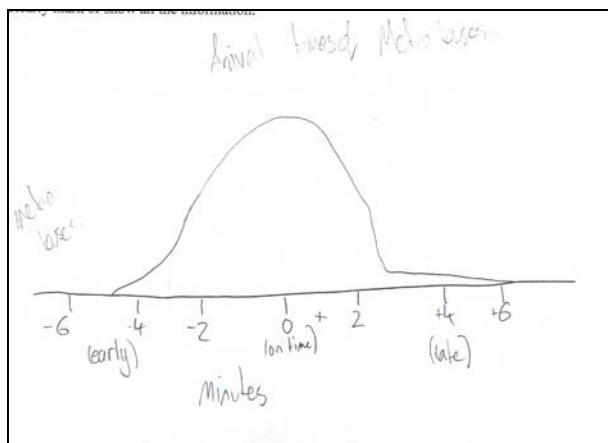
Figure 6. Student G1408E Metro bus-service.

Several students used bar charts, but in two distinctly different ways. Student L2103S used time as the independent *x*-axis, but was then faced with the problem of how to include buses that "did not arrive at all"; i.e., at infinity. This student resolved this difficulty by including this information as a separate category in the top left hand corner of the chart. This was a far superior response to ignoring or not including this category of responses. The time axis showed the student using negative number to describe buses that arrived early.
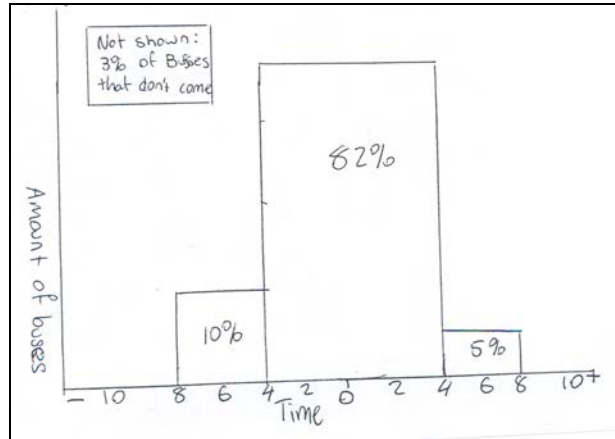
Figure 7. Student L2103S Metro bus-service.

Student N2208B used a bar chart, but the bars represent categories of whether the bus had arrived early, on-time, late, or not at all. On reflection, this chart most accurately, concisely and effectively portrays the information provided in the question.
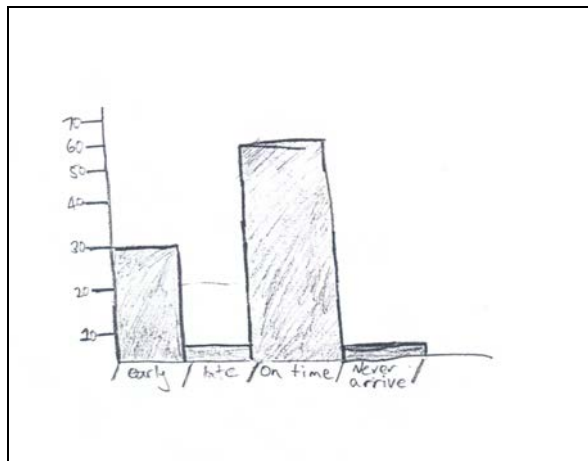
Figure 8. Student N2208B Metro bus-service.

Student N1902P provided a sophisticated response which sought to convey bus arrival times at various times of the day. The area of the discs represents the number or frequency of buses, and the student is clearly intending to show that, in non-peak times of morning and afternoon, buses are likely to arrive early.
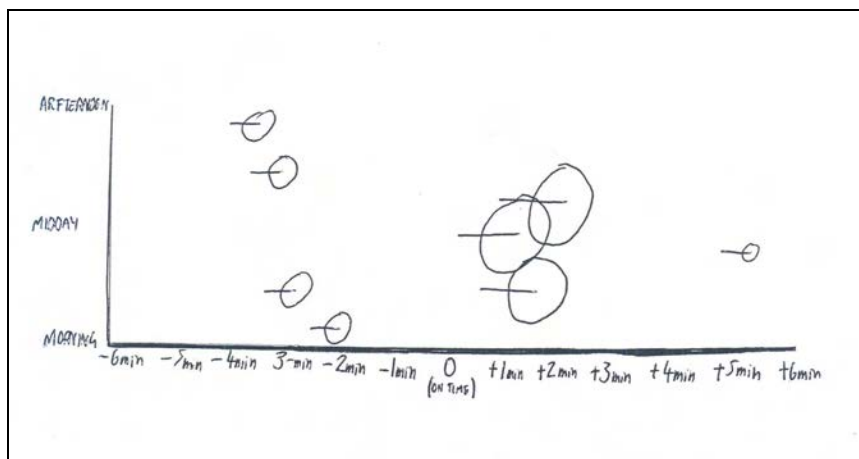
Figure 9. Student N1902P Metro bus-service.

Discussion

The first task, the cross-country run, was a complex task. To successfully complete the task students identified the key information, and simultaneously addressed the range and the area beneath the distribution representing the majority of the student times. The three component graphs must also be addressed simultaneously, as the three graphs must be linked to demonstrate the decrease in students' running times.

Tasks 2 and 3 were less complex, but were also less supported. Neither of the two tasks required students to make a comparison between graphs.

All three tasks involved constructing graphs without using data. To complete the tasks, students needed to convey a deeper understanding of the information, rather than constructing the graph applying a set procedure to a data-set. "Sense-making" has application in statistics, algebra and mathematical modelling.
The range of student responses provided a rich resource for subsequent class discussion.

The tasks did not allow student development to be demonstrated, as the first task was the most complex, but also the most highly supported. Tasks 2 and 3 were conducted in traditional examination conditions where students worked independently.

Students' performance on this series of tasks — although assessed under different criteria — may have been superior to that found by other researchers. The study of Mevarech and Kramarsky (1997) found a quarter of students correctly transform to graphic representation; this study showed three-quarters of Year 9 students could construct another representation.

A key recommendation of contemporary statistics education research is the cultivation of students' ability to communicate their understanding of key statistical concepts. This understanding can be communicated in a number of ways. Students who are not confident writers could convey their understanding in classroom discussion; students who do not have the self-confidence to participate in classroom discussion, may prefer to give a written explanation; and visual learners might find graphical representations effective.

A goal of the research teaching unit was to encourage a shift in students thinking from discrete, to continuous, distributions. This was designed to provide an essential foundation step for the introduction of continuous normal distribution, at more senior school years. Students' responses indicated that the first tentative steps to an understanding of continuous distributions were being taken, and that the construction

and interpretation of continuous distributions were being added to students' repertoire of skills.

The use of the bar chart in Task 3 — not the intention of the task — was noteworthy. The students had made a conscious decision to use a type of graph introduced outside the study unit, and used in earlier school years. If an indicator of deep understanding is the ability to apply principles to a variety of tasks, then deep understanding had been demonstrated.

Recommendation for teaching

- Include data-free, or largely data-free, scenarios when teaching statistics.
- Cultivate a sense that a graph must "tell a story".
- Use students' work as a resource for class discussion.

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Quality teaching of mathematics:
Common threads and cultural differences[*]

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It is well established that Asian students achieve highly in international monitoring tests such as PISA and TIMMS. This finding has led to a considerable interest in identifying factors that contribute towards this performance, including teaching approaches. Two lessons observed in Chinese schools are analysed using the framework provided by the AAMT Standards and the New South Wales model of Quality Teaching. The lessons observed shared many of the characteristics identified as exemplary mathematics teaching in the Australian context, although there were also some differences attributable to the cultural context. The implications of these findings for Australian teachers are discussed.

During the development of the AAMT *Standards for Excellence in Teaching Mathematics*, teachers from all over Australia proposed three domains of practice that they believed identified highly accomplished teachers of mathematics. These domains, Professional Knowledge, Professional Practice and Professional Attributes, address the diverse knowledge, skills and understanding that characterise quality teaching. Professional Knowledge is concerned with understanding students, mathematics and how students learn mathematics; Professional Practice deals with those day-to-day aspects of teaching such as planning and assessment; Professional Attributes focuses on the behaviour of teachers with respect to their colleagues, the school and wider community. The attributes identified in the AAMT Standards are applied to individual teachers, at a personal level.

A number of studies in Australia have attempted to identify quality in teaching at the classroom level, that is, what it is that good teachers do in their classrooms to develop students' understanding and develop learning e.g., Queensland School Reform Longitudinal Study, (2001); Department of Education, Science and Training; New South Wales et al., (2004). In New South Wales, a model of quality teaching is used as a tool for analysing lessons with the aim of improving teacher standards and students' learning [Professional Support & Curriculum Directorate, (2003)]. This model has three dimensions, and a number of elements within each dimension, that provide a framework for considering classroom practices. These are summarised in Table 1.

Table 1. Dimensions and elements of the NSW Model of Quality Teaching.

Dimension	Intellectual Quality	Quality learning environment	Significance
Elements	Deep knowledge	Explicit quality criteria	Background knowledge
	Deep understanding	Engagement	Cultural knowledge
	Problematic knowledge	High expectations	Knowledge integration
	Higher-order thinking	Social support	Inclusivity
	Metalanguage	Students' self-regulation	Connectedness
	Substantive communication	Student direction	Narrative

When the elements of the NSW model are considered against the framework of the AAMT standards, it appears that all the characteristics of highly accomplished mathematics teachers would be called upon to deliver lessons that addressed all three dimensions, in some way. For example, planning lessons that address problematic knowledge and substantive communication requires understanding of how students' learn mathematics; engagement and explicit quality criteria implies good planning for learning; and connectedness and cultural knowledge suggests community involvement. All of these aspects are underpinned by knowledge of, and interest in, mathematics itself. In other words, teachers who operate in ways that meet the expectations of the Quality Teaching model, will also demonstrate many of the personal characteristics identified in the AAMT Standards.

There is also, currently, considerable interest in learning from Asian countries following the high performance of students from Hong Kong and Korea, for example, in international programs such as PISA [Organisation for Economic Cooperation and Development, (2004)] and TIMMS [Mullis, Martin, Gonzales & Chrostowski, (2004)]. Western countries are interested in understanding what teachers in these countries do to achieve such high results. Popular perception of teaching in Asian countries is that it is teacher-centred and based on "chalk-and-talk" approaches. These methods would be the antithesis of the features of exemplary teaching identified in the AAMT Standards and the NSW model.

There are many studies attempting to characterise mathematics classrooms in diverse cultural settings e.g., Clarke & Keitel, (2006). The small scale study reported here was opportunistic, based on lessons observed during overseas travel in Hong Kong and China. As such, it is not generalisable, nor necessarily representative of mathematics lessons in these countries. Nevertheless, it does provide an insight into what Chinese teachers perceive as quality teaching, given that in both instances the teachers were aware of visitors to their classrooms.

The lessons

Two lessons are described, one in a primary classroom and one in a high school.

The primary classroom lesson

The primary lesson was a demonstration lesson, focussing on technology use in a mathematics classroom. The students were around 9- or 10-years old, and the lesson addressed early ideas of permutations and combinations through a problem involving choosing different ways of combining teddy bear clothing.

Before the lesson started, the teacher was laughing and joking with the students as he checked the various bits of equipment that he was going to use. The desks were arranged in rows, and each place had a set of three small teddy bears dressed in different coloured jumpers, together with two small teddy-sized bags in two more colours. The children were excited, chattering and rushing around. Then the lesson began. The teacher bowed formally to the children and they bowed back, sat down with their arms folded and focussed all their attention on him. This position they maintained for the next 40 minutes or so.

The lesson started with a short video of Deng Deng, a cartoon cat obviously familiar to the children, trying to choose a teddy bear with a bag. He had three different coloured jumpers and two bags to choose from, exactly as the children had on their desks. The teacher obviously asked children, "What would you choose?" As the children responded, they stood up, the teacher acknowledged their answer, clearly making the point that different combinations were possible. He then looked puzzled. How many different choices were possible? He placed large cut out teddies and bags randomly on the board, as a prompt. The class quietly began working on the problem, individually. Some used the teddies on the desks, others were getting out coloured pencils. Each child recorded an answer in a way that appeared to make sense to them.

The teacher quickly moved round the room and chose several children to show their solutions on the blackboard. He had obviously chosen a variety of solutions, including some that would not lead to a correct response. One child was drawing each teddy, others were using coloured chalks, making lists and so on. At this point there was clearly a discussion about the solutions and each child explained the thinking behind their response. Skilfully, the teacher selected two approaches and then demonstrated how these could lead to a generalisable approach, using both the teddies he had previously placed on the board and then a computer animation to illustrate a tree-diagram approach to the problem. He then posed the extension problem of adding in an extra bear. The children quickly responded as a class. Several related problems were then shown: routes from school to home via the park or the temple, food choices for dinner, and so on. In each case the solution was sought from the group rather than from individuals. The lesson ended with the teacher asking the children if they had enjoyed themselves, to which they responded enthusiastically. The formal end of the lesson came when the teacher and children bowed and thanked each other. At this point, the children lost their serious demeanour and began chattering and laughing with each other.

The secondary classroom lesson

This lesson took place in a government secondary school in Hong Kong, in a second year classroom. The school was a participant in a learning oriented assessment project that aimed to develop teachers' assessment skills. The lesson focus was rates and ratios,

and three teachers chose to team teach it, an unusual situation for the class. The start of the lesson included a formal greeting and a brief explanation of why there were three teachers. An initial problem was posed:

> I took 1.5 hours to run 16.5 km on one day and 2 hours to run 20 km on the next day. Had my performance improved? Explain your answer.

The students were unused to being asked to explain their thinking and it was obvious that, for some, this was difficult. After some time in which each student worked on the problem in silence, the teacher collected the responses and then chose several students to explain their responses. The students' answers varied. One student admitted he had no idea how to do the problem; another pointed out that it was unrealistic because no-one could maintain the same rate for two hours; others explained their response in terms of the amount of time taken or the distance run; and others used rates as a justification. The teacher asked students to comment on these responses, gradually establishing the idea of a rate.

A second teacher then took over the class. He extended the idea of a rate into the topic of apartment rentals. In Hong Kong, apartments are measured in square feet so the students were asked to work on several problems, from their text book, that involved comparing cost per square foot. This involved some large numbers! Very little time was given for completing the problems before the solutions were given and discussed.

At this point the class was beginning to be restless, shown by yawning and shuffling feet. There was no disruptive behaviour. The third teacher took over. He was the specialist physics teacher and showed a video comparing the speed of a snake attack with that of a Kung Fu master. The students were engaged immediately. They sat up and watched the video with great concentration. Finally, the data shown on the video was used to emphasise the concept of a rate and the comparison worked out formally on the board. Incidentally, the Kung Fu Master was quicker. The lesson ended at this point with homework being set and a formal completion to the class time.

Discussion

Both lessons were impressive to watch. The students were engaged and worked diligently on the problems posed, but also appeared to enjoy themselves. The relationships between the teachers and the students were relaxed and it was evident that students had no fear of making mistakes in the ways that they responded to the questions. The subject matter presented was appropriate for the students concerned and the stimulus material, such as the cartoon and the teddies and the video, certainly struck a chord in the students. In terms of the model of Quality Teaching, both lessons had Significance, characterised by the Background Knowledge (the cartoon, video), Connectedness (using familiar situations such as food for dinner and rental rates) and some elements of narrative (the story of the cat). A Quality Learning Environment was also evident in the students' Engagement, which appeared genuine. The High Expectations and the Social Support ,that was implicit in the way that the students answered questions, also suggested a Quality Learning Environment. In terms of Intellectual Quality the lessons were impressive. Both lessons had a single focus (permutations and combinations; rates and ratios) and this was explored in different ways showing Deep Knowledge. Both lessons centred on Problematic Knowledge, through their starting points, and Substantive Communication was also present.

There were, however, some significant differences from Australian lessons. The pace of the lesson was much faster. I remember thinking at the time of the Primary lesson that it would probably have taken two lessons to reach the same point. I was told later, by a teacher, that they taught to the best student rather than trying to support the weaker students, as is the culture in Australian classrooms. Although there was considerable discussion in both classrooms, this was always mediated by the teacher: children did not talk to each other. It seemed that the job of the student in these classrooms was to work hard at making sense of the lesson by yourself, rather than with the help of peers. Students had little control over what was happening, although there was freedom for them to get an answer in their own ways. These lessons were far from the "chalk and talk" image of Chinese teaching, although the teacher did stand at the front for most of the lesson, and controlled all classroom activity.

The teachers obviously knew their students well, and understood how to engage them in learning. The lessons were well structured, with a clear beginning, middle and end and were tightly focussed. A variety of approaches was used to address the same concept, allowing for differences among learners in the class. Many of the qualities of highly accomplished teachers were demonstrated by the teachers concerned.

Some aspects were missing, however. There was little student direction in either lesson. All the problems were posed by the teacher and, although students could use any route to obtain an answer, the teachers ultimately decided which solution strategies to pursue. Both lessons started with a problem, although that problem was relatively closed, so it is questionable the extent to which higher order thinking was needed. The lessons aimed to develop understanding of some underlying and important concepts, rather than creative problem-solving skills. There was little peer interaction. Students were not asked, or expected, to work together, although there was some low level conversation in the Hong Kong secondary school lesson.

Implications for teachers

What can teachers learn from these examples? Australian teachers could consider the narrow focus of the lesson, but the deep way in which the concept was considered. Although there was variety in the lessons, this did not distract attention from the core idea of the lesson. There was also little time wasted on behaviour management. Students did not move about the classroom, and neither did the teachers, to any great extent. The lack of movement cut down possibilities for poor behaviour, and also avoided time being spent on organising groups, for example. There was, maybe surprisingly, little emphasis on practice: students did not work through a large number of examples. Nor was time spent on checking homework, although homework was given in the secondary lesson.

On the other hand, the Chinese students were not developing the skills of working in teams or wrestling with a problem that was genuinely messy or ill-defined. They were not asked to justify their answers, apart from the starting activity in the secondary lesson, which was obviously unusual. The Chinese students appeared to be developing conceptual understanding but with limited opportunities to use this knowledge in a meaningful way.

The fast pace of the lesson also meant that, inevitably, some students did not fully grasp the idea. In the Australian context it is the teacher's role to ensure that all students understand; in the Chinese context it appeared that it was the students' responsibility to learn. This difference implies a divergence of teaching philosophy based on the local

culture. Whether that philosophy is transferable, or even whether transfer is desirable, is a matter for debate.

It seems clear, however, that many of the qualities of exemplary teaching identified in both the AAMT Standards and the Quality Teaching model are found in good teachers and exciting classrooms in other cultures. Those aspects that were not observed may be so culturally bound that they are applicable only in a specific context. This thought reminds us that we should beware of making simplistic judgements about teachers and teaching. All teachers, in whatever cultural situation, should avoid pressure to adopt a particular teaching style because it is apparently linked to high achievement. Teaching is a more subtle, and messier, task than such simplistic approaches imply.

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Year 7 students' understanding of area measurement[*]

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This paper reports on a study of Year 7 students' understanding of area measurement. 43 students completed a written test just before and immediately after studying areas of rectangles and triangles. Twelve of these students, representing the range of ability in the sample, were interviewed as they solved three area tasks. Results indicate that many students confuse area and perimeter, use slant and perpendicular height interchangeably, and do not understand the basis of the formula for the area of a triangle. Implications for teaching are discussed.

Introduction

Area measurement is an important topic in school mathematics. It has numerous real-life applications and can be applied to mathematical concepts like multiplication of fractions, enlargement and similarity. Area measurement is also used by many teachers and textbooks to demonstrate number properties such as the commutative law of multiplication and algebraic results like the expansion of binomial expressions (Schultz, 1991). A good understanding of area concepts is also essential when learning integral calculus.

Area measurement is based on partitioning a region into equally sized units which completely cover it without gaps or overlaps. Although the idea of a unit is fundamental, it is often neglected as teaching tends to focus on numerical results, ignoring the spatial relationships involved in measuring areas (Kordaki & Portari, 2002). However, students need substantial experience in covering regions by drawing unit squares and noting the regularities that occur (Outhred & Mitchelmore, 2000). With practice, students can recognise that the array of rows (or columns) are *necessarily* equivalent and can therefore be regarded as a composite unit so that rows (or columns) can be repeated instead of drawing each individual square.

It cannot be assumed that the row-column structure of rectangular arrays is automatically recognised and understood by all students (Battista, Clements, Arnoff, Battista & Borrow, 1998). However, many textbooks either pay little attention to covering activities or provide regions which are already partitioned, so that all students have to do is count the squares (Carpenter, Coburn, Reys & Wilson, 1975). Such tasks can be easily done by simple addition and obscure the essential multiplicative relationship of the rectangle's length and width to its area (Battista, 2003). But the process of tessellating a region with a unit must be well understood if students are to develop a relational understanding of area measurement, especially in the correct application of the area formulae (Woodward & Byrd, 1983).

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[*] Paper accepted by peer review

Researching students' understanding of area concepts

Students at all levels experience difficulties in dealing with area concepts, a phenomenon documented in research with primary children (e.g., Reynolds & Wheatley, 1996), middle school students (e.g., Comiti & Moreira Baltar, 1997) and pre-service teachers (e.g., Baturo & Nason, 1996). Studies have sought to identify misconceptions associated with area measurement. Hirstein, Lamb and Osborne (1978) interviewed 106 elementary grade children. They observed five commonly held area misconceptions, including: a tendency to make judgments about the area of a rectangle by examining the length of one of its sides; basing conclusions on informal observations when partitioning and recombining shapes to compare their areas; point-counting, or counting all regions equally regardless of their geometrical shape when using a grid; double-counting units around the corner of a rectangle; and counting tick marks rather than counting units.

Dickson (1989) interviewed 20 students aged 9 to 13 on four separate occasions, before and after a teaching program, about their understanding of area concepts and their ability to find the area of rectangles. She reported that most students could correctly describe the meaning of the word "area" and that just over one third of the students, who already knew the formula $A = l \times w$, used it almost exclusively, even when it was inappropriate to do so. Another third learnt the result during instruction, and the remaining students continued to use non-multiplicative approaches, despite having been taught the area formula.

Kidman (1999) interviewed 36 students, 12 each from Years 4, 6 and 8 to investigate strategies used in area measurement. She found that many confused area and perimeter: 19 students used additive approaches and perceived the area of a rectangle as the sum of its dimensions, a phenomenon that was consistent across all grades. Students who employed additive rules used a number of strategies when comparing the areas of two rectangles during the interviews including an insistence on using a ruler to measure sides, and a preference for vertical rather than horizontal alignment of rectangles. Those who used additive rules also mistakenly believed that doubling the length of both sides of a rectangle would double its area.

Method

The present study uses a multi-method design combining written pre- and post-tests with a semi-structured clinical interview. The sample consisted of two Year 7 classes, one each from two coeducational, comprehensive high schools in middle-income suburbs of Sydney. In one school, the class was of mixed ability, with 21 students, while in the other, the class of 22 students was the bottom set of 6 classes. Each class teacher chose 6 students (3 boys and 3 girls) to reflect the spread of ability and these 12 students constituted the interview sample.

The written tests

All 43 students completed a 20-minute written test of five questions on two separate occasions, just prior to being taught the area topic and immediately afterwards. A 30 cm ruler and a transparent, square-centimetre grid sheet were made available to students but calculators were not permitted. Students were expected to record their calculations and explain their thinking throughout the test.

Question 1 asked for a definition of the word "area". Question 2 displayed a 5 cm by 3 cm rectangle with tick marks each centimetre around its perimeter and students had to

find its area and explain their method. Question 3 showed a right-angled triangle of sides 3 cm, 4 cm, and 5 cm with tick marks at 1 cm intervals along the two perpendicular sides and students had to calculate its area and justify their solution strategy. Question 4 gave an L-shaped figure, labelled "not to scale", with integer values given for 4 of the 6 sides. Students had to calculate its area, setting out their working. Question 5 required students to make an accurate drawing of two shapes with area 24 cm^2, one a rectangle and the other not.

The interviews

Semi-structured clinical interviews were videotaped immediately after the area topic. Each student was interviewed individually for approximately 15 minutes while they attempted two tasks. First, the interviewer (author) gave each student a 10 cm by 8 cm cardboard rectangle and asked them to find its area, explaining their method. Students could use a 30 cm ruler and a transparent, plastic grid. In task 2, the ruler and grid were removed and the interviewer gave the students two more cardboard shapes (a right-angled triangle of sides 10 cm, 12 cm and approximately 15.5 cm, and a parallelogram of base 10 cm and perpendicular height 8 cm) and asked them to predict whether either shape had the same area as the rectangle. The students were encouraged to explain their reasoning and the triangle and parallelogram were discussed in the order which the students considered them.

When discussing the triangle and rectangle, the interviewer asked how the students could compare the two areas. Students typically suggested superimposing the shapes, aligning their edges, or measuring them. The interviewer allowed the students to try their chosen method and interpret their results. When discussing the parallelogram and rectangle, the interviewer again began by asking how the students could compare the areas. Those who wanted to superimpose the shapes were asked for alternative ideas, while those who wished to align or measure the sides were permitted to do so and interpret their measurements. The interviewer next asked if there was another way to compare the areas without measuring them and students who now suggested superimposing the shapes were permitted to do so and talk about what they saw. A pair of scissors was provided for students to cut and recombine the parallelogram and place it over the rectangle. Having done so, the interviewer asked students to explain what they saw.

Results

The results of the written tests and the interviews are reported separately.

The written tests

Most students defined area as the "space inside" a figure (pre-test = 53%; post-test = 72%). Others expressed similar ideas, including "how big a shape is" (14%; 5%) and "how many units [cm, m, etc.] fit into it" (5%; 5%). However some described area as "length times width" (19%; 12%), or defined it in terms of perimeter (7%; 5%). Most could calculate the area of the rectangle in question 2 (72%; 93%) by measuring the sides or counting the tick marks and multiplying the numbers (53%; 47%), drawing a grid inside the rectangle (19%; 44%), or using the transparent grid to add squares (9%; 5%). In question 3, about half the students were correct (44%; 49%) while those who were not drew a grid inside the shape and added the squares, but could not count the

fractional parts that resulted (21%; 40%). Some failed to divide by 2 (12%; 5%), and some multiplied all of the side lengths together (12%; 0%).

Question 4 proved more difficult with few achieving the right answer (14%; 23%) and many finding the perimeter; some did so correctly (5%; 37%), but others omitted the lengths of the two unlabelled sides (30%; 21%), or did not attempt the task (19%; 2%). Most students accurately drew the rectangle in question 5 (58%; 79%), while some did not measure the sides correctly (7%; 7%) and, as in question 4, others drew a rectangle with perimeter 24cm (19%; 14%). Drawing a non-rectangular shape was more difficult; those who were correct (26%; 30%) drew a right-angle triangle or created a composite shape by drawing 24 unit squares one row at a time, but many who tried were unable to do so accurately (33%; 14%). Here too, students' predilection for perimeter persisted (21%; 30%).

The interviews

All students found the area of the rectangle in task 1, either by measuring and multiplying its sides (6 students) or by overlaying the grid, counting squares along two adjacent edges and multiplying (6 students). Only one student could explain without assistance why multiplying length by width guaranteed the correct answer; two more students were eventually able to explain it after prompting, but the remaining nine could not. For task 2, five students correctly stated that the area of the triangle was smaller than the rectangle. They superimposed the shapes and described how the protruding pieces of the triangle could be cut off and repositioned inside the rectangle without covering the entire space. Three students claimed the triangle was larger "because it looks bigger" but, after superimposing, they explained the task. Two students were convinced that the triangle was larger "because it's taller" and the remaining two students could not decide: one because he claimed he first needed to measure each shape, and the other "because they're different shapes".

Seven students stated that the parallelogram "looks about the same size" as the rectangle, while the others were unsure. The students either aligned the corresponding edges (7 students) or measured the sides (5 students). However, in doing so, all students compared the *slant* height of the parallelogram with the *perpendicular* height of the rectangle, thereby falsely concluding that the parallelogram had a greater area. When they recombined the parallelogram to fit it exactly over the rectangle, only four students could eventually explain what had occurred; the other eight remained completely mystified.

Discussion

Three closely related misconceptions which emerged from analysis of the test and interview data are now discussed.

Area–perimeter confusion

Students often confuse area and perimeter (Kouba, Brown, Carpenter, Lindquist, Silver & Swafford, 1988). In test question 1, students labelled the rectangle's sides 5 cm^2 or 3 cm^2 or, more commonly, gave its area in linear units as 15 cm. They confounded their language during the interviews and spoke of perimeter when they should have referred to area, and vice versa. For example, students referred to the "area of a side" or said that the side was "eight square centimetres long". Sometimes the students would correct

their words, but often they gave the impression that they did not realise what they had said.

Many based their written answers on perimeter rather than area. This occurred in questions 2 and 3, but was more prevalent in questions 4 and 5, which had a greater degree of difficulty. Students tried various methods to calculate the perimeter of the rectangle: some measured the sides and added the lengths; others used a variation of *counting around the corner* (Hirstein, Lamb & Osborne, 1978) in which they omitted the corner squares and obtained only 12 cm; and some simply counted the tick marks. Similar approaches were applied to the triangle in question 3.

In question 4, the more complex L-shaped figure was too demanding and many reverted to perimeter, sometimes accounting for the missing sides but often simply adding the given lengths, as shown in Figure 1. In question 5, they drew shapes with a perimeter of 24cm rather than an area of 24cm^2. These students may be relying on what Tierney, Boyd and Davis (1990) call *signposts*, or familiar ideas on which to focus when they feel disoriented by strange mathematical settings, even if the signpost (perimeter) is irrelevant to the task (finding area).
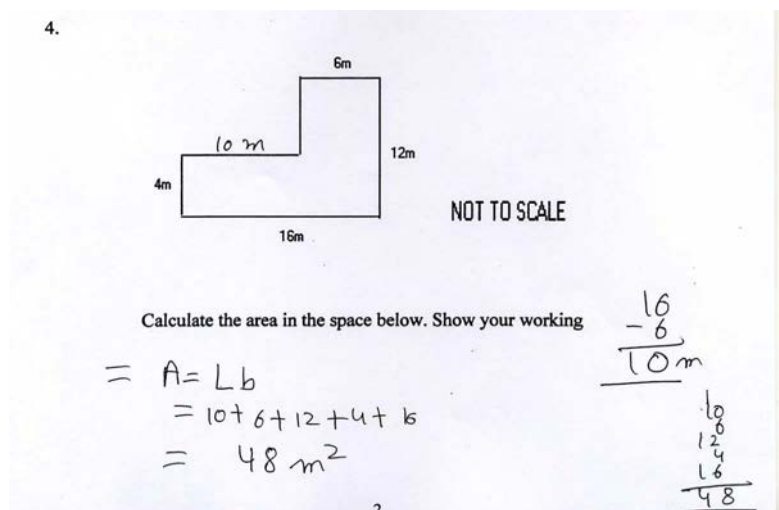
Figure 1. Area–perimeter confusion.

Perpendicular-slant height confusion

Related to area–perimeter confusion was the students' inclination to make decisions about areas by comparing any height, regardless of whether it was at right-angles to the adjacent side or not. For instance, some thought that the parallelogram in interview task 2 was larger than the rectangle because it had a longer side, but they contrasted the perpendicular height of the rectangle with the slant height of the parallelogram. There was also some confusion about the precise meaning of "height" and it seemed a flexible notion for some because they focused solely on the edges of the shape.

Interviewer: Do they have the same height, those two shapes [the rectangle and parallelogram]?

Student: When it's like that [slant height of parallelogram aligned with perpendicular height of rectangle], it's a bit bigger. But when it's like that [parallelogram recombined to form a rectangle], it does.

Interviewer: So what would you say, the rectangle and parallelogram, they do or they don't have the same height?

Student: The height, they do now, but they didn't before.

Responses to interview task 2 revealed that students did not appreciate the importance of using perpendicular height to calculate areas of triangles and parallelograms, and this made it difficult for them to understand how two shapes, despite ostensibly having different dimensions, could still have the same area.

Student: That's difficult! Well, it's still 10 at the bottom and 8 [perpendicular] at the sides, but it was 9 [slant] before. Somehow the area looked bigger before, but now it's smaller. [Pause] I don't know, that's difficult, I can't think of an answer for that. That piece that we took [a right triangle cut from one end of the parallelogram and recombined to form a rectangle], we made the height straight so that we could make a rectangle but I still don't understand how it got back to 80, we lost 10. [The student then measured the perpendicular height of the parallelogram] That's 8, so when I measured it there [slant height] it was 9. So we've kind of cut off a centimetre and put it over there.

Interviewer: So it's 9 on the edge and 8 in the middle. Can you explain that?

Student: It's just how the shape is, I guess.

Interviewer: Can you explain it?

Student: No.

The student is confounded by the "loss" of 1 centimetre as the parallelogram (of slant height 9cm) is recombined to form a rectangle (of perpendicular height 8cm). This perpendicular-slant height confusion is probably the reason why so many students could not explain how the two shapes had the same area because it seemed to them that the dimensions of the parallelogram were greater than the rectangle.

The students' exclusive reliance on the slant height of the parallelogram for calculating its area might also be interpreted as an over-generalisation of the $A = l \times w$ formula. As Tierney et al. (1990) found, many students do not recognise the underlying array structure of the formula and often apply it indiscriminately. For example, some multiplied all of the lengths together to find an area—perhaps another feature of area–perimeter confusion—while others found the area of the triangle by multiplying 3 by 4 by 5. In the interviews, some students claimed that it did not matter which sides of the triangle were used to calculate its area, so long as any two lengths were multiplied together and the result halved. It is worth noting that students who arbitrarily multiplied the lengths of shapes like this tended to define area as "length times width" in the test.

Poor grasp of the rectangle–triangle relationship

Those who were prepared to multiply any two lengths of the triangle to find its area may also have been exhibiting a limited understanding of the fact that the area of a triangle is half that of the rectangle which shares its base and perpendicular height. The following exchange took place after a student had correctly calculated the area of the triangle in interview task 2.

Interviewer: Why do you halve it? [The product base $\times$ height for the area]

Student: I was taught that way.

Interviewer: Would it matter which lengths you multiply?

Student: I'm not sure.

Interviewer: Why did you choose those lengths?

Student: I don't know.

Students' inadequate appreciation of the rectangle–triangle relationship was further demonstrated in test question 3 where, if they understood the relationship, the task of calculating the area of the right-angled triangle would have been relatively straightforward. However, many did not use the rectangle connection, choosing instead to construct a grid and count squares. But, certain squares were incomplete so it was difficult to recognise the precise fraction represented by each part-square, particularly as some students did not draw their grids accurately. Thus most could not obtain the correct answer. A few students went to great lengths when trying to calculate the total number of grid squares that comprised the triangle, as Figure 2 shows.
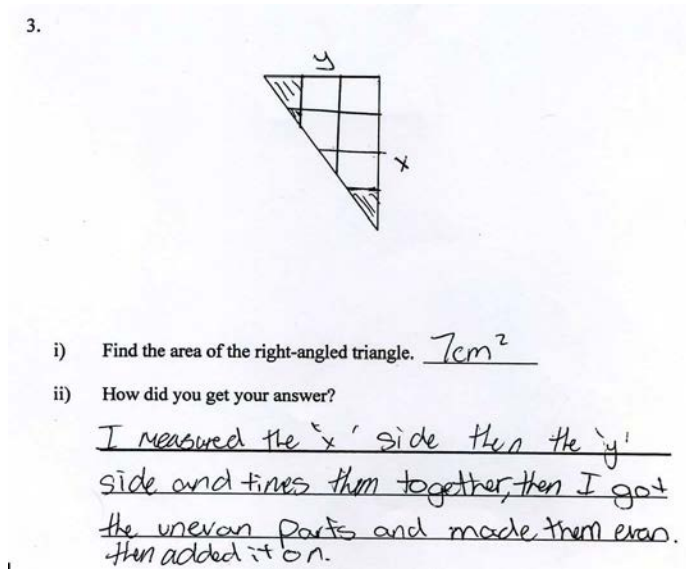
Figure 2. Poor grasp of the rectangle–triangle relationship.

Conclusion and implications for teaching

The results of the present study confirm the fairly grim assessment of students' abilities in area measurement reported in the literature. Many students made substantive errors in the test and interview tasks and, even when they were correct, most could not provide an adequate explanation for the procedures they used.

It is clear that area teaching, which focuses too much on formulae at the expense of conceptual understanding, is unlikely to be successful. Even Year 7 students need substantial hands-on experience in partitioning regions and constructing grids. It is important that students' attention is drawn to the structure of the grid during such activities but this is more difficult when students do not have to construct the grids for themselves (Reynolds & Wheatley, 1996). For instance, students in one school had grid books and spent much time drawing shapes and counting squares to find areas using the ready-made grid which probably accounts for their increased use of primitive counting techniques in the post-test.

Practical activities like those used in the interviews could be profitably employed to help students deal with likely misconceptions such as area–perimeter and slant–perpendicular height confusion. In doing so, teachers should avoid telling students what to think; rather, students must be encouraged to explain what they see in their own words. In this way, their misunderstandings are more likely to be revealed and more easily rectified.

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Students constructing interactive learning objects for conceptual development in mathematics

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Somerville House

The following paper provides an outline of an ASISTM project involving Somerville House, Moreton Bay College, Brisbane State High School, Clayfield College, Anglican Church Grammar School, Queensland University of Technology and The University of Queensland. This project aims to enhance student engagement in mathematics and technology through the creation of interactive mathematics learning devices. Integral to the project is the students constructing Interactive Learning Objects for conceptual development in mathematics. The students are taking a topic in mathematics and developing an interactive Learning Object to demonstrate and deepen or consolidate their understanding of this topic. In taking on the role of the teacher, the students will create objects in order to instruct and assist other students in the learning of specific mathematics content. What the students create will also be used as an indication of their preferred learning style. This will support teacher reflection upon their own pedagogy leading to better connection between teaching and learning. Identification and development of preferred mathematical learning styles underpin the project, as do the principles of collaboration amongst peers in order to engage a more varied pedagogy in the classroom.

Introduction and background

In understanding and accepting the simple premise that everybody is active sometimes, and reflective sometimes, it follows that it may be in the best interest of students and teachers of mathematics to seek to incorporate these values into learning, planning and pedagogy. This ASISTM project aims to do so by asking students to be more active in their learning of a mathematical concept and by being even more active in demonstrating that knowledge. Subsequently, the teacher and the student are both required to be reflective of the process and the resultant models, and be mindful of how this may affect teaching and learning of mathematics into the future.

There is no doubt that individuals have the capacity to see and experience the world in different ways. Howard Gardiner's much talked about Multiple Intelligences is one framework which may be used to understand that persons learn and act in differing ways. Focusing on children in the middle years of schooling, some of these intelligences seem to be more prominent than others. A quick survey of 106 11–13 year-old females at Somerville House showed that the Interpersonal, Musical and Bodily Kinesthetic were the most used Intelligences, with Intrapersonal and Mathematical/Logical being the least favoured. Contrasting to this would be the generally accepted presumption that much mathematics teaching in Years 7–12 focuses

on a style that predates even our own generations: one that is founded on didacticism and rigor. The intention of this project is not to dismiss the traditional modes of teaching and learning mathematics, but rather to investigate whether there are also alternatives to cater to the immediate needs of the current generation. The concept of learning styles, particularly mathematical learning styles, addresses this question as we may gain valuable insight as to how our students think and how they would prefer to learn. The idea that individuals learn differently is not new but can still be ignored in many mathematical classrooms. The use of the very student-friendly Multiple Intelligences framework allows students to understand that they have preferred ways of thinking and expressing themselves. We can then open up to the students the possibility that these learning styles are linked indelibly to their personalities. Once this is achieved, we can encourage students to think about how they may use this knowledge to help them understand mathematics or, indeed, any subject.

For the purposes of this project, Learning Objects can be described as any resource that can be used in the teaching of concepts or content. These objects can be created, as in the case of this project, by using technology and involve platforms such as Excel, MovieMaker, Word, Dreamweaver, PowerPoint or Esiteach. However, they can be as simple as an instructional poster, booklet or model. Learning Objects may provide a way for students to express their own preferred learning styles. For example, use of auditory material may suggest that the students have a preference for the Musical intelligence while the use of numerous pictorials may indicate a strong visual learning style. In this project, students will create Learning Objects for a particular mathematics topic, with the focus on technology as the method, or pathway, of creation.

Assuming that any Learning Object created by students is a reflection of how they understand particular mathematical concepts, it may be a logical step to also assume that this may be useful to the teacher in determining their own pedagogy. Whilst we have a class full of individuals, it is also safe to say that these individuals fit into certain categories, from academic to social. Collection of data pertaining to how a certain group of students best learn a mathematical concept may have benefits for the teacher when next approaching a similar cohort with the same, or related, content matter. It is the aim of this project to determine how individuals and groups of students, in their own estimation, best learn mathematics and, then, to adjust teaching styles accordingly in order to become better, and more relevant, mathematics teachers.

An identified need in the teaching profession is the area of greater peer collaboration, support, and sharing of information and resources. The ASISTM project aims to develop ties between schools and universities, as well as creating links to the professional world. The project relies heavily on the interaction between the cluster schools and the sharing of teaching ideas and experience in order for any data to be relevant beyond the bounds of each specific class. There is also a pressing need to make mathematics more relevant to the changing world and the potential career opportunities that exist. The creation of stronger links with tertiary education and the IT community may assist in this area.

The teaching program

The project is a team effort where all cluster schools take part in its complete life-cycle. After the initial concept development, each school will trial, or is already trialing, the program as outlined with a specified cohort (Year 7–10). The cluster schools will submit the data and resultant student work for viewing by all partner schools. Feedback and professional development will be shared across academic staff, both relative to the

student work and, subsequently, as to how this will affect future pedagogy. The final resource bank website will be able to be accessed and added to by each cluster school as determined by their specific needs and work. Consideration of each participating school now follows.

Somerville House

Somerville House is running the program across the entire Year 7 cohort. There are five phases as follows:

1. Teacher professional development including, learning styles, assessment of mathematics and learning styles as well as the use of IT (Excel, Moviemaker, Flash, Esiteach, PowerPoint) in the mathematics classroom. Professional Development was undertaken as part of the cluster or within the Somerville communities follows:
    (a) Learning Styles — Shelley Dole (University of Queensland)
    (b) Mathematics Assessments and Rubrics — Shelley Dole
    (c) Use and creation of IT — Greg Egan (industry professional), Sally Mack (IT/Science/Mathematics teacher and independent creator of Learning Objects using IT — Somerville House)

2. Introduction of students to possible software that they may use in the creation of their Learning Objects. Students will be provided with an overview of each program so that they have the scope to choose the format of delivery that best suits their needs. Each platform can be taught separately from mathematics curriculum, or can incorporate the teaching of mathematical content.

3. Introduction of students to the concept of Multiple Intelligences. Students are encouraged to identify their personality traits that may influence their learning at school and to link these to Howard Gardiners' 8 Multiple Intelligences (Logical/Mathematical, Verbal/Linguistic, Bodily/Kinesthetic, Visual/Spatial, Intrapersonal, Interpersonal, Musical, Natural). The students take an online test, then observe and log how they may "use" their top three resultant Intelligences in the learning, or understanding, of particular topics.

4. The students must create a Learning/Teaching Object focusing on such abstract mathematical questions as: "Why do we flip and multiply?", "What does the equals sign mean?", "What is a percentage?". From recommendations of our Senior School teachers, these notions were highlighted as areas that require more depth of understanding, amongst our mathematics students, in later years.

5. The completed Learning Objects will be evaluated in two ways. For students the Learning Objects will be assessed to measure the extent to which students have consolidated, or gained, a deeper understanding of the mathematical content. For teachers the Objects will be viewed with the intention of determining whether there are possible "better" ways to teach this content, to other groups of students in the future.

Brisbane State High School

Ten classes in Year 10 at Brisbane State High School have undergone a seven week project in their mathematics classes. The project aimed to produce a Learning Object in a mathematical area of their choice. Students were encouraged to select a topic that was either being studied currently in Yr 8 or Yr 9, or an area of mathematics that they did not fully understand, but which they wanted to know more about. Each class was given examples of Learning Objects in both Vegas and PowerPoint application format. They were then given an opportunity to build a simple Learning Object in each medium and taught the basic ICT skills required with using this software.

Students were able to work individually or in groups of up to three people. Each student completed a learning log which detailed their activity in each lesson. Students also provided a concept map of how they intended to get across their message, before they began making the Learning Object. Students were expected to have at least 3–5 minutes of footage if they chose to use the Vegas format. Students were given a four week period in order to put together the Learning Object. The quality of the Learning Object was judged predominantly on how effectively they conveyed the mathematics and success of the project will be determined by the increased perceived level of understanding within the mathematical area chosen.

Moreton Bay College

Moreton Bay College is implementing the ASISTM project in one Year 10 Preparatory Mathematics B class and one Year 9 Extension Mathematics class.

1. As with all schools involved in the project, Moreton Bay College took part in professional development that included:
    (a) IT (Word, Excel, PowerPoint, Easiteach, Flash, MovieMaker)
    (b) learning styles
    (c) interactive whiteboard and associated software (TeamBoard)
    (d) assessment of mathematics.

2. Students are to be introduced to the various software that will be utilised in the creation of Learning Objects. Students will be introduced to each piece of software in their mathematics class or be introduced to it in other subjects, particularly IT. Interactive Whiteboards will be extensively used in class. The students will be able to choose which software will be used in the creation of their Learning Object. The students, then, have a choice of whether to use the same software, or use another piece of software, in developing another Learning Object.

3. Students are to be introduced to the concept of Multiple Intelligences and how this may affect their learning. They are to undertake an online quiz and use the results of identifying their favoured learning styles, and choice of relevant software, to develop Learning Objects.

4. The students are to pick a mathematical concept that they are interested in and develop a Learning Object based on this choice. The audience of the Learning Object is their peers. However, students may also choose the alternative of creating a Learning Device to teach the topic to a younger age group.

5. The major focus of the assessment of the Learning Object is whether the intended audience obtained an understanding of the mathematical concepts that were delivered in the Learning Object. In addition, do the developers of the Learning Object also have a deeper understanding of the mathematical concepts? Finally, would the teachers of the students who developed the Learning Objects change their methodology with future classes of students?

Clayfield College

As part of their study in ITS (Information Technology Studies), Year 11 students develop ICT products for identified clients and their needs. Such products are usually developed using Macromedia Flash. Hence, in the interests of cross-curricula cooperation, the ITS staff are directing and assisting their students to develop products (Learning Objects) for Year 8 mathematics students (the clients). Thus, the needs for two subject disciplines, and their respective stakeholders, are being met by means of the one project.

Conclusion

Currently, the cluster schools are carrying out the project in the varied forms but are continually meeting and sharing their experiences. Some of the key points of interest thus far are:

1. The varying degrees of student motivation according to their age and gender.

2. Convincing students, teachers, parents and administrators of the value of the project and of the possible future benefits of not only improving mathematical understanding but also of improving mathematical pedagogy, students' academic self-awareness and collegial links.

3. Finding the time in the already crowded curriculum to deliver the project.

4. Deciding how to best assess the success of the project on an individual student basis as well as within an overall framework.

5. Teachers understanding the required IT well enough to assist the students in the creation of their modules.

Further dialogue between the cluster schools is essential for the success of this project, but the value has already been seen in the intellectual sharing of ideas from the five main participants. It is anticipated that this may be the cornerstone of a movement, in at least one school (Somerville House), toward a school program that deals with teaching our students how to think and learn more effectively, according to their individual strengths and weaknesses.

Chameleons in the classroom: Middle years teachers of mathematics

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Deakin University

What are the practices of effective teachers of mathematics? and What are the characteristics of effective mathematics teachers? are questions that have become prominent in recent times, particularly with respect to the Middle Years. In this paper some of the research that provides directions to possible answers to these questions is described. It is intended that the paper provides educators with an overview of the findings of research that have the potential to impact favourably on student outcomes in the Middle Years of schooling. The paper also raises the issue of how much change can be expected realistically: does the classroom chameleon exist?

Studies of effective teachers of mathematics

Several studies have identified characteristics that are linked with being effective as a teacher of mathematics, as well as other subjects. As part of the AAMT professional development strategy, a paper focussing on these characteristics was commissioned (Doig, 2005), in part, to assist teachers explore their own practice. In a similar vein, the *Improving Middle Years Mathematics and Science* (IMYMS) project (an ARC linkage project with the Victorian Department of Education and Training as a partner) was developed out of the *School Innovation in Science* (SIS) initiative (Department of Education and Training Victoria, 2004) (see, Tytler, 2004, for details of these projects). IMYMS is extending the SIS strategy to mathematics, and includes an explicit framework of effective teaching and learning, based on an extensive review of the literature on effective teachers of mathematics (Doig, 2003). While these strategies are both within Australia, interest in the effective teacher is not confined to this country, and the following section provides a brief overview of the international research in this area.

Practice issues

A seminal study of effective mathematics teachers, the *Effective Teachers of Numeracy Study* (Askew, Brown, Rhodes, Johnson & Wiliam, 1997) found that the effective mathematics teachers had a "particular set of coherent beliefs and understandings which underpinned their teaching of numeracy. Their beliefs related to (a) what it means to be numerate, (b) the relationship between teaching and pupil's learning of numeracy, (c) presentation and intervention strategies" (p. 1).

Teachers in the Askew et al. study were interviewed about their educational orientations to teaching, mathematics and styles of interaction with students. The results of these interviews led to the defining of three models of orientation that explained how teachers approached the teaching of mathematics. These orientations were *Connectionist*, *Transmission*, and *Discovery.* All but one of the highly effective teachers

were classified as connectionist, while teachers holding other orientations were all classified as being only moderately effective.

Connectionist teachers were those who had beliefs and practices based on valuing students' methods, using students' understandings, and placing emphasis on making connections within mathematics. These highly effective teachers believe that pupils develop mathematically by being challenged to think, through explaining, listening, and problem solving.

In relation to teaching, the effective, *connectionist* teachers believed that discussion of concepts and images is important in exemplifying the teacher's network of knowledge and skills, and in revealing pupils' thinking, and that it is the teacher's responsibility to intervene to assist the pupil to become more efficient in the use of calculating strategies. The highly effective teachers in this study believed that being numerate requires a rich network of connections between different mathematical ideas and an ability to select and use strategies that are both efficient and effective. They also believed that: almost all pupils are able to become numerate, and that pupils develop mathematically by being challenged to think, through explaining, listening, and problem solving.

The study also found that teachers' beliefs and understandings of the mathematical and pedagogical purposes of classroom practices were more important than the actual practices themselves, and that having an A-level or a degree in mathematics was not associated with being highly effective. The study confirmed other UK and US research that suggested neither mathematical qualifications nor initial training are factors strongly correlated to highly effective teachers of mathematics and this also was confirmed later by the *Leverhulme Numeracy Research Programme* (Brown, 2000) in the UK.

Background issues

The *Third International Mathematics and Science Study* (TIMSS) conducted in 1995–1996 collected both achievement data from students, and background information from school principals, teachers, and students. These data were linked so that the influences on the high-achieving students could be isolated. One group of students in the TIMSS was those in the middle years: that is, those who were 14-years-old. Martin and his colleagues (2000) reported comprehensively on the findings of this investigation and, in summary, their findings were that home background factors (for example, parents' levels of education, student aspirations to attend university) were the most significant distinguishing difference between high- and low-achieving students. However, factors that relate directly to the student's school experience were also strong indicators of the student's likely success on the TIMSS items. In Australia, Hollingsworth and her colleagues (2003) reported that a lack of emphasis on complex content matter in the syllabus was hampering student higher achievement.

Classroom issues

Research indicates that there exists a small set of classroom factors that appear to provide effective learning and work consistently across countries. The most striking of these factors would appear to be that "Schools [sic] where eighth-grade students were expected to spend time on homework in a range of subjects had higher average achievement in science and mathematics" (Cogan & Schmidt, 2003, p. 11). In their recommendations for reform of United States middle schools, based on the TIMSS results, Cogan and Schmidt (2003) make these five points:

1. That the curriculum should be coherent across the grades for all topics;
2. Fewer topics should be taught in the middle school;
3. Topics for study should be taught at an appropriate grade level;
4. Teachers should have clear and coherent standards of achievement to which to teach; and
5. Review and repetition in lessons should be reduced.

Policy issues

Several studies confirm that improving teacher quality is the most effective option for policy seeking to improve student learning outcomes (e.g. Darling-Hammond & Ball, 1998). Wenglinsky (2002) used NAEP data to examine factors affecting student learning in eighth-grade mathematics. He found that the factors included student background, teacher quality (e.g. a major in mathematics, professional development, classroom practices) and class size. Unlike Martin et al. (2000), Wenglinsky concluded that the impact of teaching can be said to equal SES, and even possibly somewhat greater.

Wenglinsky's research indicates that there is more return from investments that improve teacher quality and capacity than in reducing class size, such as attracting well-qualified mathematics teachers, preparing them well in methods for teaching mathematics and providing high-quality continuing professional learning. In Australia, Monk (1994) used the LSAY database to look at the effects of subject matter preparation of secondary mathematics and science teachers on student performance gains. Teacher content preparation was positively related to student learning gains. The effects of content knowledge were strengthened when accompanied by course work in pedagogy, leading Monk to state that "it would appear that a good grasp of one's subject area is a necessary but not sufficient condition for effective teaching" (p. 142). The implications of this research are clear, yet there have been indications that some states and some Australian universities have, in fact, been reducing subject matter preparation requirements in their teacher education courses over recent years.

A related message from international research on system factors is that the greatest return, in terms of impact on student learning outcomes, comes when policy makers give priority to investments that directly influence teacher quality and professional development.

Darling-Hammond (2000) found that variation in mathematics achievement on NAEP tests across the 50 US states was attributable more to variations in policies affecting teacher quality, than factors such as student demographic characteristics, class size, overall spending levels, or teacher salaries. The major policy variations affecting teacher quality were state requirements for teachers to be licensed and to have a major or minor in mathematics from university. She claimed that:

> the effects of well-prepared teachers on student achievement can be stronger than the influences of student background factors, such as poverty, language background, and minority status. And, while smaller class sizes appear to contribute to student learning … the gains are most likely to be realised when they are accompanied by the hiring of well-qualified teachers. (p. 39)

Further, after a major review of research on factors affecting student learning outcomes, the highly influential report of the National Commission on Teaching and America's Future, *What Matters Most* (1996), had come to the conclusion that the most

important influence on what students learn is what their teachers do. Its central recommendations for systemic reform called for:

1. the development of a new infrastructure for professional learning organised around standards for teaching;
2. new career paths that reward teachers for evidence of professional development; and
3. a focus on creating school conditions that enable teachers to teach well.

The characteristics

From the short overview above it is clear that, despite minor differences in emphases, there is agreement internationally about what are the characteristics of effective schools, teachers, and classrooms. The issue is, of course, that is it possible for any one teacher, or school, to have all the identified characteristics? To demonstrate the enormity of the answer to this question, the following lists contain the key, common, characteristics of effective teachers, grouped by who or what is being asked to change.

Teachers

The characteristics common to effective teachers are that they:

1. have sound content knowledge;
2. view assessment as a key facet of their teaching and learning programs;
3. model enthusiasm and interest in their subject by example;
4. value students' methods, and use students' understandings;
5. provide choice, share control; and
6. exhibit a caring attitude to students.

Teacher actions

The points below are the actions that effective teachers take more often than other, less effective, teachers. Effective teachers:

1. explain the rationale for tasks and assignments;
2. use a variety of teaching strategies;
3. engage students in making sense of what they are learning;
4. use both cognitive and affective teaching methods;
5. use advance organisers, and provide opportunities for practice;
6. create and implement sequences of lessons rather than single one-off activities;
7. place emphasis on making connections within and across learning areas;
8. continually audit their students' needs;
9. interrelate different modes of assessment with decisions about curriculum and instruction; and
10. include values, beliefs and social structures as part of their teaching.

Classroom environment

As outlined earlier, classroom environments affect student outcomes, and the effective classroom has identifiable qualities. That is to say, classrooms of effective teachers are places where students:

1. have focussed, sustained, and in-depth opportunities to learn;
2. are challenged to think through explaining, listening, and problem solving;
3. are encouraged to be autonomous and self-regulatory;
4. have varied structure, form, and context of their learning experiences;

5. are provided with appropriate feedback;
6. are engaged in their own learning; and
7. the classroom is linked to the broader community.

Other issues

Not all aspects of effectiveness are the sole responsibility of the classroom teacher: some responsibility rests within the school environment in which teachers work. The two major aspects that research suggests are the most important are leadership and the school's professional community. The characteristics of effective leadership and professional community are expanded bellow.

Leadership

Effective teachers are supported by their school leadership, and the characteristics of school leadership that encourages effective teaching are where it:
1. is transformational;
2. provides vision and facilitation;
3. provides a positive working environment for teachers;
4. facilitates organised collective learning;
5. employs a whole-school approach to school policies and practices;
6. communicates school policies and practices to the wider community;
7. informs and involves parents and the wider community;
8. provides a coherent program from primary through to secondary school;
9. encourages staff professional development; and
10. provides time for staff to plan.

Professional community

Those teachers who are deemed effective have at least some of the following characteristics in their daily practice with colleagues. They endeavour to:
1. support their colleagues;
2. collaborate with colleagues;
3. share their norms and values;
4. engage in reflective dialogue;
5. share clear and coherent standards of achievement; and
6. engage in professional development activities.

Conclusion

While not exhaustive, the lists of characteristics of effective teachers, and their practice, given above, are certainly exhausting! Who among us could claim all the "good" features? What would such a pedagogical paragon be like? Can we be expected to become classroom chameleons, and change readily?

It is my belief that if we examine ourselves, and our practice, many of these "effective features" are already present; but some are not, and some are not possible for us to change. However, as professionals, we need to be ever working towards more effective practice. So, what is the solution? Perhaps we need to push for policy initiatives, such as those outlined earlier, which would help us become more effective? And, perhaps, we can use the notion of Merrily Malin (1998), who suggested that the most effective teachers are those who know their students well.

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The games we play: Assessing algebraic thinking through arithmetic activities[*]

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Students in the later years of primary school, or early years of secondary school often have fragile understandings of number facts, inefficient fact recall and computational strategies, and a limited understanding of number relationships or the meaning of equals. Hence, they are unprepared for the transition to algebra, which marks, or mars, the early years of secondary schooling. This paper describes four activities that develop deep arithmetic understandings that are essential for students' engagement in formal algebra. The activities also provide a context for mathematical discussions and rich assessment potential.

Introduction

Although algebra is introduced in the first years of secondary schooling, preparation takes place from the first years of schooling, (e.g. Curriculum documents from America (NCTM, 2000), Britain (DEEQCA, 1999) and Australia (BOSNSW, 2002) include a section on Patterns and Algebra). Difficulties experienced by students in algebra can be shown to be difficulties in arithmetic understanding (Booth, 1984) often arising from their mathematical experiences, rather than as a result of students' cognitive capacities (MacGregor & Stacey, 1997). Algebraic thinking begins with the arithmetic experiences that children encounter from the earliest years of their schooling. These experiences are those that encourage a "conceptual orientation" rather than a "calculational orientation" (Lubinski & Otto, 2002), and develop the rich understanding of arithmetic (Tall & Thomas, 2002) that helps to bridge the conceptual gap (Linchevski & Herscovics, 1994) between arithmetic and formal algebra.

Algebraic thinking — about numbers, their relationships, the relationships between the operations on numbers and the structure of the number system — begins with the exploration of particular numbers in order that patterns may be discerned, abstracted and generalised. When arithmetic is learnt purely as a set of computational skills — where unique answers can be obtained through the correct application of particular procedures — students come ill-prepared for their introduction to formal algebra (Malara & Iadorosa, n.d.). To move from the realm of computational arithmetic to the realm of "reasoning about unknown or variable quantities and recognising the difference between specific and general situations" (van Ameron, 2003), requires a conceptual change across many aspects of arithmetic (Matz, 1982; Linchevski & Herscovics, 1994).

This conceptual change can be bridged if students' arithmetic experiences have been enriched by their meeting situations such as: where "open" expressions (e.g., 3 + 5) are acceptable answers and representative of a number as well as a procedure; where they

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[*] Paper accepted by peer review

learn that the equal sign signifies equivalence; where the arithmetic operators (plus, minus, times, divide, etc.), and the equal sign, are used as relational symbols as well as, depending on context, instructions to carry out a computational procedure to obtain a result. These experiences also assist students to develop efficient ways to recall, and use, simple number facts and procedures. They can then use these to build skills in executing more complex computations, and to identify and elaborate further arithmetic relationships.

Although the knowledge and skills of "basic arithmetic" are introduced in the primary years, together with the fostering of students' facility in their use, many students reach secondary schooling with a limited ability to recall facts efficiently and where they rely on inefficient computational strategies, such as counting on by ones. The problem for teachers of students in the first years of secondary schooling is thus one of establishing sound pre-algebra knowledge (Linchevski & Herscovics, 1994) before beginning, on a more formalised and abstract-theoretic approach, to pattern recognition and generalisation in mathematically useful ways.

The cultivation of these understandings requires that students be exposed to a variety of representations of numbers, and oft-repeated activities, that encourage the use of flexible number representations throughout their schooling. These activities should enable students to revisit the "basic" mathematic facts and skills, as well as encourage and foster the efficiency of recall and application of these facts to an ever-widening range of examples. This is not a call for the reinstatement, or continuance, of the rote learning of tables of number facts, developed without any meaning for the students, and recited *en masse* each day as the school day begins. Teaching suggestions as to ways in which students' facility can be enhanced need to be varied, engaging for students on a number of levels, and provide ways of assessing students' mathematical understanding that goes beyond a rapid, correct response to an essentially trivial question (e.g., Burnett & Tickle, 2003).

This paper describes four activities that have been used successfully with students in the first years of secondary school (and later years) to assess, and develop, their number fact recall and computational skills, their understanding of number relationships, and provide a reason why they should spend their time "learning their tables." The activities not only provide "drill and practice" but also a context where students can pose conjectures, provide explanations, and justifications, for answers or solution methods, and can be assessed in powerful, albeit informal, ways. Their thinking is "made visible" (Burnett & Tickle, 2003). Thus, these activities also provide opportunities for teachers to make judgements about students' mathematical skills and the quality of their mathematical knowing.

Making judgements "on the run" about students' understanding is something that teachers do — using cues from their students. These cues include "body language," vocabulary use, linguistic modality and complexity, time spent on calculations, a willingness to participate in a conversation, and the overt and covert reliance on peers. Assessment of this kind is able to consider, not only on how much mathematics a student can "get through," nor how fast a student can answer or complete a task, but also the quality — the breadth and depth — of the mathematics and the insights that students offer.

The following sections describe each activity, and identify the mathematical content potential of each, as well as the working mathematically aspects, and points where assessments might occur.

The games we play: Activities 1 and 2

Activity 1

Here is a number. It can be written in many ways. List as many as you can.

Students work in groups, probably no more than 4 to a group. Groups can be arranged by ability, mixes of ability, or socially. Different groupings promote different learning experiences that teachers can control, according to the values and relationships, as well as the educational, outcomes they wish to achieve. The same number can be given to each group — or different numbers. Time limits may be imposed explicitly, or left to the teacher's assessment of the situation as the progress of each group is monitored.

Students do not use calculators, in the initial stages. It may then be suggested that calculators be used in order that students discover more inventive representations.

This activity has been tried with first year pre-service teachers, as well as with students in upper primary, and junior high school. Often the first responses are to list a series of addition pairs, and then a series of subtraction pairs. Suggestions to individual groups ,as to other possible representations, are often needed to move students beyond mechanical repeating the same of arithmetic operations. In many cases this then prompts far more original thinking.

In the form described, this activity poses an "open question," much like Sullivan's suggestion of taking an answer and asking for the possible question that could result in that answer (Sullivan & Lilburn, 1997). Often, such open questions may have either too broad a scope, or serve only to accommodate too narrow a range of different responses for students to perceive the mathematical connections intended by the teacher (Watson, 2005). Putting conditions on the question by suggesting that students develop expressions using two, three, or all four basic arithmetic operations, encourages further exploration. Many students then ask if they may use square roots, powers, fractions, brackets, demonstrating a wide repertoire of mathematical operations. Here is where a calculator may prove useful, even if not "broken"[8].

Students then share their responses. If the responses are written on large sheets of paper they can be displayed around the classroom and added to over a period of time. Students, of course, need to check the work of their peers; no-one is allowed to get away with an error! This then encourages debates as requests for justifications occur.

Activity 2: Five ways (adapted from Kirkby, 1994)

Given five digits (selected from digits 1–9 inclusive) arrange some, or all, in an expression to obtain a given number (usually two-digit). No digit can be used more than once in any particular expression. This is a more constrained version of the first activity. Similar classroom organisation can be used and similar mathematical ideas are needed by the students, and can be assessed by the teacher.

Discussion of Activity 1 and Activity 2

These two activities involve the students on three levels, each level being indicative of the quality of mathematical understanding.

The first level is the knowing and recalling of number facts, and the appropriate use of mathematical operations. Here, the teacher can discern whether students are using

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[8]   There are many versions of this; e.g., Duncan Keith's interactive site:
http://www.woodlands-junior.kent.sch.uk/maths/broken-calculator

efficient strategies for mental computations — or the inefficient pen-tapping, finger-counting-by-ones strategies, that often go undetected.

The second level involves the demonstration or the development of number sense — the fact that numbers can be represented in different, unclosed ways — and the equivalence meaning of the equal sign. These activities may be the first time students have encountered the idea that an open expression is a way of writing a number, and even, that numbers can be written in many different ways.

The third level is the extension of students' knowledge to the use of a wider variety of mathematical operations (e.g., squaring, square roots, even factorials), different number patterns and the relationships between numbers, and the subtleties of mathematical syntax (e.g. the surface rearrangement of expressions such as $3 + 5 - 8 + 1 = 1 - 8 + 5 + 3$ but not $1 + 3 - 5 + 8$).

All of these understandings are necessary before students can access more formal algebra, which relies, in part, on the manipulation of expressions to obtain new mathematical insights. Students at this third level, or who begin to work at this level, once being introduced to it through these activities, can demonstrate their mathematical creativity, and begin to explore the number system in imaginative ways. Here is when conjecturing, explaining and justifying ideas begins in earnest. For the teacher listening to students articulating their ideas reveals the quality — the depth and breadth — of their understanding.

The games we play: Activities 3 and 4

The next two activities described are more "game-like" in that they involve an explicit element of competition. The focus is particularly on addition, subtraction, place value and estimation. Computational techniques employed, strategies for estimating the larger or smaller of two numbers and, often most significantly, what students do when adding or subtracting decimals can be observed, and hence assessed, "on the run."

Activity 3: Multi-digit (Goddard, Marr & Martin, 1994)

This activity requires a little more elaborate preparation than the first two activities. Each student needs a photocopied sheet (Figure 1), and the teacher needs a ten-sided die (or equivalent random-number generator). The aim of the game is to obtain the largest possible total, by recording the results of each throw of the die in each column of each row, in successive rows. Once one row is complete, the numbers are recorded in the next row. Each column represents the usual units, tens, hundreds, thousands, etc. As shown in Figure 1, there is a small version and a large version. After the totals have been added, students are asked for the largest obtained. Near totals, especially those from the large version, are written on the board for discussion and decision-making. These big numbers offer a further challenge in their naming.
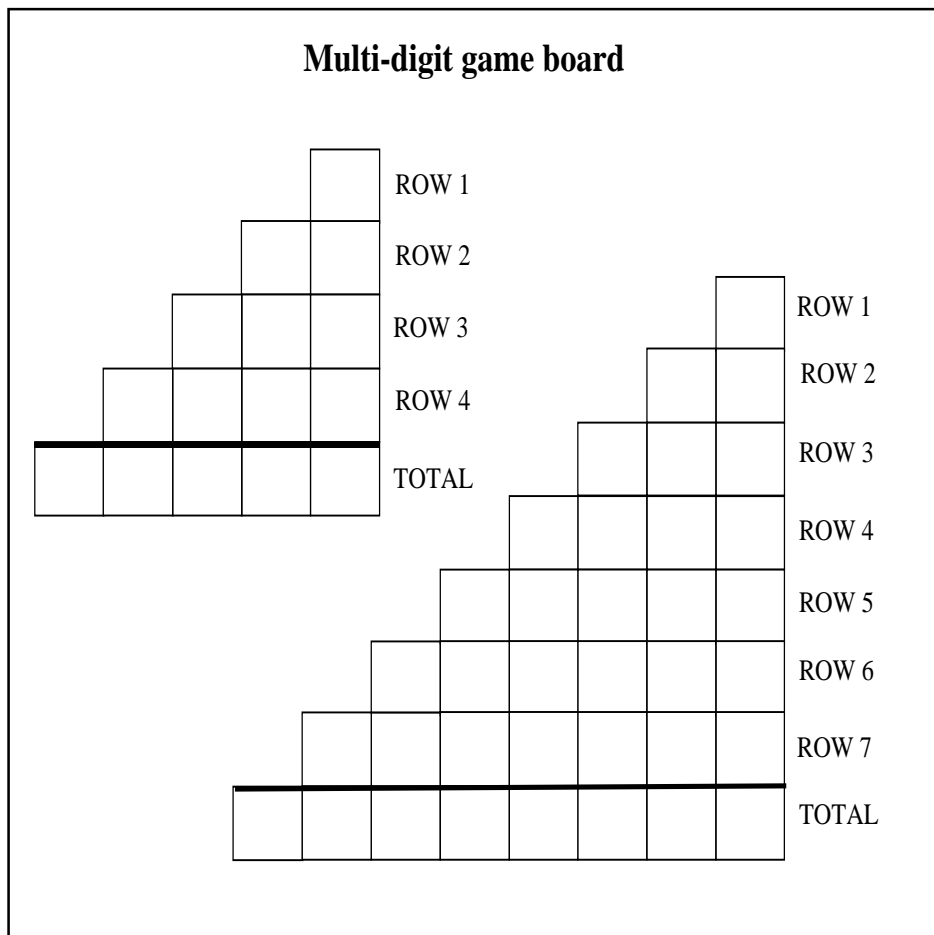
Figure 1: Multi-digit (Goddard, Marr & Martin, 1991 p. EN11).

Activity 4: Dicing with decimals (Marr, Anderson & Tout, 1994)

There are several variations of this activity. Only the simplest is described here. Students need a photocopied "game sheet" (Figure 2), and the teacher a six-sided die. In seven throws of the die, students record the result of each throw in either the left-hand column (units) or the right-hand column (tenths). Only one digit is to be recorded in each row. The aim is to obtain a total as near as possible to 9.9. As the game sheet has four grids, students play the game four times. Each time, they not only total their scores, but calculate the difference between their score and 9.9. After four turns, the differences are totalled, and the student with the smallest difference "wins."
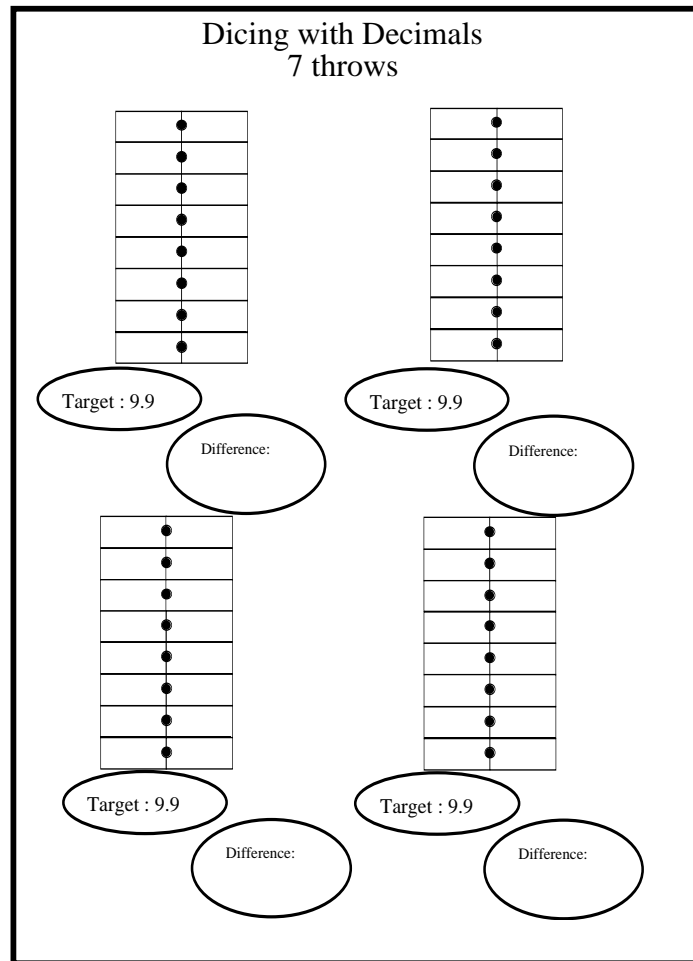
Figure 2. (after Marr, Anderson & Tout, 1994, p.16).

Discussion of Activity 2 and Activity 3

As well as the mathematics of addition, subtraction, place-value and estimation, addressed in these two activities, the concepts of chance and absolute value are alluded to and efficient mental computation strategies can be developed. Activity 3 also provides an introduction to very large numbers, and their various names and representations. By discussing their computation strategies, by "arguing" over who's total was largest, students learn to articulate, and thus clarify their ideas; and the teacher has the opportunity to listen to the students' thinking, and to make judgements about the quality of their mathematical understanding, particularly about place value and decimals.

Conclusion

Of far greater significance than the "drill and practice" provided by these activities is the fostering of mathematical thinking and the consequent rich assessment data afforded the teacher, through careful observation and listening to students. Listening to students, as they debate amongst themselves, or as a class, or ask questions, reveals misconceptions and language use that can be valuable to learning and teaching mathematics.

Observing students closely you can watch them add, or subtract, for instance, using inefficient strategies, such as counting on by ones, or reciting entire tables of number

facts, instead of their being able to recall instantly one specific fact. These are usually the students who are slow to complete their totals, who rely on others to help, or who really miss their calculator. Discussing mental strategies, and ways to remember and recall number facts, helps these students improve these skills, over time. Their development may become apparent during repeated sessions of the activity. Students are often willing to ask how others arrived at their total quickly. This gives students the opportunity to articulate their own understandings, evaluate those of others, and so provide a context for mathematical debate.

Although each of the activities is competitive to some extent, students appear very willing to help each other out. This is particularly true of activities 3 and 4, where, because of the chance element, there is no guarantee that the same student will win each time: a student who "always gets the right answer" can make a bad decision about placing a number (particularly in the larger format of Activity 3) and not get the highest total! There is also opportunity with these activities for students to participate by becoming the die-thrower — time-out from the mental effort, but still being part of the class. The teacher can also become a participant and, of course, need not win — much to the delight of the students!

With all of the activities, debates ensue about results of calculations — whether or not a particular result is possible. Calculators are best avoided, but do help resolve some arguments, particularly when the teacher leaves most of the adjudicating to the class, or as a way of having the less confident students enter the debate. One knows that the activities appeal to students when they want to complete a game — even after the recess bell has sounded!

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Preservice teachers' perceptions of their mentoring in primary mathematics teaching[*]

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A literature-based instrument gathered 147 final-year preservice teachers' perceptions of their mentors' practices related to primary mathematics teaching. The five factors that characterise effective mentoring practices in primary mathematics teaching had acceptable Cronbach alphas, that is, Personal Attributes (mean scale score=3.97, SD *[standard deviation]=0.81), System Requirements (mean scale score=2.98,* SD=0.96), *Pedagogical Knowledge (mean scale score=3.61,* SD=0.89), *Modelling (mean scale score=4.03,* SD=0.73), *and Feedback (mean scale score=3.80,* SD=0.86) *were .91, .74, .94, .89, and .86 respectively. This survey instrument may have applications for mentoring in secondary mathematics and can be re-designed to investigate mentoring practices in other key learning areas.*

Mentoring is prominent in education systems throughout the world (Hawkey, 1997; Power, Clarke & Hine, 2002; Starr-Glass, 2005) and mentors (i.e., supervising teachers or cooperating teachers) in professional experience settings (i.e., practicum, field experiences, internships) are well positioned to assist preservice teachers in developing their practices (Crowther & Cannon, 1998). Mentors' responsibilities for developing preservice teachers' practices are increasing as mentoring continues to amplify its profile in education (Sinclair, 1997). Primary (elementary) teachers in Australia generally work across all key learning areas (KLAs) and hence, in their roles as mentors, are expected to facilitate quality mentoring to preservice teachers across these KLAs. However, primary teachers will not be experts in all KLAs as research shows some areas receive considerably less attention than others (e.g., science [Goodrum, Hackling & Rennie, 2001] and art [Eisner, 2001]). As the curriculum is so diverse for primary teachers, they may need assistance in their roles as mentors with particular mentoring practices focused on subject-specific areas (Hodge, 1997; Hudson, 2004a, b, 2005; Jarvis, McKeon, Coates & Vause, 2001), which also appears to be the case for mentoring in mathematics education (Jarworski & Watson, 1994; Peterson & Williams, 1998).

Similar to teaching practices, professional development in mentoring practices may enhance the mentor's knowledge and skills. Also, similar to teaching practices, mentors operate in their own environment, where they may or may not receive further ideas for developing their practices. Yet, mentoring cannot be left to chance (Ganser, 1996) and needs to be purposeful in order to be more effective with explicit practices (Gaston & Jackson, 1998; Giebelhaus & Bowman, 2002; Jarworski & Watson, 1994; Jonson, 2002). Guidelines for subject-specific mentoring can aid the mentor's development by

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[*] Paper accepted by peer review

increasing confidence for raising issues, and providing topics for discussion and observation of specific teaching practices (e.g., see Jarvis et al., 2001; Hudson & McRobbie, 2003). Although there are various models for mentoring (Allsop & Benson, 1996; Colley, 2003; Jarworski & Watson, 1994; Jonson, 2002; Herman & Mandell, 2004), there is little literature on subject-specific mentoring in mathematics education for preservice teachers.

A five-factor model for mentoring has previously been identified, namely, Personal Attributes, System Requirements, Pedagogical Knowledge, Modelling, and Feedback (Hudson & Skamp, 2003), and items associated with each factor have also been identified and justified with the literature (see Hudson, Skamp & Brooks, 2005). The five factors, and the development of the MEPST instrument, are well articulated in the literature (see Hudson et al., 2005) for which this survey (Appendix 1) provides a direct link.

This study explores and describes 147 Australian preservice teachers' perceptions of their mentors' practices in primary mathematics education within the abovementioned five factors linked to a literature-based instrument (Appendix 1). This study aims to determine the transferability of the science mentoring instrument (Hudson et al., 2005) to the development of an instrument for mentoring preservice teachers in primary mathematics teaching. It also aims to articulate existing mentoring practices linked to this instrument on preservice teachers' mentoring of primary mathematics teaching.

Data collection method and analysis

The "Mentoring for Effective Mathematics Teaching" (MEMT) survey instrument (Appendix 1) in this study evolved through a series of preliminary investigations on Mentoring for Effective Primary Science Teaching (MEPST; Hudson, 2003; Hudson & Skamp, 2003; Hudson, 2004a, b; Hudson et al., 2005), which also identified the link between the literature and the items on the survey instrument. The MEPST survey instrument, which focused on the five factors (i.e., Personal Attributes, System Requirements, Pedagogical Knowledge, Modelling, and Feedback), was altered to reflect mentoring in primary mathematics. That is, the word "science" was replaced by the word "mathematics". A pilot study was conducted on 29 final-year preservice teachers by administering the MEMT survey instrument at the conclusion of their professional experiences (Hudson & Peard, 2005). Analysis of this pilot test indicated the possibility of a relationship between the MEPST instrument and the MEMT instrument; however further investigation was needed to verify results. For this study, 147 preservice teachers' perceptions of their mentoring were obtained from the five-part Likert scale (i.e., strongly disagree=1, disagree=2, uncertain=3, agree=4, strongly agree=5) MEMT instrument. The data provided descriptive statistics for each variable, which also provided an indication of the statistical relationship between variables and within each of the factors.

The preservice teachers' completed responses (*n*=147) represented 64% of the total cohort within a mathematics course at a Queensland university. Cronbach alpha scores (acceptable if >.70; see Kline, 1998), mean scale scores (i.e., mean scores of each item associated with each factor and then computed with a factor mean score; see SPSS13), and eigenvalues using SPSS factor reduction, which indicated the number of possible components (factors; i.e., eigenvalues >1; see Kline, 1998) and percentage of variation for each eigenvalue, aided in determining reliability and the potential transferability of the MEPST instrument to the MEMT instrument.

Results and discussion

These preservice teacher responses (109 female; 38 male) provided descriptors of the participants (mentors and mentees) and data on each of the five factors and associated attributes and practices. Responses were gathered at the conclusion of their final professional experience (i.e., practicum, field experience).

Backgrounds of participants

Twenty-five percent of these mentees (*n*=147) entered teacher education straight from high school, with 93% completing mathematics units in their final two years of high school (i.e., Years 11 & 12). Seventy-seven percent of mentees had completed two or more mathematics methodology units at university, and 86% had completed three or more block professional experiences (practicums) with 54% completing four professional experiences. There were no professional experiences under three-weeks. Ninety percent of mentees taught at least four mathematics lessons during their last practicum with 81% indicating they had taught 6 or more lessons. Most of the classrooms for the mentoring in mathematics were in the city or city suburbs (69%) with 31% in regional cities and in rural towns or isolated areas.

Mentees estimated that most mentors (male=22, female=125) were over 40 years of age (55%) with 28% between 30 to 39 years of age, and 16% under 30. Mentees also noted that 86% of mentors modelled one or more mathematics lessons during their mentees' professional experiences, with 59% modelling five or more lessons during that period. Finally, 41% of mentees perceived that mathematics was their mentors' strongest subject in the primary school setting.

Five factors for effective mentoring in mathematics

Each of the five factors had acceptable Cronbach alpha scores greater than .70 (Kline, 1998), that is, Personal Attributes (mean scale score=3.96, *SD* [standard deviation]=0.91), System Requirements (mean scale score=3.31, *SD*=0.90), Pedagogical Knowledge (mean scale score=3.58, *SD*=0.94), Modelling (mean scale score=4.01, *SD*=0.78), and Feedback (mean scale score=3.76, *SD*=0.88) were .91, .77, .95, .90, and .86 respectively (Table 1). Data from items associated with each factor were entered in SPSS13 factor reduction, which extracted one component only for each factor. Associated eignevalues accounted for 59–69 percentage of variance on each of these scales.

Table 1. Confirmatory Factor Analysis for Each of the Five Factors (n=147)

Factor	Eigenvalue*	Percentage of variance	Mean scale score	SD	Cronbach alpha
Personal Attributes	4.13	69	3.96	0.81	.91
System Requirements	2.05	68	3.31	0.90	.77
Pedagogical Knowledge	7.19	65	3.58	0.94	.95
Modelling	4.70	59	4.01	0.78	.90
Feedback	3.64	61	3.76	0.88	.86

* Only one component extracted for each factor with an eigenvalue >1.

The following provides further insight into specific data on the attributes and practices associated with each factor.

Personal attributes

When analysing the mentees' responses on their mentors' "personal attributes," a majority of mentors (89%) were supportive towards their mentees' primary mathematics teaching. In addition, 86% of mentors appeared comfortable in talking about mathematics teaching (Table 2).

Table 2. "Personal Attributes" for Mentoring Primary Mathematics Teaching (n=147)

Mentoring Practices	%*	M	SD
Supportive	89	4.35	0.85
Comfortable in talking	86	4.25	0.88
Assisted in reflecting	**73**	**3.87**	**1.01**
Instilled positive attitudes	69	3.92	0.88
Listened attentively	67	3.67	1.07
Instilled confidence	64	3.75	1.08

* %=Percentage of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

However, less than one quarter of mentees believed that the mentor aided the mentee's reflection on teaching practices (73% agreed or strongly agreed to this practice), instilled positive attitudes (69%), and listened attentively to their mentees (67%) and instilled confidence (64%) for teaching primary mathematics. Table 2 provides mean item scores (range: 3.67 to 4.35; *SD* range: 0.85 to 1.08) and percentages on mentees' perceptions of their mentors' Personal Attributes in rank order.

System requirements

Items displayed under the factor "System Requirements" presented a different picture from the previous factor. The percentages of mentees' perceptions of their primary mathematics mentoring practices associated with System Requirements were all below 50%, that is, 44% of mentors discussed the aims of mathematics teaching, 41% of mentors discussed the school's mathematics policies with the mentee, and only 29% outlined mathematics curriculum documents (Table 3). Implementing departmental directives and primary mathematics education reform needs to also occur at the professional experience level, yet the data indicated (mean item scores range: 2.71 to 3.15; *SD* range: 1.14 to 1.24, Table 3) that many preservice teachers may not be provided these mentoring practices on System Requirements within the school setting.

Table 3. "System Requirements" for Mentoring Primary Mathematics Teaching

Mentoring Practices	%*	M	SD
Discussed aims	44	3.15	1.14
Discussed policies	41	3.06	1.18
Outlined curriculum	*29*	*2.71*	*1.24*

%=Percentage of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

Pedagogical knowledge

Mean item scores (3.31 to 3.84; *SD* range: 1.08 to 1.24, Table 4) indicated that the majority of mentees "agreed" or "strongly agreed" their mentor displayed "Pedagogical Knowledge" for primary mathematics teaching. However, in this study, more than 20% of mentors may not have mentored pedagogical knowledge practices (see Table 4 for rank order percentages). For example, in the planning stages before teaching, 64% of mentors assisted in planning and 67% discussed the timetabling of the mentee's teaching and assisted with mathematics teaching preparation (71%, Table 4).

Table 4. "Pedagogical Knowledge" for Mentoring Primary Mathematics Teaching

Mentoring Practices	%	*M*	*SD*
Discussed implementation	77	3.84	1.08
Assisted with classroom management	73	3.77	1.08
Guided preparation	71	3.69	1.14
Assisted with teaching strategies	68	3.73	1.16
Assisted with timetabling	67	3.74	1.16
Assisted in planning	64	3.61	1.04
Provided viewpoints	61	3.51	1.17
Discussed problem solving	57	3.51	1.08
Discussed questioning techniques	57	3.45	1.11
Discussed content knowledge	52	3.31	1.24
Discussed assessment	52	3.50	1.19

* %=Percentage of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

Teaching strategies need to be associated with the assessment of students' prior knowledge, yet nearly half the mentors were perceived not to discuss assessment or questioning techniques for teaching mathematics (52%). Many mentors also appeared not to consider content knowledge and problem-solving strategies for teaching mathematics (57%) and providing viewpoints on teaching mathematics was not considered a high priority (61%, Table 4). This implies that many final-year preservice teachers may not be provided with adequate Pedagogical Knowledge in the primary school setting to develop successful mathematics teaching practices.

Modelling

Modelling teaching provides mentees with visual and aural demonstrations of how to teach and, indeed, mean item scores (3.81 to 4.30; *SD* range: 0.83 to 1.19, Table 5) indicated that the majority of mentors were perceived to model mathematics teaching practices. Even though more than 75% mentees perceived they received modelled practices for teaching mathematics including modelling a rapport with their students (85%), modelling the teaching of primary mathematics (79%), displaying enthusiasm for teaching mathematics (78%), and using language from the mathematics syllabus (78%), more than a quarter of mentees indicated their mentors had not modelled a well-designed lesson or effective mathematics teaching (Table 5).

Table 5. "Modelling" Primary Mathematics Teaching.

Mentoring Practices	%	M	SD
Modelled rapport with students	85	4.30	0.83
Modelled classroom management	82	4.11	0.97
Demonstrated hands-on	81	4.03	1.04
Modelled mathematics teaching	79	4.14	0.90
Displayed enthusiasm	78	4.02	1.00
Used syllabus language	78	3.97	0.89
Modelled a well-designed lesson	73	3.81	0.99
Modelled effective mathematics teaching	71	3.83	1.19

* %=Percentage of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

Feedback

Mean item scores (3.31 to 4.18; *SD* range: 0.97 to 1.38, Table 6) indicated that the majority of mentees "agreed" or "strongly agreed" their mentors provided "Feedback" as part of their mentoring practices in primary mathematics teaching. Yet, surprisingly, mentees perceived that 82% of mentors observed their mathematics teaching with only 63% articulating their expectations for the mentees' teaching of mathematics. More surprising is that 4% of mentors provided oral feedback without observation. Fifty-nine percent were perceived to provide written feedback and only 55% of mentors reviewed lesson plans, which is necessary to provide feedback before teaching commences for enhancing instructional outcomes (Table 6).

Table 6. Providing "Feedback" on primary mathematics teaching

Mentoring Practices	%	M	SD
Provided oral feedback	86	4.18	0.97
Observed teaching for feedback	82	4.08	1.00
Provided evaluation on teaching	81	3.97	1.08
Articulated expectations	63	3.55	1.16
Provided written feedback	59	3.48	1.38
Reviewed lesson plans	55	3.31	1.25

* %=Percentage of mentees who either "agreed" or "strongly agreed" their mentor provided that specific mentoring practice.

Further discussion and conclusions

There appeared to be transferability of the MEPST survey instrument (Hudson et al., 2005) to the MEMT instrument, which was supported by acceptable Cronbach alpha scores and descriptive statistics (Table 1). Ninety-three percent of these preservice teachers had completed at least three professional experiences (practicums) and nearly four years of a tertiary education degree in teaching before responding to this survey on their final-year Mentoring for Effective Mathematics Teaching (MEMT). The MEMT instrument appeared to provide a way to collect data for articulating mentees' perceptions of their mentors' practices in primary mathematics teaching occurring in various Queensland schools. Even though the Likert scale differentiated the degree of mentoring (e.g., strongly disagree to strongly agree), the quality of these mentoring practices needs to be investigated further. Anecdotal evidence suggests mentors vary

their mentoring practices considerably, and as there are national standards for teaching and assessing mathematics (e.g., NCTM, 1989, 1991, 1992, 1995), a set of standards for mentoring practices for mathematics appears a logical sequence.

The growing literature is more clearly defining mentoring practices (e.g., Colley, 2003; Christensen, 1991; Jarworski & Watson, 1994; Jarvis et al., 2001; Jonson, 2002; Herman & Mandell, 2004; Wilkin, 1992), with mentees claiming that the in-school context is pivotal to their development as teachers (Gaffey, Woodward & Lowe, 1995; Jasman, 2002). "Generalist" primary teachers will not be experts in all subjects in primary school, and some may not have adequate knowledge, skills, or confidence for teaching primary mathematics. Mathematics education is considered a priority by Australian education departments (e.g., Education Queensland; NSW Department of Education and Training [DET]); yet there are primary teaching mentors who may either not have the skills for effective mathematics teaching to mentor effectively or lack knowledge of effective mentoring strategies. There should be more emphasis on the mentoring of mathematics particularly as considerable importance is placed on this key learning area.

For mentees to receive equitable mentoring in primary mathematics teaching requires the provision of a set of specific mentoring attributes and practices for mentors. Such a set of "standards" may aid mentors to focus more specifically on their mentoring and may aid mentees in determining what to expect from their mentors. It may further promote the specific development of mentor-mentee relationships. However, mentors and mentees must work together and negotiate their roles and responsibilities (Jonson, 2002; Nolder, Smith & Melrose, 1994), and such standards would need to be flexible in order to cater for the diversity of practices and needs. Just as teachers can always improve their methods of teaching, so too can mentors improve their methods of mentoring (Boss, 2001; NSW DET, 2003), and those who receive professional development on mentoring have a greater impact on their mentees (Giebelhaus & Bowman, 2002). If preservice teachers are to receive quality mentoring in primary mathematics teaching then many teachers, in their roles as mentors, will require further professional development. The form this education takes will require rethinking, as experienced primary teachers can be reluctant to be educated on their mentoring practices (e.g., Hulshof & Verloop, 1994).

The mentoring indicated in this study only focused on the mentors' practices and attributes, therefore, further research would be needed on mentees' involvement in the mentoring processes. Nevertheless, the inadequate mentoring outlined in this study may be initially addressed through specific mentoring interventions that focus on effective mentoring (i.e., attributes and practices associated with the five factors: Personal Attributes, System Requirements, Pedagogical Knowledge, Modelling, and Feedback; e.g., science mentoring intervention [Hudson & McRobbie, 2003]). As each item associated with the MEMT instrument (Appendix 1) is linked to the literature, a mentoring intervention can be based around these items. A well-constructed mentoring intervention may provide professional development for mentors to enhance not only their own mentoring practices but also their teaching practices. A mentoring intervention may aid induction processes for early career mathematics teachers, particularly for those who do not receive adequate mentoring support for their teaching of mathematics (e.g., Luft & Cox, 2001). Additionally, the MEMT instrument can be used (by tertiary institutions or departments of education) to gauge the degree of mentoring in primary mathematics and, as a result of diagnostic analysis, plan and implement mentoring programs that aim to address the specific needs of mentors in order to enhance the mentoring process. Although the MEMT instrument was

administered to preservice primary mathematics teachers, it has the potential to gather data about mentoring practices for preservice secondary mathematics teachers.

Utilising the mentor's time efficiently is crucial for developing the mentee's practices for effective primary mathematics teaching, and this is further justification for educating mentors. The mentor's involvement in facilitating the mentee's learning for more effective primary mathematics teaching cannot be a random process; instead it must be predetermined and sequentially organised so that the mentor's objectives are focused, specific, and obtainable. This means educating mentors on such practices whether for a preservice teacher level or a beginning teacher induction level. This study outlines that in broad terms, effective mentoring requires mentors to: display personal attributes, provide guidance on system requirements, model effective mentoring (which also requires modelling effective teaching practices), and provide pedagogical knowledge and feedback towards enhancing teaching practices. Educating mentors aims at ultimately targeting the development of preservice teachers' practices, and hence a way to enhance primary students' learning experiences and opportunities towards developing higher standards of mathematics education.

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Appendix 1. Mentoring for effective mathematics teaching

The following statements are concerned with your mentoring experiences in mathematics teaching during your last field experience (practicum). Please indicate the degree to which you disagree or agree with each statement below by *circling only one response* to the right of each statement.

Key

SD = Strongly Disagree

D = Disagree

U = Uncertain

A = Agree

SA = Strongly Agree

During my last field experience (i.e., practicum) in mathematics teaching my <u>mentor</u>:

1. was supportive of me for teaching mathematics.	SD D U A SA
2. used mathematics language from the current mathematics syllabus.	SD D U A SA
3. guided me with mathematics lesson preparation.	SD D U A SA
4. discussed with me the school policies used for mathematics teaching.	SD D U A SA
5. modelled mathematics teaching.	SD D U A SA
6. assisted me with classroom management strategies for mathematics teaching.	SD D U A SA
7. had a good rapport with the students learning mathematics.	SD D U A SA
8. assisted me towards implementing mathematics teaching strategies.	SD D U A SA
9. displayed enthusiasm when teaching mathematics.	SD D U A SA
10. assisted me with timetabling my mathematics lessons.	SD D U A SA
11. outlined state mathematics curriculum documents to me.	SD D U A SA
12. modelled effective classroom management when teaching mathematics.	SD D U A SA
13. discussed evaluation of my mathematics teaching.	SD D U A SA
14. developed my strategies for teaching mathematics.	SD D U A SA
15. was effective in teaching mathematics.	SD D U A SA
16. provided oral feedback on my mathematics teaching.	SD D U A SA
17. seemed comfortable in talking with me about mathematics teaching.	SD D U A SA
18. discussed with me questioning skills for effective mathematics teaching.	SD D U A SA

During my last field experience (i.e., practicum) in mathematics teaching my <u>mentor</u>:

19. used hands-on materials for teaching mathematics.	SD D U A SA
20. provided me with written feedback on my mathematics teaching.	SD D U A SA
21. discussed with me the knowledge I needed for teaching mathematics.	SD D U A SA
22. instilled positive attitudes in me towards teaching mathematics.	SD D U A SA
23. assisted me to reflect on improving my mathematics teaching practices.	SD D U A SA
24. gave me clear guidance for planning to teach mathematics.	SD D U A SA
25. discussed with me the aims of mathematics teaching.	SD D U A SA
26. made me feel more confident as a mathematics teacher.	SD D U A SA
27. provided strategies for me to solve my mathematics teaching problems.	SD D U A SA
28. reviewed my mathematics lesson plans before teaching mathematics.	SD D U A SA
29. had well-designed mathematics activities for the students.	SD D U A SA
30. gave me new viewpoints on teaching mathematics.	SD D U A SA
31. listened to me attentively on mathematics teaching matters.	SD D U A SA
32. showed me how to assess the students' learning of mathematics.	SD D U A SA
33 clearly articulated what I needed to do to improve my mathematics teaching.	SD D U A SA
34. observed me teach mathematics before providing feedback?	SD D U A SA

Is our mathematics program working?
How do we know?

Ken Lountain

Department of Education and Children's Services, SA

Programs designed to improve mathematics learning are sometimes hindered by a lack of evidence about their effectiveness. Too often, evaluation is "tacked on" in the latter stages of projects or after their completion. Maths for Learning Inclusion is a South Australian program aimed at improving mathematics curriculum. The program operates in eight clusters of schools, supported by specialist teachers in primary mathematics. Maths for Learning Inclusion features a mixed-method evaluation which is investigating not just "whether" but "how" and "in what contexts" the program is effective. This seminar will examine the development and implementation of the program evaluation plan and explain why various components; teacher questionnaires, focus groups, interviews, student learning outcomes; were included and how they contribute to a picture of the program's impact on mathematics teaching and learning in the primary years.

Introduction

Programs designed to improve mathematics learning are sometimes hindered by a lack of evidence about their effectiveness. Too often, evaluation is "tacked on" in the latter stages of projects or after their completion.

Maths for Learning Inclusion is a South Australian program aimed at improving mathematics curriculum. The program operates in 8 clusters of schools, supported by specialist teachers in primary mathematics. Maths for Learning Inclusion features a mixed-method evaluation which is investigating not just "whether" but "how" and "in what contexts" the program is effective. The Program was developed and is coordinated by the Learning Inclusion Team within Curriculum Services in the South Australian Department of Education and Children's Services (DECS).

A rigorous ongoing evaluation of the Program has been in place from the beginning and has provided the program management team, the cluster coordinators, school leaders and teachers, and school communities with valuable ongoing information that has helped shape the direction of the Program.

The evaluation component of Maths for Learning Inclusion is an intrinsic element of the Program design which is contributing to a culture of critical reflection and responsive action across the Learning Inclusion Team, the eight clusters, and 44 schools of over 200 teachers and nearly 4000 students.

Evaluation of the program is being supported by an Evaluation Consultant, Gill Westhorpe who specialises in a form of evaluation called *realist evaluation*. In basic terms, Realist Evaluation processes enable evaluators to focus on the question: Which elements of the program are having which effects in which circumstances?

How does the program work?

The Maths for Learning Inclusion Program is aiming to systematically improve the engagement and learning outcomes of learners from low socio-economic backgrounds, and Aboriginal learners, through enhancing the capacity of primary teachers in the effective and inclusive teaching of mathematics.

The project also sought to test a specific program model — the provision of a full time Cluster Coordinator to a cluster of relatively disadvantaged schools — and to use the opportunity to learn as much as possible about how that model worked to create change.

In program logic terms, the structure of Maths for Learning Inclusion suggests that:

- employing a Coordinator to work with a cluster of schools, and
- supporting the Coordinator and the staff with centrally-provided professional development programs will
- lead to a local plan being developed, which will
- underpin a range of program activities within the cluster, and that those activities will
- lead to changes in teachers' attitudes, confidence and skills (these are program mechanisms at the teacher level), which will
- underpin changes in teachers' teaching behaviours (which are both short term teacher-level outcomes and mechanisms for change in student learning behaviours), which will
- lead to improved student engagement (which is a primary mechanism at the student level), which will
- underpin improved student learning outcomes.

The program was initially funded for 18 months, commencing in mid 2005 and concluding at the end of 2006. It has since been extended for a further 12 months.

Why are we evaluating the program?

Program model and objectives

The central point of the evaluation is to find out whether the logic upon which the program is based is supported by what actually happens; i.e., is progess being made towards the stated objectives of the program? Is the appointment of eight specialist teachers in mathematics, to work across clusters of disadvantaged schools and supported by a professional learning program, resulting in positive change for teachers and improved experiences and outcomes for students?

Within this central evaluation concern, there are several subsets:

Accountability for resources

The program is funded using Commonwealth resources that are intended to improve learning outcomes for learners from low socio-economic backgrounds. Decisions about how to use such funding are taken very seriously. The program is funded using significant resources — almost $2.1m over 2.5 years — and it is important that we try to gauge the effectiveness of the Program's various components and are able to report to all stakeholders about the Program's effects.

Enabling responsiveness

Programs that engage in evaluation only after a program is completed or in the final stages are unable to change what they are doing to respond to the data they collect. The ongoing Maths for Learning Inclusion evaluation has been designed to enable decisions about central elements of the program — how the cluster coordinators are working in schools, the content of professional learning for cluster coordinators and teachers, the allocation of resources to various priorities, the hiring of cluster coordinators, the use of assessment mechanisms — to be made on a dynamic and responsive basis.

Learning more about addressing educational disadvantage

In learning about which aspects of Maths for Learning Inclusion are having which effects, for whom, and in which contexts, we can increase the education community's knowledge about the interventions that are likely to have positive effects on, for example;

- teachers' attitudes towards, and expectations of, learners from low socio-economic backgrounds and Aboriginal learners;
- teachers' confidence in their ability to make a difference;
- teachers' ability to work effectively with a curriculum framework for planning, programming and assessment;
- teacher pedagogies that engage, and improve outcomes for, learners from low socio-economic backgrounds;
- teachers' and students' attitudes towards mathematics learning
- the learning outcomes of students from low socio-economic backgrounds and Aboriginal students

This is a rare opportunity for the information from one program to benefit policy-makers, district and school leaders, teachers and students across our public system.

Applying what we learn to future initiatives

Too often, there is a risk that what we learn is not applied to what we do in the future. While Realist Evaluation emphasises the importance of context and — in this evaluation we are being cautious about making causal links — we are beginning to learn about which elements of the program are having which effects in which circumstances.

This is knowledge that can be applied to the design of other programs, intended to bring about curriculum change and the improvement of learning outcomes for students from low socio-economic backgrounds and Aboriginal students. The Learning Inclusion Team is working with the Curriculum Superintendent, Primary Years and other stakeholders in Curriculum Services, on planning how to use evaluation data from Maths for Learning Inclusion in systematic ways to inform other curriculum initiatives, especially those directed to improving the equity as well as the quality of teaching and learning in DECS schools.

How does the evaluation work?

The evaluation design is a mixed methods case study drawing on two evaluation approaches: realist evaluation (what works, for whom, in what contexts, and how) and program logic (specifying a logical sequence of steps in program implementation and the ways that these are expected to result in desired program outcomes). The evaluation also uses the notion of "program contrasts" (elements of program context, design or

service delivery which vary between clusters and which are hypothesised to make a difference to program outcomes).

Data sources for the first full year of the evaluation include:

- PATMaths tests administered to students in Grades 3, 4 and 5 in Terms 1 and 4;
- a Pre- and Post-Program Questionnaire for teachers, addressing aspects of teacher knowledge, skill and attitude, their perceptions of leadership participation and support, Cluster Coordinators expertise, and access to professional development and peer support, again administered in Terms 1 and 4;
- a shorter term-by-term questionnaire for teachers, collecting data about their participation in project activities;
- a term-by-term questionnaire for Cluster Coordinators, providing information about their activities, the adequacy of resources available to them, their approach to their role, and their perceptions of leadership participation and support;
- a Leadership questionnaire, seeking their perspectives in relation to their own participation in the project, the extent of leadership support for the project across the cluster, the adequacy of resources for the project, and relationships across the project;
- a focus group with leadership representatives from the eight clusters;
- a focus group with the eight Cluster Coordinators;
- "Most Significant Change" stories written by teachers in some clusters.

What are we finding out?

So far, the analysis of evaluation data has yielded clear evidence to support most steps of the project logic chain, including evidence to support:

- the viability of using Cluster Coordinators with groups of schools;
- the quality and variety of professional development programs provided through the Project;
- the value of a mentoring model at cluster level;
- variability of activity at local level, suggesting responsiveness to local needs and resources;
- increased confidence on the part of at least some teachers and more positive attitudes to teaching mathematics;
- changed pedagogies on the part of at least some teachers, using more activity-based and group-based learning strategies supported by concrete materials. There is also evidence of increased frequency of understanding students (their learning interests and styles, their emotional responses to learning tasks, and the contexts of their daily lives), increased use assessment for planning and for monitoring progress, and involving individual students and classes in planning and negotiating learning strategies;
- increased engagement of students in mathematics;
- improved performance on PATMaths tests for the vast majority of students by the end of the academic year.

Among the "lessons learned" so far are the following:

- Security of funding provision over a longer time period than was originally envisaged for this project is likely to increase staffing stability, increase the

"reach" of the project (as new staff move into participating schools) and allow for changes to be embedded within schools and within teachers' practice.

- The quality of professional development, provision of training at local levels, planning for a variety of professional development approaches to meet different teachers' learning needs, and planning for follow-up of professional development at local level within a few days, all contribute to the acceptability and/or effectiveness of professional development activities.

- Funding release time for teachers supports their participation in project activities, which, in turn, contributes to project outcomes.

- Providing a range of strategies and supports at least enhances teacher confidence and appears to enhance teacher skills, in some cases.

- It is possible to change teacher pedagogies. Observing others using effective strategies, supported by a variety of ideas from a variety of sources, builds motivation to try changed practice. Experiences of success then reinforce changed practice.

- Changes in attitude, skill and confidence follow from, rather than precede, changed practice.

- Activity based learning, group based learning, use of concrete materials, and a focus on the process of problem-solving, help build student engagement.

There is also some evidence to support particular hypotheses that were incorporated within the overall logic model, including the value of mentoring, the importance of the credibility of mentors, the value of supportive leadership, and the value of positive relationships at a variety of levels within the project.

How evaluation data is shaping program directions:

The collection of evaluation data from the very outset of the program has enabled responsive and collaborative decision-making that has modified, re-emphasised and re-orientated key features of the program as work has progressed.

The initial Program timeline of 18 months was related to funding and structural issues within DECS. The resolution of these issues, and the data drawn from the first year of the program, enabled the important decision to extend the Program for 2007, in recognition of both the progress already made and of the time that it takes to build real curriculum change.

Evaluation data has also guided the professional learning program: both the 2005–2006 focus on mathematics planning and programming because of the needs teachers identified in this area, and the 2007 emphases of inclusive curriculum practice and assessment.

Issues of central/local ownership and responsibility, identified in evaluation data, have been able to be addressed when personnel changes, among the cluster coordinator group, have created opportunities for local capacity-building and shifting key responsibilities from central to local level. Responses to such issues as the role of principals in the program, the initiation of cluster-based professional learning and the development of teacher networks in and across clusters have also been guided by the analysis of evaluation data.

The development of key program frameworks — the Key Goals and Learning Inclusion Indicators — was due to evaluation data that emphasised the need for resources that would help clusters sharpen their focus on learning inclusion in 2007.

Evaluation directions in 2007

The MLI Project is now well into its second calendar year of operation. Some evaluation strategies are being maintained in the second year, notably the Teacher Pre-Post Questionnaire and the student PATMaths tests. Efforts will be made in the second year to link student PATMaths data to particular teachers, with a view to exploring correlations between teacher practice and student learning outcomes in more detail. A revised Leadership Questionnaire, focusing on different areas, will again be administered.

The Term-by-Term questionnaires will not be used again, partly in an effort to reduce "respondent burden" and partly because the data obtained was not as useful in identifying "program contrasts" as had been hoped.

There will be a much greater emphasis on qualitative data collection in the second year of the project. Cluster Coordinators will be invited to take part in individual interviews in order to gain more detailed understanding of their strategies at Cluster level. Focus groups will also be held with groups of teachers, structured around the differences in outcomes identified in quantitative data collection in the first year (by gender, years of experience, "effective teacher" status, and so on). These groups will seek to explore both the reasons for differences in outcomes and perhaps, more importantly, the strategies that particular groups found most effective in supporting their quests to enhance their practice in teaching mathematics and supporting learning inclusion.

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Planning Professional Learning using the AAMT Standards: The PLUMS Project[*]

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Professional Learning Using the Mathematics Standards (PLUMS) aimed to demonstrate the utility of the AAMT standards as a basis for developing professional learning plans for teachers and schools. Teams of teachers from eight schools, four primary and four high schools, planned and implemented a program of professional learning based on their own school's needs. The teachers were supported by a team of Academic Partners. From the perspective of the Academic Partner, this role can be problematic. Some of the difficulties are discussed and some ways of working productively within this framework are considered.

The purpose of the PLUMS project was to explore the use of the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2002) as a framework for professional learning. These standards resulted from a research study that aimed to identify what teachers across Australia believed defined excellence in teaching mathematics. Funding for PLUMS was provided by the National Institute of Quality Teaching and School Leadership (now called Teaching Australia) to explore ways in which the AAMT Standards could provide a useful background for teacher professional development that met schools' requirements as well as individual teachers' needs.

The National Centre of Science, Information and Communication Technology and Mathematics Education for Rural and Regional Australia (SiMERR) at the University of New England coordinated the project, and Academic Partners (the authors) came from the UNE School of Education. An independent evaluation of PLUMS was undertaken by Monash University (Bishop, Clarke & Morony, 2006).

Schools were chosen according to several criteria:

- likely willingness to participate based on the Academic Partners' prior knowledge of local schools;
- the nature of the school in order to provide a range of school contexts; and
- proximity to UNE to ensure reasonable contact.

Schools were contacted by one of the Academic Partners and invited to participate. Several schools initially contacted were interested in the project but felt unable to take part because of prior commitments. Eventually eight schools agreed to become involved, including schools from public, independent and Catholic sectors.

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PLUMS activities

The PLUMS project commenced with an initial conference held at UNE, and attended by participants from seven of the eight schools. The final school, a small two teacher school, could not obtain a casual teacher and could not, therefore, attend. Both Academic Partners, the AAMT project manager and the independent evaluator were present.

At the initial gathering, participants were introduced to the Standards using a process previously developed by the AAMT Professional Officer. In particular the focus was on the domains of "Professional Knowledge" and "Professional Practice" as providing the most suitable framework for whole school planning. Teachers used a self-assessment form to identify areas that appeared under-developed. These self-assessments provided a basis for collaborating with other teachers to identify an approach that would be used within their schools.

The second part of the day gave school based teams opportunities to begin planning their projects. No limitations were placed on the nature of schools' activities, other than that they had to involve the *Standards* in some way, although practical limitations were inevitably present.

Following this initial planning day, school teams returned to their schools to complete their own school plans. The schools' identified priorities provided the basis for each school's focus, and the self assessment approach from the *Standards* conference provided a framework in which the focus could be addressed. The primary schools generally addressed a particular strand of mathematics, such as measurement, identified as needing attention from statewide testing. Two high schools focussed on the use of technology in the classroom, particularly graphing calculators. Another wanted to consider a wider range of assessment processes with a focus on assessment for learning. Despite the diversity, it was possible to relate the plans to the Standards in ways that provided coherence.

Each school implemented its own project in a different way. Some schools called on the Academic Partners extensively to provide support and direction. This was greatly helped by one partner (the first author) being seconded to UNE for the year which added credibility to the support offered. Others requested some support to provide initial impetus to their project, while others worked independently. During the project all schools received visits from an Academic Partner and the AAMT Professional Officer. The extent and focus of these visits varied from school to school. The project culminated in a final conference in December, also held at UNE at which teachers presented the outcomes of their work to the rest of the group.

Outcomes from PLUMS

Seven of the eight schools reported successful outcomes. The eighth school, a small two teacher school, was unable to complete the intended project successfully, largely because of the lack of casual staff and the pressure of day to day work in a very small, isolated school.

School plans

Many schools used the self assessment process that had been used in the workshop. All appropriate teachers completed this activity and identified aspects of their work that they felt needed attention. Teachers collaboratively decided what direction their learning would take within the context of the schools' needs.

The school project plans varied in both emphasis and detail. Assessment was a focus in four of the eight schools, possibly reflecting publication of assessment documents in NSW, or a "trickle down" effect of recent changes to HSC assessment that raised awareness of its importance throughout the system. In three primary schools, the centre of attention of this assessment activity was on the Measurement strand of the syllabus, which had been identified as a weakness in recent Basic Skills Test results. Two high schools chose the use of technology to improve pedagogy as the starting point. Technology use, specifically graphing calculators and dynamic geometry software, had been identified as an area in which teachers lacked confidence, but was of recognised importance. The final primary school wanted to improve student "tracking" with the aim of improving the Year 6/Year 7 transition and the other high school project aimed to improve the learning experience for junior high school students through open-ended activities. The plans developed addressed schools' identified needs but were focussed and refined by the use of the Standards in the planning process.

Interaction with the Academic Partners

Although they had access to the expertise of the Academic Partners, some schools preferred to work without this support. The two schools that focused on technology combined resources to bring an acknowledged expert in the use of graphing calculators and other specialised programs to provide professional development. Other schools called on the Academic Partners extensively, particularly the seconded teacher who was well known and respected in the education community.

The most effective work with the Academic Partners came about when schools had a very clear idea of what they wanted to do. They were able to articulate their needs clearly in a way that allowed the Academic Partners to support them appropriately. For example, one large primary school decided to use one of its professional development days working with both Academic Partners and the District Numeracy Officer on linking assessment and teaching in the measurement strand.

In another example, one high school particularly wanted ideas for alternative assessment. Following a full day's work with the Academic Partners at which a number of ideas and suggestions for their use were shared, this school embedded practices such as mathematics journals in its programs in ways that made a considerable difference to the students. The school only needed the initial impetus provided by access to some local expertise to get under way with one of the most successful projects.

There was fairly frequent contact among the schools and the Academic Partners, although much of this happened informally. In a small community, it is inevitable that casual meetings occur, in the supermarket for example, and these often became an opportunity for updating and exchange of ideas. Through both formal and informal contact the Academic Partners were able to keep in touch with most schools. Without this informal contact, keeping in touch may have become problematic. Little use was made of email contact by teachers. Where email was used, it was usually from a private address rather than a school. Classroom teachers in public schools in NSW did not, at that time, have personal email addresses, and it seems that the culture of email use has not yet become a part of teachers' working life. Phone contact with schools is always problematic, and often several messages were needed, by which time the issue had been dealt with in other ways. Communication between schools and Academic Partners was most successful when it involved face-to-face meetings and a block of time.

Teachers' content knowledge

Several of the project schools indicated that teachers' content knowledge had increased. In the primary area, this was in the area of some mathematics, especially the focus area identified in the projects, usually measurement. In high schools, content knowledge referred to familiarity with ICT resources, rather than the mathematics. Several schools also explicitly reported that their staff were keen to extend what they had learned into other areas of mathematics. The knowledge referred to was often linked to teaching, Shulman's (1986) Pedagogical Content Knowledge, but also addressed the mathematical underpinnings, especially in the primary area.

Teachers' pedagogical knowledge

Improved pedagogical approaches were reported by all schools. Statements about increased teacher confidence were often included in the comments, despite the fact that all the teachers involved in the project were very experienced. One teacher in one of the high schools that had technology as its focus, in an informal follow-up discussion, stated how she felt that the influence had been far greater than she had expected, and that technology was now being used extensively in all mathematics classrooms from Year 7 to Year 12.

Factors contributing to success

There were a number of common features reported by schools that appeared to contribute to their success. All schools reported an increased awareness of the Standards, which in itself was a desirable outcome. Prior to the PLUMS project all teachers had some background knowledge of teaching standards through the Professional Teaching Standards (NSW Institute of Teachers, 2005) that had been recently released but engagement with a set of sharply focused subject-based standards appeared to provide teachers with a basis for considering teaching standards in general terms, and their potential use, rather than seeing standards as solely an accountability tool. The use of the *Standards* provided a context in which teachers could reflect on their practice with purpose.

Several schools also reported that they had a "product" at the end of the project: a bank of assessment tasks or teaching materials, for example. These were shared among other teachers within the school. Having such an outcome both legitimised the project for the teachers involved and extended its usefulness to other teachers.

Most notably, the Standards provided a common language that allowed dialogue. The improved communication was important within schools to keep the project moving, but also across schools. The eight schools involved were dramatically different in their contexts, approaches and organisation. Nevertheless, they were able to share their ideas with other project members. This sharing was particularly evident when the primary and high schools came together at the final conference where they were able to discuss pedagogical issues using the same language.

Implications arising from PLUMS

Although overall the PLUMS project provided some insights into the ways in which the *Standards* could add value to teachers' professional learning, there are a number of issues that arose. In particular the role and place of Academic Partners deserves some attention. There are several programs where the presence of a "Critical Friend" or

Academic Partner is a requirement but the PLUMS experience suggests that there are some aspects that require further consideration to make the experience a useful and productive one for all.

The role of an Academic Partner or Critical Friend is a difficult one. It becomes a balance between support and direction and depends greatly on the personal relationships between the school-based and non-school based people involved. This project was greatly aided by the fact that one of the Academic Partners was seconded from a school, and was well known and respected in the local education community. The second partner had also worked with several of the teachers involved.

The failure of one school to complete its project, however, raises the issue of the extent to which the Academic Partner can, or should, make demands on a school in order to keep the project moving. Too much intervention can damage the relationship between the partner and the school and result in the school losing control over its project. Too little intervention, however, can result in stagnation.

Another issue for the Academic Partner is the boundary of responsibility, especially after the project has ended. If there has been a productive partnership, both schools and academics are likely to want this to continue. Reality, however, is that academics are not able to continue to support every school. There are funding issues as well as time. This reality can lead to some stress on both academics and schools.

Following the PLUMS project, the following suggestions are made about developing productive partnerships between school and non-school based project partners.

1. Make contact early on the project.
   The responsibility for early contact is a shared one. Who should initiate this is irrelevant but making sure that it happens will help the project to run smoothly.
2. Establish clear understanding about the extent and parameters of the partnership.
   Clarifying what the expectations are of all parties will help to avoid some problems and unrealistic ideas. Role statements were developed for PLUMS but they were general in nature and did not specify realistic demands. It is also important that all parties recognise the limitations of their roles. Partners cannot (and should not) drive the project.
3. Ensure adequate time to work with the partner.
   A block of time is more useful than a 10-minute visit.
4. If you want a partner to take on some specific task, make it clear what your needs are.
   Partners will do everything they can to support and help but can only be effective if what they do meets the needs of the school or teachers.
5. Keep open communication channels.
   It is easier said than done when everyone is leading a busy professional life to maintain communication. It can, however, prevent a project from foundering. The partner is there to act as a sounding board, and will usually be happy to hear what you are doing and to share ideas with you.
6. Recognise that the end of the project means the end of the formal partnership.
   If both parties want to continue their professional relationship that is wonderful, but this should not be seen as a requirement on either side.

Concluding thoughts

The PLUMS project demonstrated the utility of the *Standards* at all levels of schooling as a framework for professional learning. This went beyond their use for accreditation, and potentially provides a route to improving mathematics education without the

personal demands of the HAToM process. Many teachers who would not consider working towards becoming a Highly Accomplished Teacher of Mathematics will happily engage with the Standards as a support for professional learning that meets their school's needs.

In addition, working with an Academic Partner or Critical Friend can become a rewarding experience for all parties if some guidelines are laid down early on in the project. Using the Standards can provide a framework and a common language for all parties to use to establish goals and expectations.

PLUMS indicates that the mathematics education community has only begun to scratch the surface of using the *Standards*. With the increased interest in teaching standards and teacher registration we should be promoting the *Standards* as a useful tool for setting expectations and goals for professional learning.

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Mathematical literacy: It's all Greek to me![*]

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Teacher-practitioners of mathematics, by virtue of their interactions with their students, tempered by their own beliefs about mathematics and mathematics teaching and learning, are powerful agents for the development of their students' mathematical literacy. Traditionally mathematics has been taught to students who were well-grounded in English (often Latin as well), with English as their mother tongue. How times have changed! This paper examines some of the issues that have arisen in the acquisition and use of mathematical language, and describes a number of strategies that may assist teachers to improve the reading, writing, listening and speaking skills of their mathematics students.

Introduction

"Mathematical literacy" in the context of this paper (throughout Australia and the United Kingdom) does not refer to the meaning generated in the United States; that is, what we call "numeracy". Mathematical literacy is the ability to use and make meaning of mathematical language, text types, terms and symbols with clarity and precision through (and when) speaking, listening, reading and writing within a variety of contexts. Student access to and engagement with mathematics is a direct result of their level of mathematical literacy.

I am a product of Dutch immigrant parents, a childhood and adolescence spent learning four musical instruments as well as music theory, and teacher-training and tertiary studies in mathematics and French. My approach to teaching mathematics is very much a language acquisition approach, focussing on the necessary coding and decoding: akin to foreign languages and music.

The language of mathematics has many of the attributes of a foreign language. The role of "teachers" may be described as "cultural brokers" (Delpit, 2006) who have the responsibility and opportunity to help make the connections between the familiar and the unknown. To be "maths-literate", students need to de-code as well as construct for themselves the language of reasoning, the language of the mathematics curriculum and activity-specific language (or language of the classroom) (Gawned, 1990).

The language of reasoning is the ultimate goal of mathematics teaching and learning. It refers to the language that the teacher and the students use in mathematics-based problem-solving contexts, and it develops out of the language of reflection and only after descriptive and comparative language is well-established. Causal relationships are central to the language of reasoning, so such language may include comparisons, predictions, inferences, considerations as well as the verbs and adverbs of possibility (*might, maybe, if … then, suppose*).

The language of the mathematics curriculum is the language, often symbolic, that is specific to the discipline of mathematics, and can only be learned in the context of

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mathematics. Students need to learn this language in order for them to succeed in mathematics.

Activity-specific language is based in one or two predominant language genres; that is, descriptive and procedural. A classification activity may require students to use requisite labels, terms, definitions or properties: all elements of the descriptive genre. However, if the students were asked how and why they classified as they did, then the procedural genre would be used, employing sequencing steps and explaining and justifying the outcomes of the task.

The language of mathematics is not "natural" language, in that it is not a "native tongue". It does not operate within the same semantic or syntactic scheme required to process everyday speech and it does not use the same grapho-phonetic system. The example below (from Simpson, 2003) illustrates particular mathematical semantics, an "unnatural" syntax and the use of English, Hindu-Arabic and Greek in combination.

Sketch the graph of $y = \sin\left(\theta - \dfrac{\pi}{4}\right)$ for $0 \le \theta \le 2\pi$.

When analysed, as suggested by (Bechervaise, 1994), the language of mathematics incorporates:

- some words that are sourced from "real world" language (e.g., add, shape, size, match)
- some words that have a redefined meaning in mathematics (e.g., table, index, multiply, volume, root, differentiate), and
- others that are specifically mathematical (e.g. intercept, logarithm, sine) .

(Donovon, 1990) observed that the beginning of mathematical understanding commences at birth, and develops as children experience their world as a place of order, pattern or prediction (or the converse for many children who find themselves in environments that lack order and predictability). In these pre-school years, usually through the medium of play, the children become involved in activities such as labelling, sorting, classifying, sequencing, comparing, measuring and problem solving. The *Early Years Phase Descriptors* (*Early Years Curriculum Materials: Phase Descriptors*, 2006) reflect these activities in the development of early mathematical understandings; "Investigating and communicating ideas about: quantities and their representations, and attributes of objects and collections; position, movement and direction; and order, sequence and pattern." (p. 20).Unfortunately this prior knowledge is often segmented into distinct, often unrelated units of mathematical learning when the child begins "formal" education; and the total concept of mathematics, particularly the language of mathematics, as being a functional part of everyday life is destroyed.

Strategies

Our challenge and obligation as teachers is to support the use and understanding of mathematical language rather than oversimplifying or overgeneralising and "de-maths-ing" the language. This may require approaching the teaching of mathematical language from a LOTE (language other than English) perspective; that is, explicitly investigating vocabulary, syntax and semantics in context. "The current best practice in second language teaching suggests that language should be taught in context when it is needed

for communication. This means that the language of mathematics should be concurrently taught with the mathematics." (Moore, n.d.)

Modelling the use, both oral and written, of correct mathematical language is the most effective strategy for teachers to use to ensure the development of the mathematical literacy of their students. Moore (n.d.) emphasised that not only do teachers need to provide a model for mathematical language; they also need to provide opportunities for students to practise it. Initially this practice will involve only a small input from the student. The amount of input will increase until students are reading, writing and speaking about mathematics thereby practising the newly acquired language. Teachers need to be aware of the pitfalls associated with early foreign language development, particularly the temptation to make literal translations. For example, "sacré bleu!" a French exclamation of incredulity, literally translates to "sacred blue", which does little to assist in the meaning-making of a particular text. Similarly, when students come across the instruction "construct a table", the literal translation does little to support the student to collect and handle data appropriately.

Another commonly identified language acquisition problem (for both English as a mother tongue and foreign languages) is the over-generalising of "rules"; for example, "I baked you a cookie but I eated it" which can lead to over-generalisations in mathematical language such as "square rooting", "timesing" and "plussing". This use of language is not cute or acceptable, and the appropriate language should be modelled in a positive fashion; for example, responding "Oh, you baked me a cookie but you *ate* it? Dear me." or "So you *multiplied* the amount by four?". This phenomenon of over-generalising can also occur when students use symbolic mathematical language; for example, $4 + 3 = 3 + 4$ so $4 - 3 = 3 - 4$. This misunderstanding needs to be quickly addressed, and the use of calculators can be quite convincing to students in showing that the two expressions are not equivalent.

An emerging phenomenon, particularly in secondary schools, is the infiltration of "texting language" as an alternative written vernacular of the classroom. I had a Year 11 Mathematics A student write an entire budgeting assignment using text "lingo", such as: "Ill budget $20 ea wk 4 trnsprt" = "I will budget $20 each week for transport." Rather than embracing this written form, teachers need to explicitly teach the language of mathematics: "find the value of x" rather than "fnd d valU of x". Although this texting language is widely accepted and understood amongst students in their "real" world it has no place in the discipline of mathematics.

Writing tasks in mathematics are valuable for both teacher and learner. However, it is the type of task that determines whether or not any mathematics learning takes place. (Pengelly, 1990) strongly advocated that "mathematics is not about writing journals, reading books or describing what was done in a mathematics session. Mathematical learning comes about as a result of engaging in the *mathematising* process." (p. 15). There is little value in using a learning journal for mathematics when entries resemble these:

"We did group work for some of the lesson, and then exercise 3B".

"We did integers today."

The kinds of writing tasks that foster mathematics learning are those that expose misconceptions and gaps in understanding when the learner tries to write a definition or an explanation (or the question style is altered). In this way, the "learner is engaged in high-level thinking and the active construction of meaning." (MacGregor, 1990).

The following student samples were selected from the trial of test items for my doctoral study. The wording of these tasks and the subsequent student responses provide a greater insight into the students' understanding of concepts (misconceptions

and gaps) as well as their ability to express these understandings. The usual syntax used by teachers for these kinds of concepts includes:

3. Evaluate: $\dfrac{1}{2} \times \dfrac{1}{2}$ _ x _

4. What is the length of this brick?

6. Continue this number pattern: 1, 4, 7…

12. Evaluate: $8 \div 3$ or $\dfrac{1}{3} \times 8$

3. Explain why half of a half is less than a half.

because half of a half is 2.5 and just half 5.0

Figure 1. Breeanna, Year 6.

3. Explain why half of a half is less than a half.

Half is alread less, so if half is halved then it became lesser.

Figure 2. Huy, Year 8.

4. What is wrong with this statement? "I am going to weigh the length of this brick?"

You can't weigh the length of a brick, you have to mesure a brick

Figure 3. Breeanna, Year 6.

4. What is wrong with this statement? "I am going to weigh the length of this brick?"

The length is supposed to be amount.

Figure 4. Tara, Year 6.

6. Explain why if you start at one and count up in threes then you cannot reach 12.

because time tables start at 0.

Figure 5. Daniel, Year 8.

6. Explain why if you start at one and count up in threes then you cannot reach 12.

Because it goes : 1, 4, 7, 10, 13 there is no 12 in it, It just skips it.

Figure 6. Tim, Year 7.

12. Show one-third of 8 using a picture.

Figure 7. Breeanna, Year 6.

12. Show one-third of 8 using a picture.

Figure 8. Huy, Year 8.

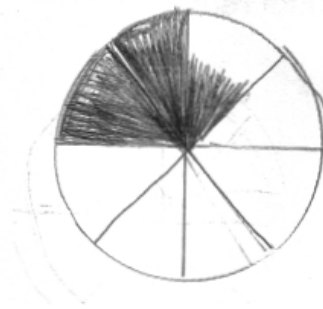12. Show one-third of 8 using a picture.

Figure 9. Heather, Year 7

Morgan (as cited in Wagner, 2004, p.409), as a result of her extensive study of secondary school mathematics writing, identified the need for students to become more aware of their language practice. Typical students' mathematical writing tends to be a poor reflection of their mathematics, even if they are proficient users of mathematics.

Story (word) problems bring natural and mathematical language structures together and require students to operate in both representational systems simultaneously. When engaged with, these word problems need to be analytically read. This is not an intuitive skill, and students need to have modelled for them and have opportunity to practise strategies for doing this analysis. The typical use of complex sentences, conjunctions and natural language in word problems can often lead to ambiguity. However, when students in the class come up with different interpretations of a problem statement, this can become a "teachable moment" engendering discussion that focuses on both the language and the mathematics involved. (Mitchell, 2001) The increased need for explicitness in written language is one reason writing is seen to be useful for the teaching or learning of mathematics — the process of developing clear communication using written words forms a bridge to the use of mathematical symbolic expressions. In contrast with natural language, mathematical representations achieve their value mainly from their precision, their (relative) lack of ambiguity. When a type of mathematical object is defined, it must be clear for any given object whether or not it is an instance of that type — otherwise the type is not well defined. Mathematical expressions in general must be well-defined and in part derive their analytical power (as well as their limitations) from this precision (Mitchell, 2001).

For instance, a class is given a task as follows:

Draw a diagram to illustrate each of these words:
Quadrilateral, parallelogram, rhombus, rectangle, square.

One student draws a square for each word. Are they incorrect? What did the teacher really have in mind? To what extent did the combination of natural and mathematical language contribute to the mismatch of student performance and teacher expectation?

The mathematical literacy challenges that students face in reading and writing mathematics are also evident in oral communication and interpreting spoken mathematical language. As with foreign language acquisition, engagement in oral and aural language demands point-in-time decoding and coding. When using the language of mathematics, this situation can be further complicated by the use of homophones such as *discrete* and *discreet*, and "*shine x*" and *sinhx*. Easdown (2006) described a technique called "spotlighting" the purpose of which is to "prime students" and to give them an "appropriate frame of reference". The aim of spotlighting is to avoid even momentary confusion because that can mean that some students are not actively listening to the important material (development of new concepts) and they lose the thread of the lesson. Thoughtful spotlighting takes little time and saves confusion. For example; "You will be collecting and displaying discrete data. Remember that means that it represents categories such as eye colour."

Another area that causes confusion with students is the apparent anomalies in the written and spoken versions of index notation. For example, 6^2 is often said as "six squared". The confusion arises when students investigate area and vocalise (or have modelled to them by their teacher) 6 cm^2 as "six centimetres squared" rather than the correct version "six square centimetres". As shown in Figure 10, these expressions represent two different quantities. This kind of accuracy is important in retaining the unambiguous nature of mathematical language.

"six square centimetres"
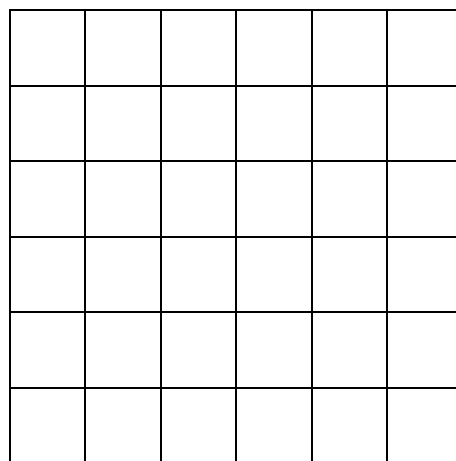
"six centimetres squared"

Figure 10

Some early mathematical conceptual understandings are confused by the inappropriate use of language. In particular, incorrect oral language used to vocalise decimal and common fractions. For example, many teachers verbalise 2.6 as "two point six". There is no such number. What they have communicated are the words associated with the symbolic representation of the number "two and sixth-tenths". The use of "point" does little to foster place value facility. Another example of unhelpful teacher modelling is when common fractions such as $\frac{3}{8}$ are verbalised as "three over eight". This refers to the positioning of the numerals not the fraction being represented which is "three-eighths".

The aim of using the correct terminology and modelling descriptive and procedural language in mathematics is that teachers and students will be able to engage in mathematics discourse, thereby raising the intellectual quality in the classroom. The language specific to a given subject is termed the "discourse" of the subject. Mathematics classroom discourse is about whole-class discussions in which students talk about mathematics in such a way that they reveal their understanding of concepts. Underlying the use of discourse in the mathematics classroom is the idea that mathematics is primarily about reasoning and not memorisation. "Mathematics is not about remembering and applying a set of procedures but about developing understanding and explaining the processes used to arrive at solutions." (*Mathematical Classroom Discourse*, n.d.). The NCTM (as cited in Wimer, Ridenour, Thomas & Place, 2001, p. 90) suggested that teachers could encourage student participation in classroom discourse through (a) using higher-order questioning (those that promote analysis, synthesis and evaluation), (b) explicitly asking for students' clarification and justification of their ideas, (c) appropriate use of mathematical language, and (d) listening and responding to students' ideas.

Ferreira and Presmeg (2004) contended that the language used inside the classroom has a significant influence on what students learn, and the oral communication between teacher and students plays a crucial role in this regard. While classroom "discussions" are nothing new, the theory behind classroom discourse stems from social constructionist views of learning where knowledge (new understandings) is created internally through interaction with other students and the teacher. Teachers' questioning, listening and responding approaches in the classroom have been suggested to characterise their pedagogical practices, and to reflect their beliefs about mathematics and its teaching and learning. Ernest (as cited in Ferreira and Presmeg, 2004) suggested that teachers holding "instrumentalist" beliefs perceive mathematics as an accumulated set of facts and utilitarian rules and skills; instrumentalist teachers see themselves as instructors. "Platonist" teachers view mathematics as a static and unified body of knowledge that is discovered, not created; Platonist teachers see their role as that of an explainer. Teachers holding a "problem-solving" perspective of mathematics perceive it as "a dynamic, continually expanding field of human creation and invention" (Ernest, as cited in Ferreira and Presmeg, 2004). One would be hard-pressed to find instrumentalist or Platonist mathematics teachers (in the majority?) who would be effectively utilising classroom discourse due to the nature of their beliefs and their resultant practice.

Conclusion

The quest for "mathematically literate" students is neither unreasonable nor unattainable. Every discipline has specific language demands and mathematics, science and music also have a repertoire of symbolic representations that students need to use and interpret in order to learn aspects of those disciplines. The use of discipline-specific language needs to be explicitly taught within the context of the discipline because the ambiguous nature of natural (everyday) language can lead to misunderstandings. For example, imagine if an experiment had as part of the procedure to add "a bit of acid". This ambiguity could have series consequences as there is quite a difference between "2 ml dilute hydrochloric acid" and "5 ml concentrated sulphuric acid".

The responsibility of teachers of mathematics is to model correct terminology and language of reasoning, and to encourage (insist) students to use the same in their teacher-student and peer interactions. Our aim is to scaffold students to develop their mathematical language from novice usage to, at least, apprentice usage. The language of mathematics may have the hallmarks of a foreign language yet it can and should be learned.

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A grassroots approach to improving numeracy[*]

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*Following a successful approach to improving literacy, a group of schools in one
district in South Australia developed a plan to raise numeracy outcomes among
their students. The three-year Southern Numeracy Initiative was planned
collaboratively by teachers in the district. Changes in performance were monitored
across the years of the project to provide feedback to schools and to the system.
Despite transfers of teaching staff and changes in the schools involved there was
significant improvement in numeracy performance across every grade. Strengths
and weaknesses of this grassroots approach to improving standards will be
discussed.*

The Southern Numeracy Initiative (SNI) began with district discussions in 2003. At that
time all the leaders in the Southern Vales district, an area south of Adelaide covering
schools from the beach to the hills, were asked to determine, "What are the big rocks in
our schools?" Numeracy became one of the areas pinpointed and interested leaders
came together initially through the auspices of the Quality Teacher Program to look at
how we could and would address our Numeracy concerns.

These discussions led us to look at models of success across the world and a range of
assessment strategies that we could possibly use to determine if we were making a
difference. Many of our schools had been involved in the Southern Vales Early Literacy
Project and liked the model developed by Hill and Crevola (1998). It was important to
us that all schools had the same aims and ideals so we eventually developed the
following statement for the participating schools:

> The aims of the Southern Numeracy Research Initiative are:
> * To develop and implement successful teaching and learning practises to
>   improve numeracy.
> * To focus on the pedagogy of explicit teaching balanced with open-ended tasks.
> * To challenge teachers to explore their beliefs and understandings about how
>   children develop their understanding of mathematics, and how this can be
>   supported through the teaching program.
> * To implement the design elements as part of the school's mathematics program.

The Southern Numeracy Initiative framework will be based on the nine design
elements for effective teaching and learning (Hill and Crevola, 1998) underpinned by
the model of *Towards a Profile for Numeracy Improvement* (Smyth, McInerney &
Hattam, 2004), as used in the recent South Australia research projects. The nine design
elements are shown in Figure 1.

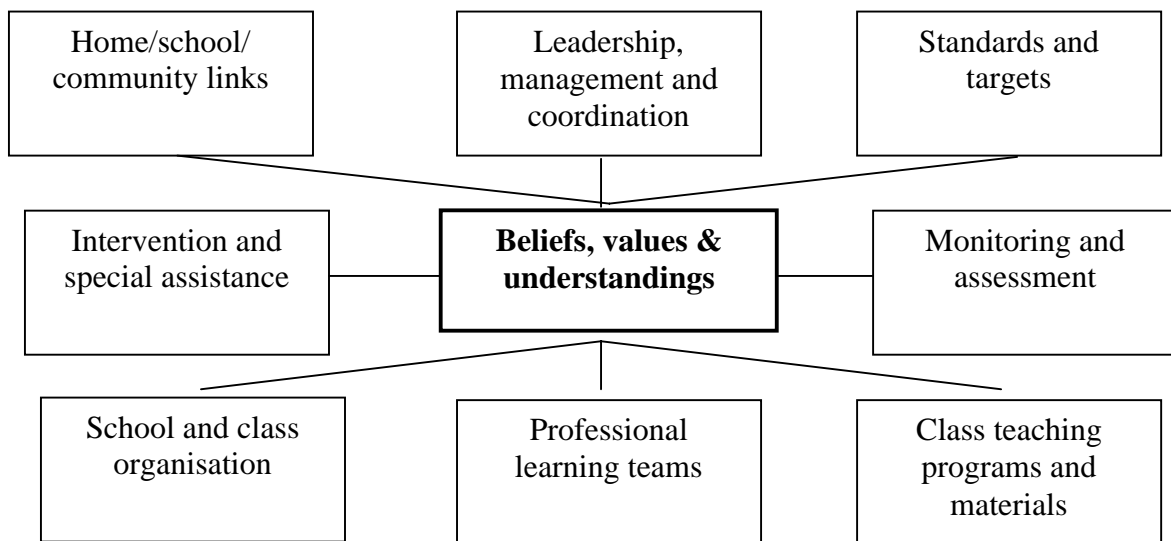---

[*] Paper accepted by peer review

Figure 1. Design elements for effective teaching and learning (Hill & Crevola, 1998).

Common areas within the nine elements

In implementing the 9 design elements there are 3 key areas that will be common in all sites implementing the initiative:

- beliefs and understandings
- standards and targets
- monitoring and assessment.

At the end of the 3 year project we will report in all of the nine design elements. At leader and teacher network meetings there will be opportunities to share how different sites have implemented different aspects of the elements.

Planning

Altogether 24 sites from pre-school to high school agreed to work together to achieve these aims. This involved 5500 students across all grades of schooling. We divided the responsibilities of the group into three areas: Training and Development, Data and Promotion. The Training and Development group were responsible for the professional development program, the Data group undertook the monitoring program, and the Promotion group was responsible for advancing the SNI project in our schools and elsewhere.

How to assess became an area that caused huge debate and many meetings to come to a resolution. We decided that we would begin our Initiative anyway and asked Doug Williams of Black Douglas Professional Education Services to launch our work. Discussions occurred with Doug about what he had seen as the best assessment strategies. Doug's comments seemed to make sense, "Why not assess while you are teaching?" That statement led us to connect up with a Tasmanian project that had had success with their assessment techniques with high school students (Callingham & Griffin, 2001). We determined we would test out their assessment strategies into the primary schools.

We set up an assessment regime whereby the end of the project we would be able to compare students' growth over two years. This meant we assessed each Year 3, Year 5, Year 7 and Year 9 student, once a year in term 2 for the period of the Initiative. To assess our Junior Primary students we used *I Can Do Maths* (Doig & De Lemos, 2000) and the Preschool Directors began work on creating and collecting Learning Stories (Harley, Perry & Hentschke, 2006) and in turn a matrix of development. To add to these data sets we engaged the help of the Manager of Student Achievement Information in DECS, to help us crunch the data from our "I Can Do" tests as well as attitudinal and school audits, which were conducted yearly.

To manage all of this we developed a document that showed what commitment was expected from project participants. This included expectations for schools, the district, leadership and teachers, as well as a list of the data that would be collected. The purpose was to ensure that all schools and participants received the same message. The initial Expectations and the Protocols are shown in the Appendix

Activities

Supporting teachers was a key feature of the Initiative. We asked teachers what they wanted and tried to find people who could run workshops and professional development days to address these needs. Over the three years we engaged many different speakers to talk to staff and leaders of current thinking in Numeracy and how to improve student outcomes. These people came from around the country as well as South Australia, and were based in universities, consulting services and the South Australian Department of Education and Children's Services.

Teachers liked hands on activities and ideas for teaching. Over the years we addressed topics such as ideas for teaching number, financial literacy, robotics, assessment processes and intervention strategies. Intervention became a particular focus in the final year of the project. Nearly 20 different presenters were involved, some more than once, across the period of the project.

Each year we also held a celebration where schools could showcase what they were doing. This was held at the local football club and the party pies and spring rolls became quite famous! These celebrations were well attended and schools went to considerable trouble to showcase their work, with displays and demonstrations by students. We invited members of the DECS hierarchy to speak to the group as well, which helped to make teachers feel that what they were doing was recognised.

Outcomes

Because we collected a considerable amount of data along the way, we were able to track progress from year to year. All Year 3, Year 5, Year 7 and Year 9 students completed at least one assessment task as part of their classroom activity each year. We chose *Street Party* and *Come in Spinner* (Callingham & Griffin, 2000) as appropriate tasks. These were modified for Year 3 students but some questions were the same across all the tasks to allow for a single scale across every grade. The tasks were marked by the students' own teachers and each school entered its data into a spreadsheet that was then sent to a central location for collation and analysis.

Over the period of the SNI, results varied from year to year. The best results overall were obtained in the Round 3 assessments, when the project had been running for two years, probably reflecting a peak in enthusiasm. From the first round of results to the last, however, there was a sustained improvement across all grades, shown in Figure 2.
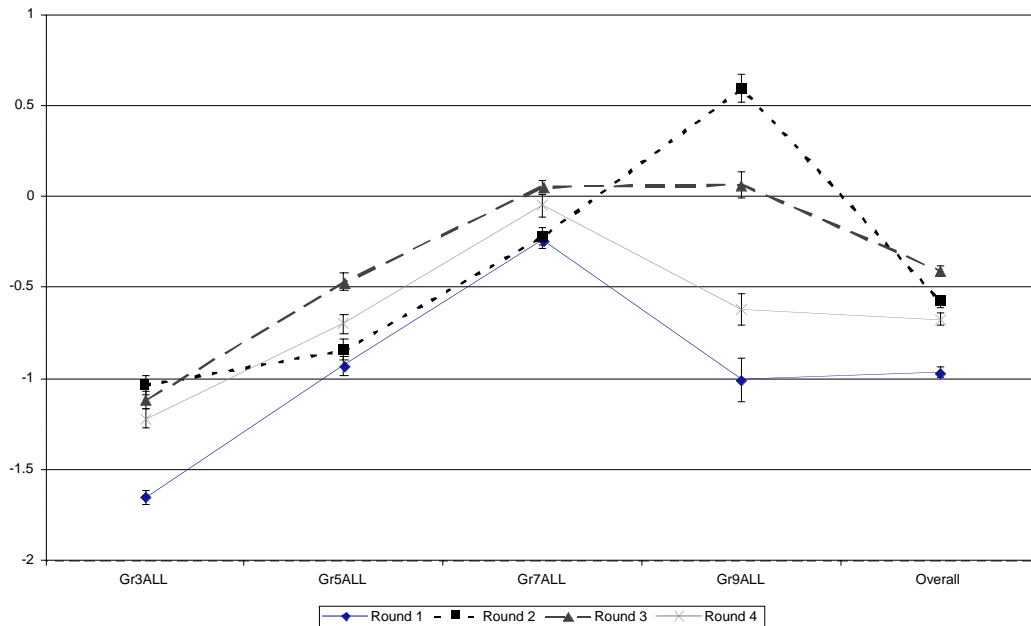
Figure 2. Results by grade and overall for four testing periods of SNI.

Successes and difficulties throughout the initiative

Although we saw an overall improvement, this was only part of the story. Our successes and difficulties were many.

One of our initial difficulties was a changing district. At the beginning of 2004 our district was split in two with two new District Directors leading these districts. Losing one main leader who had worked to develop the Initiative and then getting two new leaders on board was a challenge.

Changing staff always creates difficulties in a long term project, and changing leaders in our schools also created turmoil. Every year we had to contend with new staff and new school leaders but a positive from these changes was the commitment from existing leaders who were able to talk about the changes in their sites and why they felt SNI was a valuable group to be a part of. Some people from within the SNI were promoted to other positions both in and out of DECS. Where the promotion occurred within the District this did not pose a problem, but we did lose some key people along the way. The good thing was that there were others waiting to take over the roles, helping to build management capacity in the District.

Running the project was not always easy, however. The SNI was run by teachers with no additional time release. Data collection and entry was undertaken by school administrative assistants on top of their own work. On the other hand, we were in a position to understand our own schools' needs and to try to be responsive to these. We could not impose anything, and adherence to the expectations and protocols was obtained by the enthusiasm of committed teachers. The early development of a strong framework, and the expectations and protocols were essential to keep us on track over the three years of the SNI.

One of the biggest challenges was the implementation of the assessment tasks. The tasks had been deliberately chosen to support the teaching approaches we wanted to see. In schools where the leaders did not see this connection, and in turn the teachers did not see the connection, the tasks became an arduous task that lost all their potential.

Conversely when the tasks were used as part of the teaching program and teachers understood their importance, the results were fantastic! In the high schools, timetabling was an issue and which students would complete the assessment. Many high school teachers could not come to grips with the thought that everyone was involved in teaching numeracy no matter what the curriculum area.

There were, however, also many successes at all levels. At the district level, the improvement in Numeracy was reflected to some extent in the State Literacy and Numeracy testing. At the site level many teachers commented on the improved level of their students. At the student level there were many stories like this personal experience of a 10-year old boy who hated numeracy with a passion and felt he was no good at all. After working with the teacher and putting in place the hands-on assessment tasks, and demonstrating to the teacher the usefulness of the tasks, the child's attitudes turned around. He can do mathematics and loves it!

Overall on reflection of the Southern Numeracy Initiative, much was achieved. Numeracy became a much needed focus in our schools. Teachers were exposed to a range of top quality training and development that, but for SNI, would not have occurred. As individual sites we were able to reflect on our site's results and, in some instances, where money was available to set up intervention programs that have really made a difference to our students and our teachers' views of teaching mathematics. Despite the difficulties and the very hard work involved in driving a large project from within schools we feel that the Southern Numeracy Initiative was very worthwhile.

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Appendix

Southern Numeracy Initiative
Overview of Expectations for 2004

2003   Preliminary investigations occur across the Southern Vales District

Individual sites commit to Southern Numeracy Project

2004–2005   Participate in Southern Numeracy Project

1. School Structures:
- Identify Numeracy as a priority within the strategic plan and resource accordingly (HR & assets)
- Timetable for staff to attend one professional development day in Week 0, 2004 (Wednesday, Thursday or Friday) with Doug Williams.
- Commit one pupil free day (June 3 or 4) to professional development with Doug Clark.
- Assign a minimum of five staff meetings per year to numeracy professional development.
- Form an internal numeracy management team.
- Collect and collate data as per project outline.

2. District Commitment
- Learning Band Coordinators will be involved in the project

3. Leadership Commitment:
- Nominate one leadership person to be part of the site's management team.
- Nominate a leader to attend Southern Numeracy Project meetings once a term.
- Attend minimum of two professional days per year.
- Encourage staff to join professional discussion sessions.

Please note: small sites and preschools may choose to combine and have one/two representatives to represent the entire group.

4. Teachers:
- Attend Southern Numeracy Professional Development sessions (minimum of 2 days)
- Attend numeracy/cluster staff meetings with a numeracy focus (minimum of 5 per year)
- Participate in Southern Numeracy Professional Discussion sessions as appropriate
- Collect and collate data as per project outline.
- Maintain on-going data collection and observations of individual's progress

5. Data

The following Data will be collected for this project
- LaNS
- "I Can Do" assessment — Rec, Yr 2
- Street Party — Yr 5, Yr7, Yr9
- Attitudinal audit — staff and students

PROTOCOLS FOR SOUTHERN NUMERACY INITIATIVE

The overarching purpose of the Southern Numeracy Initiative is:
IMPROVING NUMERACY OUTCOMES FOR ALL STUDENTS

- The impetus for this Initiative is shared between a group of schools and preschools south of Adelaide, and will involve schools and preschools from Wallara Outer South and Southern Sea and Vines.
- Leaders of these preschools and schools are committed to the purpose of the Initiative, and demonstrate this by providing release time for personnel involved.
- The organisation of the Initiative is managed by the Numeracy Management Group, which consists of teaching and leadership representatives from all schools and preschools involved in the Initiative.
- A Code of Cooperation exists for ensuring the processes involved are representative of all stakeholders.

CODE OF COOPERATION

- The Management Group is responsible to Wallara Outer South and Southern Sea and Vines LLEG.
- Meetings for all stakeholders are held at least once per term.
- Decision making by stakeholders will be by consensus, either at the whole group meetings, or by e-mail response.
- A Numeracy Management Group exists to facilitate communication processes for all stakeholders, and various organisational tasks as required.
- Representatives on the Numeracy Management Group will consist of a chair of group, chair of each subcommittee, a Learning Band Coordinator from each district, Recorder, representation from Preschool sector, Primary and Secondary. Representation on this group will be shared across the two districts. District Directors will be asked to join this group at least once per term.

COMMUNICATION
- Communication between all stakeholders is primarily through:
  - Meetings
  - Minutes of these meetings, circulated to all stakeholders
  - E-mail via svng@nexus.central.sa.edu.au,
- At meetings of all Initiative site representatives a range of tools**, will be used to ensure the voices of all stakeholders are heard, and decisions made reflect shared understandings and commitment.
- All representatives will work proactively to resolve grievances as they arise. Where grievances remain unresolved, a District Officer will lead the resolution process.

DATA
- All Data will be managed according to DECS and State Government policies and directives.
- No aggregated data relating to schools or the Numeracy Initiative will be published or released to other bodies (including project schools) without the expressed permission of the parties concerned.
- All Initiative representatives will act respectfully and maintain confidentiality of sensitive information (e.g., data from schools, individual students).

WORKING PARTIES
- As the Initiative develops, non-exclusive working parties will be established to focus on different aspects. These working parties will exist for the duration of their responsibilities, as decided by a full meeting of the Initiative representatives.
- Working parties are responsible for keeping all stakeholders informed via svng@nexus.central.sa.edu.au to foster collective ownership of the various aspects of the Initiative
- Working parties will report to the Numeracy Management Group meetings

** David Langford tools, the protocol tool, etc.

This Code of Cooperation was developed by the Protocol Working Party in 2003 and ratified by all representative stakeholders Feb, 2004, updated 2005

The reality of intellectual quality in mathematics classrooms[*]

Kay Owens

Charles Sturt University

Preservice teachers evaluated their teaching of space lessons in early primary school by focussing on three aspects of the Intellectual Quality guidelines from the Quality Teaching and Learning in NSW Public Schools *document: substantive communication, deep knowledge and deep understanding. The impact of expected class behaviour, the learning experiences as planned, and the age of the students formed the reality of the classroom. The teachers struggled with the guidelines in terms of their relevance for the stage of the students. However, the teachers' expectations limited their effectiveness. This case study informs our understanding of activity theory and argumentation in primary mathematics classrooms.*

Classroom communication

Mathematics lessons are frequently seen as a time for students to open a textbook and answer the questions. Few students are satisfactorily challenged by textbook exercises (Bell & Birks, 1990). They are regurgitating a method told to them by the teacher or they cannot make sense of the questions. The alternative is quality learning with students constructing their own knowledge. The reality of this approach is much more related to the cognitive challenges that students meet. These challenges need to be presented to students in a way that they will develop positive affective responses (Goldin, 2004; William, 2002). To achieve this positive response, classroom communication is critical. Both the teacher and the other students need to engage in conversations about problems. Teachers need to provide adequate support in terms of reviewing relevant approaches to the problem solving, assisting students to make links to existing knowledge and strategies, and setting up group support that will engage students and encourage students.

We would assume that in many classrooms, there has been a shift from the above description given by Bell and Birks (1991) and, indeed, Johnson and Cupitt (2004) did find that many teachers were using group activities and concrete materials. However, they found that engaging students in mathematical inquiry and investigation is much more challenging for teachers. Teachers need to develop skills to increase communication through open-ended questioning, setting appropriately structured tasks, and developing group discussion that leads to students developing the concepts. Students should be initiating conversations, making comparisons, asking the hard questions, justifying their answers and explaining more thoroughly. Successful teachers carefully followed students' descriptions of their thinking, and encouraged more complete explanations and deeper thinking. Teachers allowed for interruptions from

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[*] Paper accepted by peer review

students when explaining in order for students to explain, or to own, new strategies (Hufferd-Ackles, Fuson & Sherin, 2004).

Classroom communication patterns (Bauersfeld, 1988; Clarke, 2004) that are not very effective are those in which the teacher asks a question, the student responds briefly and the teacher immediately praises (or otherwise evaluates) the response (NSW DET, 2003). One successful approach is to provide an initial task that gets all students started, then tasks that encourage all students to present what they know, or could easily develop, and for others to query their ideas before giving the students a harder challenge (Williams, 2002).

Wood (2003) considered teachers' questions to be critical during periods in class when students were reporting back after the task. The reporting back should not just be a report of what the students found out. It should be strategy reporting. Wood (2003) recommended that questions such as "How did you decide this? Are there patterns? Is there a different way you can do this?" encouraged strategy reporting. More successful mathematics lessons required inquiry and argument. These communication tools could be prompted by questions such as "How are the two things the same? Does this make sense? Does it always work? Why does this happen?" The conversations in the classroom need to be about the substance of the lesson and be about making distinctions, critically examining each others' ideas, forming generalisations and asking questions (NSW DET, 2003).

Communication and culture are central to activity which starts with the child. Mellin-Olsen (1987), in his classical discussion of activity in mathematics education, referred to students' thinking-tools and communicative tools used to investigate and thus develop mathematical concepts and processes. This mental activity belongs to individuals and signals of the activities guide teachers to provide the students with educational tasks. The study reported here attempted to consider some of these tools, and how teachers may generate their use to achieve learning.

The study

We provided preservice teachers with exemplar lessons that were expected to change the classroom communication rules, the means of learning, and the fine grain intention of the lessons from focussing on knowing shape names to explaining classifications of shapes (Cole & Engeström, 1991). The teachers participated in conversations about quality constructivist teaching as presented above and were given copies of the references. The focus of this study was to analyse what two of the preservice teachers considered critical aspects of the classroom learning experiences, and what they viewed as bringing about change in the rules of the classroom.

The two preservice teachers (I will refer to them as teachers) taught a Kindergarten class as a team over nine sessions, spread over a number of weeks. Six months later, I observed one of the students teaching and asked whether she felt the experience of concentrating on substantive communication had affected her teaching. Evidence for the emerging communication patterns and their thinking are supported by quotes from the teachers' reflective journals.

Teacher analysis of the classroom communication

The teachers pre-assessed the students' shape recognition during the first activity, in which the students were making pictures with shapes. The teachers asked questions like, "What shapes do you think you will use to make a motorbike?" They noted that

students were able to trace around a shape in a picture, identify shapes in their classroom environment, and they made "complex and highly different pictures." Teachers were aware of their use of praise. They commented: "teachers provided feedback and elaborated on students' answers and teachers attempted to foster discussion with students without much success".

In the second lesson on shapes in the environment, the teachers said they gave good examples and a variety of explanations of the activity, and they made it sound exciting to do the rubbings of shapes in their environment. Their questions during the conclusion were: "Where did you find this shape? Did anyone else find this shape? Tell me about the shape." The teachers responded to the finding of a hexagon, which was a new shape to the students, "by listening and drawing the hexagon on the board and discussing it with the students." They noted that few students would respond to the question "Tell me about it" and that they rarely communicated with each other or asked each other questions. They felt this was reflective of the general classroom milieu of sitting quietly, listening, and putting up your hand to answer the classroom teacher's question as well as lack of group work.

They decided that students should be introduced to the trapezium, rhombus and hexagon next lesson. Students made shapes with their bodies. The teachers noted that:

> students were… actively engaged in the activity making their bodies into the forms of shapes, correctly displaying the appropriate number of sides and corners… The students were self correcting by standing back and observing… [During the whole class reporting session] communication was one sided with the teacher providing information to students and asking questions… One child appeared to dominate the direction the shapes took, and other students took on a passive role in shape making. IRE [initiate, respond, evaluate] communication took place during the whole lesson.

The teachers recognised a continuing lack of substantive communication.

The fourth lesson was introduced by revising the names and basic properties for the rhombus, hexagon and trapezium using shape cards. No one could say what the hexagon was, so the teacher named it and described its basic properties (e.g. number of sides and corners). Then the rhombus was held up and the conversation between children (C) and teacher (T) continued:

C1:   It's like the bottom of the chair
C2:   The signs, the lollipop ladies signs.
T: (holds up rhombus). What do we call this shape?
C3:   It's a square.
C4:   It's a rectangle.
T: That's great but this shape is called a rhombus (describes properties)

Then the teacher demonstrated making shapes with plasticene snakes including "Ms B's Wageagon" (the name given by the teacher to an unusual shape). With the plasticine snakes, "students created complex shapes giving them names that related directly to the shape as in a ZigZagagon" which the child said was because of its "head and legs … curved ones and sigsgs and bumpy". However, the lack of substantive communication meant the teachers set their goal to considering a different questioning technique in the next lesson.

During the fifth lesson, students created their own triangles by drawing lines (intervals) radiating from a central point, joining the ends of the intervals, colouring in the triangles, cutting them out and pasting onto another sheet. The teachers reported that they "modelled open-ended questions and encouraged the students to discuss triangles', shapes and sizes with the students at their tables. It was noted that some students have little to no understanding of triangles and their properties, and so further development is needed". Some students did not notice the quadrilaterals drawn on their paper. However, "students were observed discussing their work with others at their table … each student contributed to the discussion; however most students accepted what they were being told, rather than offering their own insight into the problem". Students were beginning to communicate about the mathematics:

> C1:   How many triangles do you have?
> C2:   Four
> C1:   You didn't make all triangles
> C2:   Yes I did
> C1:   This one has four sides
> C1:   Triangles have three sides.

However, the notes on the lesson showed that despite their good intentions, the teachers continued to ask closed questions with only one open-ended question, "What about this one?" They used some harder questions such as, "Can you point to the largest triangle on your page?" but without following up the question or having a clear purpose for the question.

The next lesson consisted of students making shapes with a piece of string. The teacher demonstrated, but the class were hard to get on track outside. However, the students began to pose questions. In the following lesson, the students cut up a large triangle into several triangles. During the explanation at the start of the lesson, one child drew a line and formed a quadrilateral. The teacher asked her to have another go and when she completed the triangle the teacher said, "Well done!" This individual activity seemed to inhibit discussion. Some students found it hard to draw triangles on the sheet. The teachers were still asking the questions and students were beginning to give short sentence answers. Some children cut any shape. Others explored the properties of triangles by making smaller triangles. The teachers also realised that "some closed questions were given, resulting in yes or no answers… The communication that was evident appeared to occur between one child and the teacher, rather than a group discussion." At this point the teachers were further focussed on questioning to foster communication.

The eighth lesson started successfully with a clear explanation, in which the students were involved in making an example picture from tangram pieces, for the whole class to observe. The students' attention was gained. Working in pairs, "some children were able to express their knowledge as they manipulated shapes in various orientations to create the correct representation of the tangram." (Some) children expressed knowledge as they realised they had misplaced shapes in their creation, and moved shapes to fit the design, so they were responsible for their own self regulation. Some "worked well together, offering ideas to each other and exploring their thoughts." During the conclusion, the class made the picture of a runner, as a whole class, except that the preservice teacher put some pieces in the wrong place. The students identified how the teacher needed to place it. For example, a child said, "Turn it around so that it points to the top."

The last lesson was an assessment: (a) the teachers asked the students to draw each of the named shapes, one at a time; (b) the students were given a sheet of shapes and asked to colour all of the named shape they could see with a specific colour, before the teacher named another shape and colour. The teachers summarised the students' achievement pointing out the variance in communication, recognition of shape properties, and seeing shapes in different orientations or when "stretched." The teachers concluded that:

> Aspects of the NSW Quality teaching framework may not have been developed successfully due to the following factors:
> - the level of understanding and learning the students were at, being the lower level Year 1 mathematics group;
> - teacher questioning needed to be further developed, and aimed more at getting the students to collaborate and contribute towards discussions;
> - teacher questioning needed to be more focused on developing and enhancing upon the knowledge that children already displayed on the content areas;
> - students were observed to be in the routine of IRE question sequence and, as the teachers, we found it hard to break the students out of this, and get them to elaborate on the questions we asked them;
> - students were not used to generating their own discussions and offering new ideas and found it uneasy to do so when asked;
> - some of the lessons were based on individual completion of activities. Given the classroom setting, it may have been more beneficial for us to complete these in group work scenarios, where deep understanding can be fostered and expressed more clearly;
> - students were used to completing worksheets and blackline masters and so, with our lessons being based on hands on exploration activities without the use of handouts, students may have been put off: they were not used to doing mathematics in this way (and had no routines for this);
> - during the teaching of the nine lesson plan sequences, the class had three relief teachers (not including both of us), which affected the children's regular routine and classroom patterns.

When one of the teachers was teaching in another school, some months later, I observed that she praised students readily and kept students on task by this technique, and by keeping the class moving. She quickly had them working in groups and the students knew the purpose of the group activity. The teacher described her teaching as using students' prior knowledge and expanding on it a little bit: "When they are doing group work, I am observing how students are interacting with each other, but there is not so much collaboration… There is not much of taking ideas from each other. I allow them to reflect and vocalise how they learnt in other areas of my teaching but not in mathematics lessons."

Discussion and conclusion

To assist in activities, students have tools: thinking tools and communicative tools (Mellin-Olsen, 1987). The thinking tools allow the student to explore or investigate examples and associated rules that make knowledge a familiar process. In this study, the thinking tools were encouraged by the wide range of open-ended physical activities such as making shapes with string, their bodies, and other equipment and by cutting a shape into pieces. It meant students were directing their understanding of the concept and developing a metaconcept of shapes and types of shapes. The concept involved the

shape in different orientations and with different lengths. The teachers extended the range of shapes from circle, triangle, square and rectangle to incorporate the hexagon, rhombus and trapezium, as well as shapes with made-up names. The words were made available by the teachers but it was necessary for the students to adequately match the words and their schema or metaconcepts. The communication from the teachers about the shapes was limited to names and the properties related to type of lines, number of sides and corners, but student descriptions were also admissible especially in describing the shapes in different orientations and when shapes were given non-geometrical names. Students were using thinking tools but there were mixed motives (developing their own concepts about shapes and using teachers' labels). Furthermore, the contextual influences of classroom routines, the extensive use of praise upon students' answers, and the teachers' attention to recognising and naming shapes, limited communication (Mason, 2003; Owens & Clements, 1998).

While teachers tended to use the symbols (shape names) from the start, this was a given rule limiting the degree of construction of the thinking rule by the students themselves. However, the naming of their own shapes, helped students to realise that shapes were part of a bigger group but, even then, the shapes were expected to have names, another rule imposed by the teachers. There were some examples given where students told each other whether a shape was acceptable as an example of the symbol (shape name). However, continuing conversation about the shapes did not develop as expected if the second child was exploring and devising their own sign system. There was a lack of discussion and self-developed description by students due, apparently, to the well established routine that the teacher asks the question, the student answers (usually with a single word), followed by praise, while the rest of the class remains silent and does not interrupt or express their own thoughts. Students, largely, felt comfortable about having a go at answering questions but their skill in working cooperatively in small groups, or talking with neighbours, was subdued. Mellin-Olsen (1987) warned against the control of language. Immediate praise, while it may have created some positive affect, restricted conversation and reduced the positive affect inherent in the mental activities of problem solving (Goldin, 2004).

This paper illustrates the struggle of two preservice teachers to achieve their goal of substantive communication and deep understanding, and the restrictions that the milieu of the classroom and their own limited pedagogical and mathematical knowledge had on achieving their goal. Nevertheless, their action research and reflective teaching began to bring about change in their teaching, and in the students' ways of learning. Consequently, there was an improvement in the students' knowledge. As teachers, we are challenged to implement learning experiences that create substantive communication for the early years of school.

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Acknowledgements

Who's doing whose mathematics?[*]

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We examine two fundamental questions about school mathematics: Who is doing the mathematics, and whose mathematics is being done? We claim that in most classrooms the mathematics that is being done belongs to the teacher or the textbook, and that the teacher is the one who does the mathematical thinking. In these classrooms students are at best dependents, with little access to the discourse of mathematics. We describe an alternative classroom in which students argue about mathematical ideas and are thus apprenticed into the discourse of mathematics. They have ownership of and power over their own mathematics.

"Maths classrooms are places where kids watch teachers do maths."
(a quote from a colleague)

Curriculum documents abound with references to desirable outcomes such as students choosing and using mathematics (AEC, 1994). Yet many classrooms remain places where content is dictated by a syllabus or textbook, where the way in which students learn is decided by the teacher, and where students become reliant on others for their learning. In this paper, using the lens of Social Activity Theory, we compare two classrooms in terms of the way in which the content, activities and interactions position students as learners.

Social activity theory

Dowling (1998) claims that the activities in the mathematics classroom, that is the texts, language and symbols used, can position students as subjects, apprentices, dependents or objects. He describes a "Social Activity Theory" (Figure 1) in which he looks at how an activity regulates who can say or do what.

		Content	
		Strongly classified	Weakly classified
Expression	Strongly classified	Esoteric domain	Descriptive domain
	Weakly classified	Expressive domain	Public domain

Figure 1. Domains of practice.

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[*] Paper accepted by peer review

Building on Bernstein's (1996) theory of classification he describes four domains of practice:

- The esoteric domain is strongly classified with respect to both content and expression. That is, it consists of mathematical topics such as algebra, arithmetic and Euclidean geometry rather than real-life contexts and, within those topics, the specialised language of mathematics is used. In the school mathematics classroom it could be described as "pure mathematics".

- The public domain is weakly classified with respect to both content and expression. It focuses on everyday contexts using everyday language. It is the mathematics that people do outside school, much of which is often not recognised as mathematics (Hogan & Morony, 2000)

- The expressive domain is strongly classified with respect to content, but weakly classified with respect to expression. That is, it focuses on content found within mathematics, such as algebra, but expresses it in everyday language and symbols. Bernstein calls this process one of recontextualisation. The idea of a "function machine" (AEC, 1991, p.197) is an example of recontextualising purely mathematical content (linear functions) into a more familiar setting, using the everyday metaphor of a machine.

- The descriptive domain is strongly classified with respect to expression, but weakly classified with respect to content. That is, it recontextualises an everyday situation using mathematical terms and symbols. Solving word problems using algebraic equations is an example of the descriptive domain.

Dowling (1998, p. 136) claims that the domain of practice in the classroom partially determines the positions that can be occupied by students. Where practice is restricted to the public domain, students have no access to the valued discourse of mathematics, and, hence, cannot be apprenticed into the discipline. He claims that this has the potential to objectify the students, denying them power and agency (Boaler, 2003) over their own mathematics. In classrooms in which the teacher frequently recontextualises mathematical content into the everyday, as occurs in the expressive domain, students are likely to become dependents, again being denied access to the valued discourse of mathematics. In order for students to become positioned as apprentices or subjects (that is to have maximum control over their mathematics), they must be able to operate within the esoteric domain.

At a textual level, message and voice correspond to the practices and positions described above (Dowling, 1998, p. 142). The range of voices in the classroom reflects, and produces, the range of positions in the classroom. An authorial voice is indicative of maximum subjectivity, in which the author or speaker has control of the mathematical activity of the classroom. Typically, this is the voice of the textbook or the teacher. The authorial voice may construct readers' or listeners' voices as apprenticed, dependent or objectified voices, depending upon the relationship being established by the author. Thus, a teacher who stands at the front of a classroom explaining a mathematical process, positions herself as author and her students as receivers. The extent to which the authorial voice invites students into the discourse of the activity will determine whether they are positioned as apprentices or dependents. A teacher who restricts activity to the public domain, and does not invite recontextualisation, objectifies the receivers.

The message of the text or spoken word is distributed via the strategies employed by the author. Expanding strategies, such as generalisation, broaden the range of the

message, while limiting strategies, such as specialisation or localisation, narrow the message. A focus on principles abstracts the discourse of the classroom, while a focus on procedures tends to particularise the discourse. The degree to which the distributing strategies expand or narrow the range and abstract or particularise the discourse will determine the degree to which students can be apprenticed into the discourse of mathematics. Thus, expanded range and abstract discourse will invite students into the discourse of mathematics as apprentices or subjects, while limited range and particularised discourse exclude students from this discourse, and construct them as dependents or objects.

Dowling's work focused on school textbooks. He analysed the School Mathematics Project (SMP) texts written for students of different "abilities", and discussed how the content, written text, pictures, drawings and symbols used in the text positioned students. He found that the texts, written for supposedly more able students, had a strong focus on the esoteric domain, invited students to express ideas mathematically and discussed general principles. In this way, the text positioned students as apprentices. Texts written for less able students positioned them as dependents. They focused on the public domain, containing a large number of pictures and cartoons rather than mathematical symbols and they focused on particular procedures rather than general principles.

In this paper we apply the ideas of Social Activity Theory to interactions in the mathematics classroom. We look at extracts from two classrooms and examine the classroom conversations in terms of domain, voice and message and, hence, discuss how students are positioned within the two classrooms.

Classroom 1

The extract below is taken from a publicly available video of an Australian Year 8 mathematics lesson filmed during the TIMSS study (Hollingsworth, Lokan & McCrae, 2003). The students have been given an exercise to investigate the conditions for congruency of triangles. They are working in groups, and have each constructed a triangle using ruler, compasses and protractor, then written down a set of instructions by which others in the group could construct the triangle. The aim is to find a particular set of minimum conditions that will guarantee that the other students in the group construct an identical triangle. In this way, the class are to discover the four conditions for congruency.

This episode occurs approximately twenty minutes into the lesson. The particular group of students speaking have completed the task, and each has successfully drawn the triangle described by the other students in the group.

SN:	Miss what should we do now?
T:	Pardon?
S:	What should we do now?
T:	Have you done all of them have you?
S:	Yep.
SN:	Yeah.
T:	Well, did you end up working out how many instructions you actually needed for each one?
SN:	Yep, I needed four.
T:	You needed four, what were they?
S:	Um, interval AB is vertical and measures five centimetres. Interval AC is horizontal (inaudible).

T: So if it's saying horizontal it's saying it's at an angle aren't you?
S: Yeah.
T: So we're saying one's vertical and one's horizontal.
S: Yeah and you um…
T: Then you're defining the angle between them aren't you?
S: Mm, I'm saying it's a right angle.
T: Yes.
S: Because interval… interval ABC is a right angle.
T: Right, so did they need all those instructions to finish your triangle?
S: Yes.
T: Yes.

Classroom 2

The extract below is also taken from a Year 8 lesson. The students are exploring the mathematical concept of gradient, and attempting to discover, for themselves, the effect of changing a in the equation $y = ax + b$. They have drawn graphs using two or three numerical values, and are reporting to other students in the class on their findings. Students take it in turns to walk to the front of the class and sketch graphs on the whiteboard, explaining to other students what they have discovered, and responding to what other students have said.

Class: Go Carly
Carly: Um, I think that what Catherine said makes sense but I think that when it comes to the values that you go up by on the graph it's really up to you and it's what you think will best show the lines on the graph. But what I found with a is that the higher the value of a the more acute the angle will be compared to the y-axis. So say, um, it was $4x$ then it will be closer to the y-axis than $2x$. 'Cause $2x$ will be here the $4x$ will be here. I found that the lower the value the closer it was to the x-axis. But the higher the value the closer it was to the y-axis. [Applause and a "whoo"]
Sarah: Just a question. I was wondering do you even need the, um … the numbers. 'Cause where it says $5a$ don't, can't you just go like $7a + 2$ or something. Like can we represent the numbers on the y and x-axis with something else? [Laughs]
Class: [Laughter, murmurs.]
Cameron: Like shapes? [Laughter] So you could like do like squares, circles, triangles…
Sarah: Yeah but this is just more easy. [Inaudible]
Teacher: One at a time please. Could we just have one at a time? Kate, what did you just say?
Kate: I was saying that if you replace the numbers with like shapes and letters and stuff it's just a complicating thing 'cause we all know the number system and it's simpler for us than all these other symbols.
Sarah: Yeah, but what I'm saying is, like, if, why we're using the number system we're really, um, pinpointing the graph 'cause then we're saying … one, one and y one, sorry, and we're just, you know, we're really pinpointing, I mean using numbers and if we like, if we can find a way to represent it with letters then we'd be able to make it whole infinity, infinity instead of just drawing on the graph. Does that make sense?
Class: [Muttering, faint "No"]
Teacher: That might be something for people to have a think about.

The domain of practice

The content in both classrooms is strongly classified. In classroom 1 the focus is on congruent triangles, in classroom 2 it is on graphs of straight lines. In classroom 1 the students use the terms "horizontal" and "vertical". The teacher intervenes at this point and suggests to the students that they are saying it is "at an angle, aren't you?" She is intervening to encourage the students to move their expression from the everyday (horizontal and vertical) into the mathematical (right angle). However, she does not explain to students, nor invite students to see for themselves, that the reason why it is better to use the term right angle is that a triangle may have any orientation. In classroom 2, the students discuss whether or not it is possible to use shapes rather than numbers on the x and y-axes. Although it is not readily apparent in this short extract, the students are suggesting that the relationships they have observed are more general than might be expressed via actual numerical values. In this way, they are recontextualising mathematics into everyday expression. The teacher responds, at the end of this part of the discussion, by suggesting that people "have a think about" an idea. In this way, she is explicitly inviting students into the discourse of mathematics.

Voice

In classroom 1 the students adopt, and are compelled by the teacher to adopt, a receptive voice. The initial question, "Miss, what do we do now?" indicates that students position themselves as dependents. Every subsequent student utterance is in direct response to a teacher question. The teacher recontextualises on behalf of the students, using phrases such as, "Then you're defining the angle between them, aren't you?"

In classroom 2 the students adopt, and are encouraged to adopt by the teacher, an authorial voice. They use phrases such as "I think", "What I was saying" and "I was wondering" frequently, suggesting that they have personal ownership of the ideas they are discussing. The teacher intervenes only to maintain order, and to suggest that they might think further about an idea. The students' utterances are long and detailed, and their struggle to personally recontextualise partially-formed mathematical ideas into mathematical language is apparent.

Message

In classroom 1, the focus of the activity is for students to discover and to generalise the minimum number of conditions for congruency, which would appear to be an expanding strategy. However, rather than discussing the concept of congruency, the students' aim appears to be procedural, to complete the task and for the entire group to successfully draw each other's triangles. Later in the lesson one student claims "I won", while the other students say "We all won", meaning that they had been successful in copying each other's triangles, rather than in identifying mathematical principles. As described above the teacher attempts to move the students' language from the particular (horizontal and vertical) to the general (right angle) but does not discuss with students why this language is more general.

In classroom 2, the students' explicit aim is to generalise. The discussion focuses on principles rather than procedures, with Carly stating that she found "the lower the value the closer it was to the x-axis". Sarah says that "if we can find a way to represent it with letters then we'd be able to make it whole infinity". By using an expanded range and abstract, if somewhat ill-expressed discourse, students attempt to enter the valued discourse of mathematics.

Conclusions

This brief discussion of two classrooms indicates that Social Activity Theory (Dowling, 1998) can provide a powerful framework with which to examine the enacted discourse of the mathematics classroom. By looking at two quite distinct classrooms, we were able to observe clear differences in the way in which students were positioned. Whereas both classrooms focused on abstract mathematical ideas and involved a high level of verbal interaction, in classroom 2 students were invited into the esoteric domain of school mathematics, used authorial voice, and engaged expanding strategies such as a focus on principles. In classroom 1, on the other hand, students were asked to use mathematical language without seeing why this was more powerful than everyday language, positioned themselves as dependents responding to teacher questioning, and saw the activity as a procedure rather than as a discovery of principles.

We suggest that a coherent theory of classroom interactions, such as Social Activity Theory, is a powerful way in which teachers can analyse and improve the discourse of their own classroom. In particular it can address the question of "Who's doing whose mathematics?" by giving students an authorial voice and positioning them as subjects or apprentices rather than dependents or objects. In this way the students are doing their mathematics rather than watching while the teacher does his or her mathematics.

The teacher of classroom 2 describes the way in which she endeavours to give students this voice in the following way:

> My aim in my mathematics classroom is for students to regard mathematics as an art which belongs to them, a means of regarding and interpreting the world, a tool for manipulating their understandings, and a language with which they can share their understandings. My students' aim is to have fun and to feel in control.
>
> At the start of each year group responsibilities are established by class discussion and generally include rules such as every member is responsible for the actions of the other members of their group (this includes all being rewarded when one makes a significant contribution to the class and all sharing the same sanction when one misbehaves), members are responsible for ensuring everyone in their group understands what is going on at all times and students have some say in the make up of their groups .
>
> My role is primarily that of observer, recorder, instigator of activities, occasional prompter and resource for students to access. Most importantly, I provide the stimulus for learning what students need, while most of the direct teaching is done by the students themselves, generally through open discussion. Less obvious to the casual observer is my role of ensuring that students have the opportunities to learn all that they need to achieve required outcomes. It is crucial that I, as their teacher, let go of control of the class and allow students to make mistakes and then correct them themselves. An essential criteria for defining one of my lessons as successful is that I do less than 10% of the talking in the whole lesson."

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Cooperating to learn mathematics: Some lessons from Year 7[*]

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Cooperative learning is often advocated as a way of improving mathematical learning. This study examined how a small class of Year 7 students, with learning difficulties in mathematics, responded to a program of intervention based on working in small groups. The aim was to improve attitudes and achievement across three domains: social, affective and cognitive. The cooperative learning strategy had a positive effect on students' attitudes in the affective domain, but made little difference to their performance in word problem tests. These findings are illustrated by three case studies of students who responded differently to the cooperative learning approach.

Problem solving methods and peer collaboration have been seen to have positive effects on academic motivation, academic self-concept and social-concept, especially with low achieving students (e.g., Campbell & Evans, 1997; Ginsburg-Block & Fantuzzo, 1998). However, there has been evidence to show that some approaches to collaborative problem solving may not necessarily be beneficial to learning (Stacey 1992). Recent documents published by the NSW Board of Studies and NSW Department of Education and Training (2003) have been particular in encouraging teachers to develop students' ability to communicate ideas, expecting that this should result in engagement and achievement in mathematics. Further, it has been suggested that some students do not understand, or have problems understanding, the basic number facts to enable them to decode word problems (Cumming & Elkins, 1994). This study considers the effect of a program of cooperative learning in mathematics on Year 7 students' attitudes towards mathematics and their subsequent achievement. Specifically, the program focused on developing the skills of decoding word problems.

The context

Fifteen years' experience as a secondary mathematics teacher, in the New South Wales public school system, had fostered a special interest in students' attitudes and their impact on student achievement and subsequent engagement in mathematics. The teacher had become increasingly aware of the number of students entering high school displaying a difficulty in interpreting word problems, especially in identifying the required operation to solve the problem. From experience, the teacher believed that, for students to progress successfully through secondary school, it is essential they develop the ability to identify appropriate methods to solve word problems because much of

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[*] Paper accepted by peer review

their formal, external mathematics assessment is presented in this format. The teacher also observed that students faced with word problems, and lacking the skills to decode them, in later years seem to lose the motivation to succeed. Such students perceive mathematics as difficult and irrelevant. They avoid the challenge to succeed student and are content to copy information from the board, or from a friend, believing this is "work." Further, they avoid taking the risks of attempting to formulate their own answers for fear of failing, yet again, resulting in an attitude of "not caring." At this stage, students often become disruptive and a behaviour problem in the classroom. These students, like all others, have the right to enjoy and succeed in mathematics. Rather than allow them to progress through the system without intervention, unsuccessful students should have the opportunity to develop skills in solving word problems in Year 7, in order to help develop a positive attitude to mathematics as they progress through their high school years.

The study was carried out in a government co-educational, comprehensive high school located in a town in the Blue Mountains, 80 km from Sydney. The class chosen for the study consisted of ten students, formed from funding provided by a newly developed departmental initiative for students with mild disabilities who required special attention to integrate into secondary school from primary school. In the classroom, the teacher had alternated between the pair seating arrangement, the blocks of four seating arrangements, and the horse-shoe arrangement. The teacher had considered the paired arrangement the most appropriate for maximum classroom management. The familiar classroom strategies incorporated teacher-focused student questioning, with theory presented on the overhead projector and exercises presented on worksheets.

The program

The program took place over a seven week period. Students undertook a word-problem test and an attitude survey at the beginning and end of the teaching intervention. A further test was conducted after the Christmas holiday break to determine the extent of retention. Table 1 provides an overview of the activities which occurred during weeks one to seven and eight weeks after the initial collecting period.

Table 1 Overview of the data collecting period.

Week	1	2	3	4	5	6	7	15
			Cooperative Learning Program 1 (CL1)	Cooperative Learning Program 2 (CL2)	Cooperative Learning Program 3 (CL3)	Cooperative Learning Program 4 (CL4)		
Activity	Questionaire Test Introduce audio-tape	Cooling off period	Word Matching	Squaresaw	One-step Word Problems	Two-step Word Problems	Questionnaire Test	Test

The program consisted of five stages. The first stage involved a questionnaire, to determine students' attitudes towards mathematics and a word-problems test, to evaluate mathematical achievement, both administered during one classroom period. The second stage was a one week familiarisation period. During this time, students continued with the normal classroom routine following the program devised from the syllabus. The third stage consisted of a four-week period where one teaching period

each week focussed on a cooperative learning lesson to develop word problem skills. This followed usual group work procedures with specific word problem teaching strategies. The students were divided into two groups with the teacher working with one group and the teacher's aide working with the other, alternating each lesson. The groups' discussions were audio-taped and observations noted by both the teacher and the teacher's aide. The fourth stage consisted of re-administering the original questionnaire and test immediately after the teaching intervention. The final stage involved the administration of the original test, approximately eight weeks after the end of the teaching sequences, after the Christmas break.

Cooperative Learning Activities

Four different strategies were used with a focus on word problems.

Cooperative Learning Program 1 (CLP 1) — Word matching

The teacher initially introduced the students to targeted words, synonyms for the arithmetic operations, by placing the symbols +, –, ×, ÷ at the top of four columns and showing these on the overhead projector. Each word was then displayed for the students to read, and then indicate to which column the word belonged. The teacher practiced these targeted words until the students became familiar with the process.

The students were divided into two groups. Each group was given a set of 76 word cards, which were divided evenly between each member of the group. Each student in turn placed a card on the appropriate symbol. When they completed this task, the teacher then allowed the group to play a game of dominoes placing similar words together.

Cooperative Learning Program 2 (CLP 2) — Square-saw

A square-saw is a word matching activity where a grid has questions and matching solutions are placed on adjacent sides of the squares (tiles). The grid is cut out and shuffled. The students are required to reassemble the grid, so that the questions match the answers. The students were shown how to complete a square-saw and then divided into groups to work together to complete the square-saws.

Cooperative Learning Program 3 (CLP 3) — Word problems

In this lesson, the teacher explained to the students that each group would be given an envelope containing a number of word problems and a table for recording the answers to the word problems. It was stressed that the aim of the exercise was to decide on the operation used in each question rather than answering every question quickly. This operation was first recorded for all problems, and then students used their calculators to work out their answers.

Cooperative Learning Program 4 (CLP 4) — Two-step word problems

This lesson was similar to CLP3, except that the problems consisted of two steps. Both operations had to be identified before the students attempted to work out the answers.

The outcomes

The questionnaire focused on the cognitive, affective and social domain. From the beginning of the program to the end of the cooperative learning lessons, analysis of the responses showed a significant positive change in the affective domain, while the cognitive and the social domains showed no significant change.

The word problem test concentrated on the operations of addition, subtraction and multiplication. Analysis showed the students had a good knowledge of the one operation involving either addition or subtraction, and this showed little change. Both the operation of multiplication and the two mixed operations of addition, subtraction and multiplication showed a short term improvement after the cooperative learning program, although this was not statistically significant.

When audiotapes of the cooperative learning lessons were analysed the students responded well to activities which they perceived as the more traditional mathematics lesson. The analysis also demonstrated interesting aspects of individual students which led the researcher to consider three students as case studies. The first case study focussed on B, who did not wish to work in a group work situation. The second case study profiled Je, who benefited from the program. The final case study considered N, who did not perform well under the cooperative learning conditions.

Individual students

Of the students, B was the most reluctant to work in a cooperative learning environment. From the very start he displayed no interest in being involved, to the extent that he endeavoured to undermine the lessons by hiding tiles and making inappropriate comments. Most of his comments were negative, ranging from, "Shut up," to his peers to, 'This is boring,' when commenting about the activities. He required a lot of coaxing from the facilitator to answer questions. He generally repeated words or made silly comments, unrelated to the question. This attitude resulted in disrupting the lesson, by unsettling the other students and creating a negative quality in the classroom. He particularly unsettled E who was very keen on doing her best in the activities, to the extent that E had to leave the group and work on her own. Interestingly, E could have quite happily worked in a group situation or on her own, as she seemed to "switch off" to the others in the group and work by herself or with the facilitator. B's involvement socially did not change throughout the course of the cooperative learning activities. His behaviour demonstrated that this type of learning strategy was not appropriate at this stage for B, or that a different kind of group structure might have been more appropriate.

J provided the total antithesis to B in the social domain. Having presented herself in the traditional classroom environment as a quiet worker who very rarely made any contribution to classroom discussion, this student excelled. She was keen to organise her group, displaying leadership skills. She worked well with the other students and helped them where she could and also gave them encouragement. At the end of the cooperative learning program her answers to the questionnaires which were focused on the social domain had become more positive. This demonstrates that for some students a cooperative learning program can be positive.

In comparing the responses to the questionnaires, N began the cooperative learning program with a more positive attitude to her social interaction than at the end. Even though N still likes to "be" in the mathematics classroom she definitely does not want students to think she is smart or to help her peers. This is interesting since N always appeared to be an outward going student who had plenty to say. The transcripts showed that in the last two cooperative learning lessons, N became quite determined in her responses, displaying irritation with her peers when she could not answer questions correctly. With N, the cooperative learning program allowed her to see her learning difficulties against the other members of her group.

Although for some students the cooperative learning program allowed them to develop, for example Je, Jo and K, this was not so for all the students. Both B and N did not gain from the program. E enjoyed working on her own and the cooperative learning program did not impact on her. Tr and M gained to some extent, however it is difficult to say whether it was the result of the program. It is important here to realise, this study involved only a small cohort and as a result cannot be applied universally.

Implications for teaching

This study targeted a group of low ability mathematics students. Therefore, it would be reasonable to suggest that the implications for teaching maybe more specific for classes with students who require development of basic skills and students who have difficulty concentrating on set tasks for periods of time, however aspects can be extended to other levels of ability.

Adequate time should be allocated to meet the needs of the students. Students should be assessed according to their level of the basic skills, or skills in general. An assessment of the objective anticipated should then be made and the optimum time needed to obtain these objectives. As this study suggests, a single lesson in a cycle may be insufficient for students to show changes in achievement.

Second, having made this assessment, flexibility is required to make available alternative pathways for students who are not meeting the requirements of the program. These alternatives may include varying the tasks or forming a small group of students with similar needs, to allow them to work on a specific task before moving on to more complex work.

Third, during the program, students should be continually assessed as to the compatibility of the group to which they have been assigned and alternatives provided for students to either join another group or spend "time out" working on their own. Some students became quite antagonistic towards other members of the group, if the work was not going well, providing classroom issues. Such aspects of cooperative learning with low ability students do not appear to have received much attention.

An assessment should be made as to the type of activities the students prefer. Different groups within the class may respond more positively to a particular form of activity, so each group should have the advantage of an activity which best suits their requirements at that specific time.

Particularly with low ability students, an experienced facilitator is required to curtail inappropriate comments, which may have a negative impact on other students. An experienced facilitator can also be advantageous in assisting with the extension of students' concepts and this can be extended into all levels of cognitive ability. It should not be considered that an experienced facilitator is only required for low ability students.

An experienced facilitator is required to maintain students on task. This is especially necessary for low ability students, where students need to be encouraged to respond positively when a question is presented to encourage the students' attention.

Finally, an experienced facilitator is required to assist students to self correct. Often students do not recognise when their response is incorrect. If the group is inexperienced, this inappropriate response to a question may result in the group being misinformed. It is important that the facilitator is knowledgeable about the concepts addressed in the lessons, and students' learning of those concepts.

Finally, in considering a cooperative learning program, there are a number of different characteristics of students which need to be considered and provision made to

meet the needs of these students. In particular in this study there were students who were:

- keen to take leadership and respond to the facilitator;
- keen, but who did not have the maturity to understand their strengths and weaknesses;
- keen, but who found the work too demanding and as a result developed negativity;
- capable of working quietly and competently within the group;
- able to work competently but who required direction;
- explicit in their wish to work on their own;
- not interested in group work and either disrupted others or refused to participate.

This provides some examples of the characteristics which could present in a cooperative learning program, and should be considered.

Concluding thought

The research discussed in this study has described the ways a program of targeted intervention using cooperative learning strategies to aid the decoding of word problems could influence students' achievement on tests of mathematical word problems and attitudes to mathematics in the cognitive, affective and social domains.

This study has drawn attention to the vast and distinctive conditions which must be considered when designing a cooperative learning program in the classroom. More investigation is needed in the achievements and attitudes of students who are mathematically challenged and who have difficulty with the type of collaborative problem solving approaches promoted by educational institutions. If this is done, collaborative problem solving could be of significant value to a much broader school population and not be seen as merely lip service to a departmental initiative.

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Concept exploration in mathematics using Spinnamaths on an interactive whiteboard

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I happen to believe that with enough experience, all mathematics teachers eventually acquire a set of pictures associated with mathematical concepts, and that these pictures are essentially the same from teacher to teacher right across the world.

The late James Taylor, out of Mariah College in New South Wales first pointed me in the direction of Excel as an effecting teaching tool. He stated, in one of his workshops at MANSWA (Leura, NSW), that Excel was "the most under-utilised program sitting on a teacher's desktop!" That night I began playing with Excel and found an incrementing device called a "spinner". The rest is history!

The spinner (and its close cousin the scroll bar) is a cell incrementing device that can be dragged onto the worksheet from the tool bar. You can specify the cell that the spinner is to refer to. When you click on the right side of the spinner for the first time a number appears. Click again and the number increments by 1. Keep clicking and the number keeps incrementing. Click on the left side repeatedly and the number in the cell will decrease.

When I first played with the spinner I was only marginally curious. Later on that night, I began thinking about the possibility of using spinners in series. Suppose, for example, I developed a series for the relation $y = mx + c$. Suppose, further, that I attached spinners to the coefficients m and c. When the series is graphed for a particular m and c, a line on the x-y plane can be drawn. However, if I begin clicking the spinners the value of m and c change and line graph responds accordingly.

When I saw my first "moving" graph I was hooked! It was not long before tangents were rolling along curves (see Figure 1) and sine and cosine curves of different frequencies were combining to beats (see Figure 2).
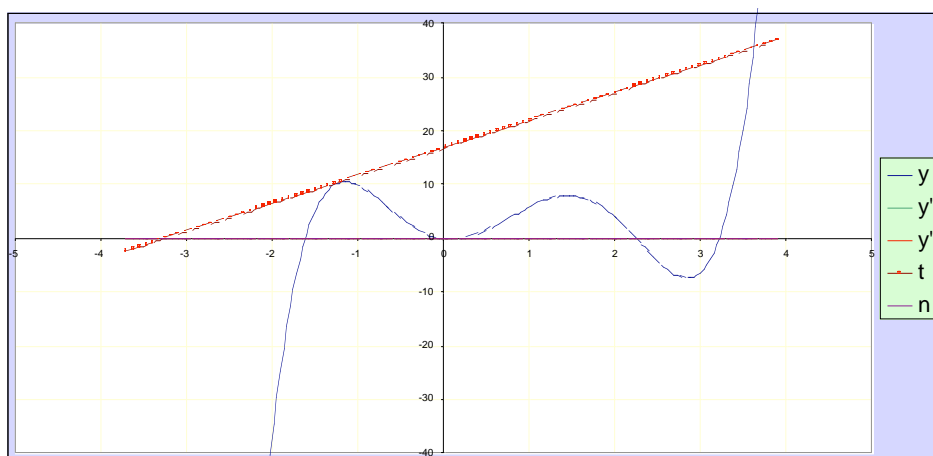

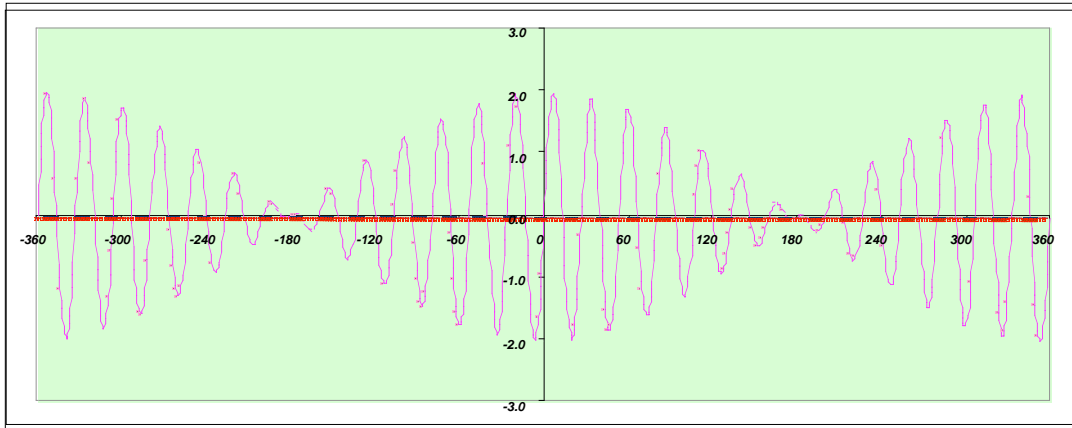*Figure 1. The tangent rolling around a curve.*

Figure 2. Explaining superposition.

There are countless investigations in mathematics where exploiting variability reveals a concept. Spinnamaths is a collection of almost 100 of these. It touches on many areas of the curriculum, including some forays into finance. Take mortgages for example. Diagram 3 shows the sheet. By varying interest rates and terms, students come to understand the dynamics of mortgages.
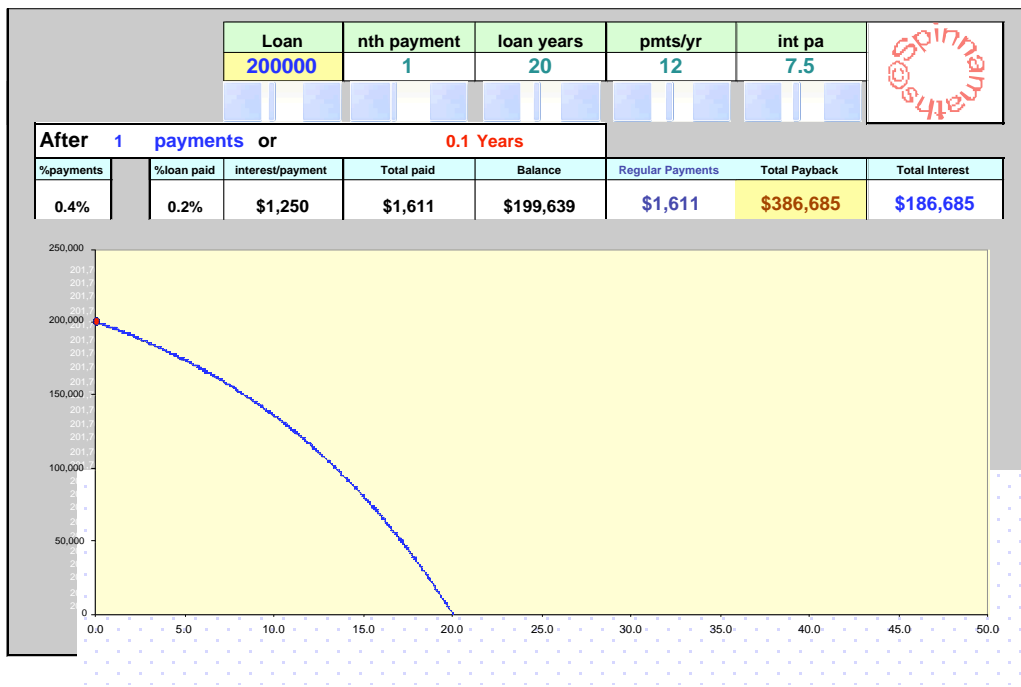

Figure 3. Examining mortgages.

What about a parabola? If I attached spinners to the coefficients could I show students the relationships between the coefficients and the graph?

What about other graphs?

What about the normalising effect on a skewed binomial probability distribution with large samples?

What about finding the roots of equations using Newton-Raphson's method?

What about translation effects on curves?

What about the conic sections and their eccentricities?

What about projectile motion with varying initial velocities and angles of projection?

What about statistical sampling? What about complex numbers and the roots of unity?

What about interest rates and mortgages?

It is in the spirit of enquiry that Spinnamaths was designed. Around ninety spreadsheets were put together into a CD and released through AAMT (see www.canberramaths.org.au). The authors (Margie Smith and I) wanted to produce the work cheaply for schools as an alternative to much more expensive packages. It is not designed to replace a graphic calculator. It is, however, designed to explain a number of mathematical concepts in a powerful way.

On interactive whiteboards (IWBs) they become a powerful tool to develop concepts and understandings.

It is my belief that all mathematics teachers, who have been around a classroom for a while, have developed a series of concept pictures. These pictures are fairly similar from teacher to teacher. They are instantly recognisable. I also know that mathematics teachers are busy people. They sometimes want to illustrate a concept quickly and efficiently, before engaging on an algebraic treatment. They do not necessarily have the time and patience to learn the nomenclature of high tech software packages. They want a visual concept explanation, without wasting a full lesson in a computer lab.

This seminar will examine some new directions in the set of sheets as well as revisit some of the more established sheets. The Interactive boards are useful in combining elements of the spinner programs to the board's software.

Challenges in formulating an extended modelling task at Year 9[*]

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The formulation phase of an extended modelling task can present many challenges for beginning modellers especially those at Year 9. However, teacher support to meet these challenges allows students to engage meaningfully with such tasks and to experience first hand how mathematical modelling is an essential form of reasoning about real situations. Data from the RITEMATHS project are used to examine teacher and student moves in successfully negotiating the formulation phase during a modelling task. In the classes where the majority of students had less difficulty with formulation, the teacher used a combination of physical demonstrations and dynamic computer simulations.

Teachers of mathematical modelling at university level possess a worldview recognising "the power of mathematics to describe, explain, predict and control real phenomena [and appreciate] that mathematics is indispensable as a way of knowing about the world in which they live… [For them,] 'everything turns into mathematics.' On the other hand, for many students at the tertiary level 'it is nearly impossible to adopt this new way of looking at the world so late in one's education'" (Lamon, Parker & Houston, 2003, p. ix). It is thus important that students develop awareness of this fundamentally different way of perceiving the world, whilst in primary and secondary school. To Mason (2001, p. 43), "exposing students to mathematical modelling is not just a form of teaching applications of mathematics, nor of illuminating the mathematics being applied. It is equipping students with the power to exercise a fundamental duty" as a citizen to appreciate, critique and use the models and modelling that permeate our modern world. Only then can students fully appreciate that mathematics is "a way of life": essential for learning, essential for life.

Data for this paper have been generated within RITEMATHS, an Australian Research Council funded Linkage project (http://extranet.edfac.unimelb.edu.au/DSME /RITEMATHS/). The research was undertaken as design based research (Wang & Hannafin, 2005). This research is focussed at the level of teacher action and student learning, in particular, teacher facilitation of students' developing their modelling competencies as they engage with an extended task in classrooms where electronic technologies are available to teachers and students. This paper is limited to the formulation phase of modelling and student and teacher moves during this.

The formulation stage of mathematical modelling separates a modelling approach from an applications approach, where real world situations are introduced merely to illustrate the applicability of mathematics (Stillman, 2004). In a modelling approach the mathematisation of the real situation, by the students using their knowledge of

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[*] Paper accepted by peer review

9   Jill was a doctoral student of the University of Melbourne when these data were collected.

mathematical models and techniques, is of interest. Model formulation is the first phase of modelling covering the process from the simplification, structuring and idealising of the real situation (Maa, 2006), through to translation into a mathematical model. The importance of this crucial phase of modelling of real world situations has been known for some time (e.g., Voskoglou, 1995) but it is still often neglected in teaching despite (or, perhaps, because of) it being shown to be the very difficult for students (Crouch & Haines, 2004).

The study

The part of the study reported here involved implementation of one task in three Year 9 classes (14–15 year olds), each from a different school B, C, or D. The teachers in these classes have been developing a lower secondary mathematics curriculum, providing opportunities for engagement in extended modelling. Intensive data were generated in the form of student scripts, video and audio recordings of the teacher, whole class and small group collaborative activity, and post-task interviews. In order to identify and document characteristic actions and the respective interactions between modelling, mathematical content, and technology, these data were entered into a NUD.IST database (QSR, 1997) and analysed through intensive data scrutiny, to develop and refine categories related to these themes. Illustrations are drawn from the analysis of the implementation of the task, *Shot on Goal* (Figure 1). In all three implementations our focus is on the formulation phase. However, this phase is revisited by some students several times after they have done some calculations using formulations, which they then decide need revision.

In *Shot on Goal*, the cognitive demand required for task formulation by Year 9 students is high, potentially leading to a blockage in this early phase of the solution if students find the level of challenge too high for them to engage with the task. To help overcome this, teachers from different schools used a variety of methods to ensure students did not have difficulty interpreting the situation. At all schools, students were asked to write, in their own words, what they thought the aim of the task was and then specify this in terms of a problem that could be solved mathematically. Peter, the teacher from school B, anticipated that specifying the aim of the task and the angle involved would be a challenge for many students. He provided a supporting physical demonstration in the classroom, involving throwing a tennis ball, from various angles, through two goal posts. Class members either watched or participated to help clarify their thinking. For example, here are the words of student Sandra:

> Sandra: It helped to explain like what we hoped to find out. Like I didn't really get what we were trying to do. And that kind of explained what angles we were trying to find.

Shot on Goal[10]*:* You have become a strategy advisor to the new soccer recruits. Their field of dreams will be the SOCCER FIELD. Your task is to educate them about the positions on the field that maximise their chance of scoring. This means: when they are taking the ball down the field, running parallel to the SIDE LINE, where is the position that allows them to have the maximum amount of the goal exposed for their shot on the goal?

Initially you will assume the player is running on the wing (that is, close to the side line) and is not running in the GOAL-to-GOAL corridor (that is, running from one goal mouth to the other). Find the position for the maximum goal opening if the run line is a given distance from the near post. What advice would you give as strategy advisor to an attacking player running down the wing towards the goal line with the ball?
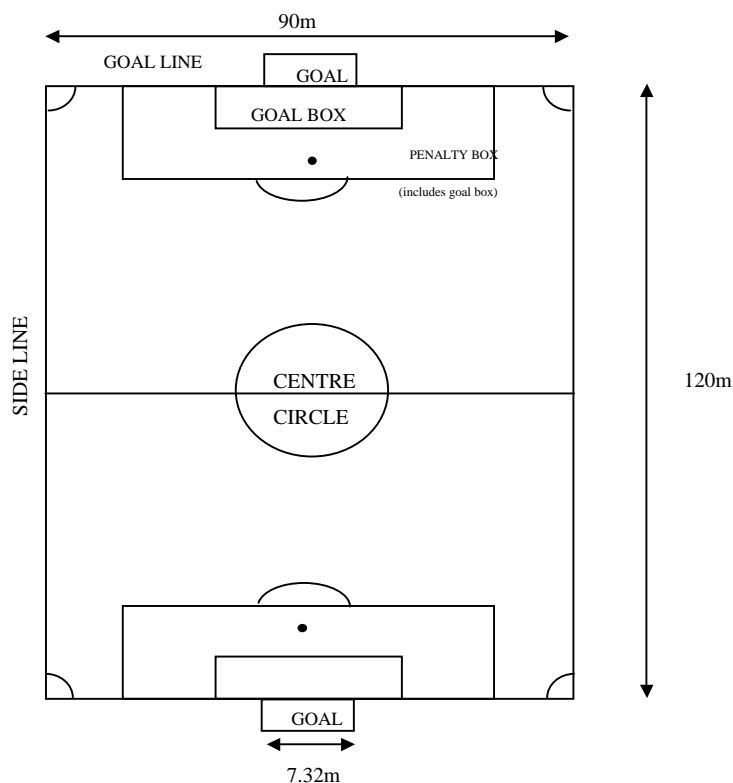
(Additional suggestions were provided as to how the work was to be set out, and for intermediate calculations, especially in the area of graphing calculator use, providing extra task scaffolding.)

Figure 1. Shot on goal task.

Students were then asked to record what they believed was the angle being sought. As has been found in all schools, several students took the point of view it was various angles made by the ball at the goal mouth as it entered the goal (e.g., as in Gary's and Leo's diagrams in Figure 2), whilst others thought it was the angle subtended by the goal mouth from the spot where the ball was kicked (see Nita's angle).

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10 This task was refined by the researchers and class teachers from a task originally designed by Ian Edwards (Luther College).
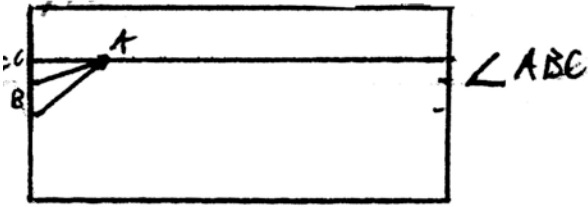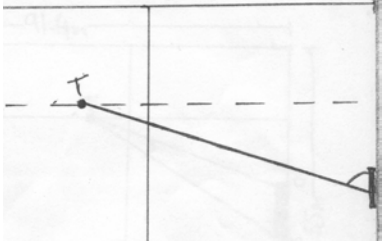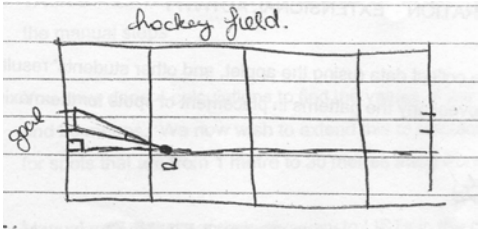
 Gary's angle	To work out on a hockey field where the best place while running down the field to get a goal with the highest efficiency. This will be the place with the largest angle.
 Leo's angle at centre of the goal	I think we are trying to work out how likely (we are) it is to get a goal at different angles.
 Nita's angle	We are trying to find the best (widest) angle to shoot a goal from.

Figure 2. Students' diagrams of angle sought and aims for the task.

This was followed by a debate about which angle was the shot on goal angle, by selected students using diagrams on the whiteboard. These angles were critiqued by the class and then all students were asked to draw their own diagram, showing the shot on goal angle. Following group discussion of suggested angles, further speakers presented the views of their table group. The likely "goal angle candidates" varied until, finally, Mei marked the correct angle as shown (Figure 3).
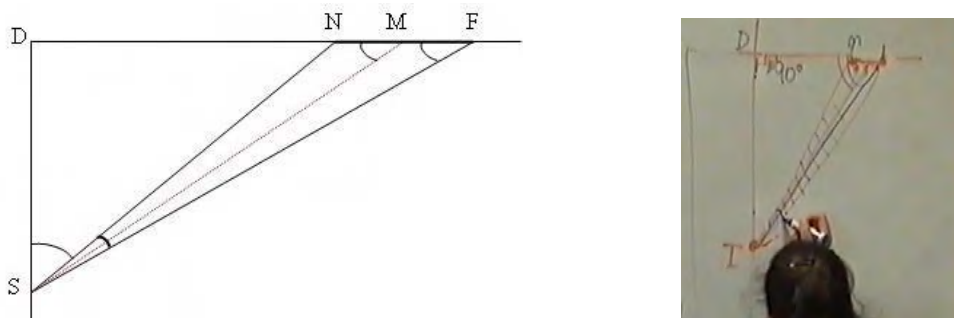

Figure 3. Shot angles suggested during debate.

At School D, the teacher, Wayne, provided a demonstration on the nearby soccer field, to help students gain direct real-world knowledge of the situation they were modelling, in a similar manner to students in Galbraith and Clatsworthy's (1990) study. There senior secondary students did this, at their own instigation, in order to extend

their knowledge of modelling artefacts, such as speed bumps. For students to be able to use knowledge of real situations, previous studies (e.g., Stillman, 2000) suggest that direct, rather than vicarious, experiences of real world situations are more effective in allowing this to occur. Some students actively participated in the soccer demonstration, whilst others watched. Wayne used a rope to indicate a particular run line to the students and asked them, one at a time, to run down the run line and stop when that thought they had the best chance of scoring a goal (see Figure 4). Students were then asked to explain why they stopped where they did. For example, student Mark stated:

Mark:	To start with, it was closer than everyone else was but I still had a fair chance of scoring, a fair angle I could get the ball in.
Wayne:	What do you mean by a fair angle?
Mark:	If I had gone further the goal wouldn't have been as big as that further down the field.

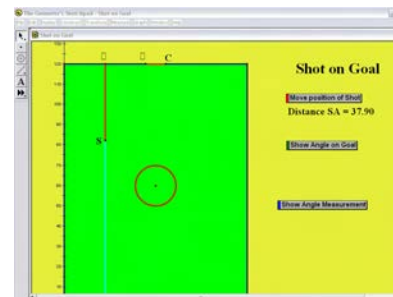A second group of students then repeated the demonstration for a run line closer to the goal.

Figure 4. Teacher moves supporting formulation of the situation.

During the post-task interview, students agreed that the soccer demonstration was an effective teacher move to help their interpreting the task situation and formulating a mathematical model. Raza's response was typical:

Raza:	It was, sort of, I guess, a way of understanding the task better… because some people like to have it, you know, instead of on paper in real life, to help them see what they are talking about.
Inteviewer:	Did you find this increased your understanding of the task?
Raza:	Yeah, I guess it probably did. Just that it translated [it] from paper to real life better.

In the classroom, as further support for mathematising the situation, Wayne showed the class an animated two-dimensional representation of the task, using a dynamic geometry application (Figure 4). This further clarified students' thinking about the task.

Raza:	Well it sort of showed us, the way that, this, this is the angle to find. And it showed us (it didn't specifically say this is the way), it just sort of helped us along to figure that ourselves.

Despite this, three groups of students, including Raza's group, considered the angle at the centre of the goal mouth as a possibility but this was quickly dismissed by all but one group, who persisted with this thinking. This last group initially found the angle at

the shot spot between the run line and the near post (∠ASB in Figure 5). However, the increasing size of this angle as the shot spot approached the goal line puzzled them, as they expected "the angle size would reduce because, as much of the goal wouldn't be facing me, it would be more of an acute angle." It also did not mesh with Mia's experience during the on-field soccer demonstration. After much discussion, they agreed that this was because they "were aiming not at the centre of the goal" and the required angle should be given by (∠ASG – ∠ASB) (Figure 5). This group persisted with this formulation for most of the modelling task, despite consulting the dynamic geometry model on the whiteboard many times, during their discussions over the three lessons. It was only after discussion with Wayne, in the third and final lesson, that they decided to double their angle calculations, to find the full angle subtended at the goal mouth. They realised that this formulation was not exactly correct, but it would allow them to progress.
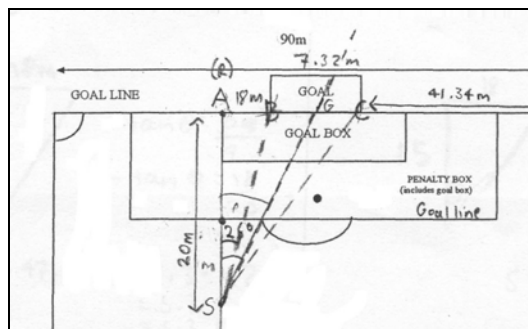

Figure 5. Mia's diagram.

Ned and Len also began by attempting to find the angle of the shot at the shot spot, as the angle from the run line to the line to the near goal post. They mixed the adjacent and opposite sides of the triangle they were using, calculating the angle at the near post. The decreasing size of the shot angle, as they calculated it from various positions on the runline closer to the goal line, did not make them question the angle they were focussing on or their calculations. It was not until Ned was mulling over the task, at the beginning of the second lesson, that it occurred to him "there is no way it could be a whole 50° angle… when the goal isn't that wide." The importance of allowing students the time to reflect, and self-regulate, their modelling process is crucial as students move from teacher orchestrated modelling to independent modelling (Leiβ, 2006).

The mathematising of the run line, as a straight line parallel to the side line, presented little difficulty for all but the most lateral thinkers. Instead of advancing down their specified run line in 1 m intervals to collect data showing the variation in the angle size with the distance the shot was taken from the goaline, Ben and Ken from School B took a stepped trajectory towards the goal.

Once the desired angle was known in *Shot on Goal*, students had to work out how they might find this angle geometrically, in order to apply formulae to carry out their calculations. The calculations were considered far less difficult, if not routine, by students who had studied the techniques involved relatively recently as in Schools B and C. However, establishing which geometrical relationships were the focus proved problematic, particularly for many students at School C. Even when they had ascertained that they were to focus on a shot angle, several students thought they should be finding a distance using Pythagoras' theorem, but could not see how this led to finding an angle. This arose as they could not decide which geometrical elements would specify the angle. Difficulties continued for many of these students as they tried to

apply trigonometric formulae, since flaws in their formulation meant they did not have right angled triangles in their representation of the required angle.

In the third school, choosing between Pythagoras and trigonometry was potentially more of a challenge, as it was some time since students had studied trigonometry. In some groups, the possibility of using Pythagoras' theorem was raised but this was readily dismissed in favour of trigonometry, as students quickly specified the elements of a geometrical representation of the situation.

Finding the required angle involved a decomposition of the angles from the shot spot to the near and far goal posts into component parts. The vast majority of students used the method of subtracting the angles from the shot spot between the near post and the run line and between the far post and the run line, that is, (∠BEP – ∠BEC) in Nita's diagram (Figure 6a). This method was promoted by all teachers. Sandra, however, described her alternative method as finding "the extra two". She partitioned the rectangle made by the run line, goal line, and line segments parallel to these (as in Figure 6b) into three triangles, two of which contained the extra bit of the angle which would not result in a successful shot. These the measures of the extra angles were then added and subtracted from 90° to find the shot angle.
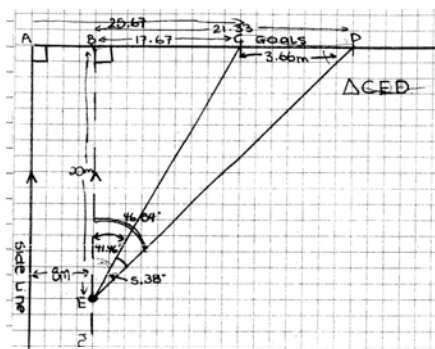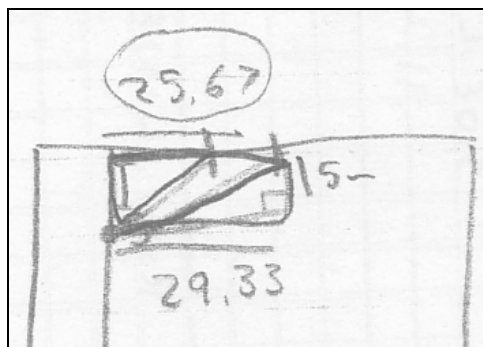
(a)   (b)

Figure 6. (a) Nita's diagram. (b) Sandra's diagram.

A third method was observed at both Schools B and D. At the latter, Christine and Molly found the angles created at each of the goal posts (∠A and ∠B in Figure 7). By subtracting ∠A from 180°, they found ∠D. Finally, they had a triangle containing two known angles and the shot on goal angle. Subtracting the two known angles, ∠B and ∠D, from 180° produced the measure of the required angle.
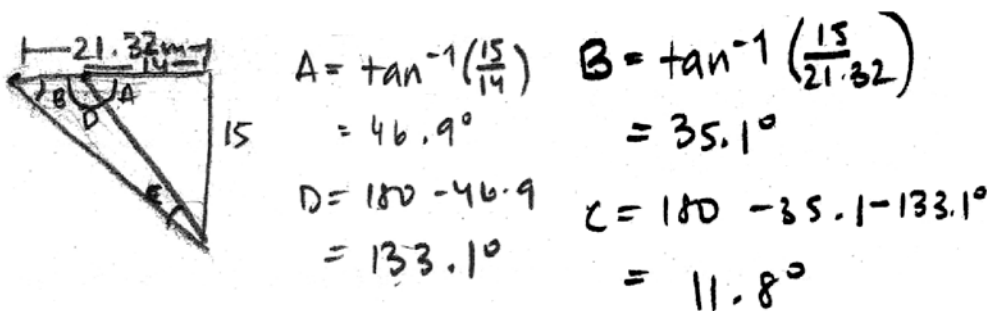

Figure 7. Alternative specification of the shot angle.

Conclusion

As Voskoglou (2006, p. 58) points out, "formulation of the problem involves… a deep abstracting process, which is not always an easy thing to do for a non expert" such as a lower secondary student. However, as has been shown, the difficulty is not an insurmountable one and teachers should, and can, support students effectively in meeting the challenges of the formulation stage of mathematical modelling. We are in agreement with Leiβ (2006) that it is necessary for teachers to ensure that beginning modellers "put themselves mentally into the situation" (p. 87) as this facilitates their development, and refinement, of a suitable mathematical model for the situation. Mathematical modelling is an essential form of reasoning about the world which all students should experience and be encouraged to develop.

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Exploring the relationships between student achievement in Working Mathematically and the scope and nature of the classroom practices[*]

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The focus of Monitoring Standards in Education 2004 random sample assessment program was on student achievement in Working Mathematically. For the purpose of the study, innovative assessment tasks were developed to embed Working Mathematically into the mathematics content. Samples of students were drawn at random from Years 3, 7 and 10 in government schools throughout Western Australia. A survey of teacher Working Mathematically practices in schools was also conducted. Possible associations between curriculum exposure, teacher's background, teaching strategies and teacher's beliefs about student's potential to work mathematically, and the levels of student achievement, were investigated. The aim of the paper is to outline the nature of the instruments used and the Working Mathematically activities associated with statistically significant effect on student achievement.

Introduction

The Monitoring Standards in Education (MSE) Program is an ongoing program established by the Department of Education and Training of Western Australia, to provide the community with information about the level of student achievement. The MSE 2004 program was the third project assessing student achievement in Working Mathematically and followed similar assessments in 2000 (Education Department of Western Australia, 2002) and in 2002 (Department of Education and Training of Western Australia, 2004).

Methodology

The MSE 2004 instruments

The main focus of the program was on assessing Years 3, 7 and 10 student achievements in Working Mathematically. As a part of the program, relationships between the nature and scope of Working Mathematically activities, and the level of student achievement, were explored. For the purpose of the program, innovative assessment tasks and a teacher questionnaire were designed and a teacher survey was conducted.

Assessment tasks

The assessment instrument reflected the Western Australian *Outcomes and Standards Framework* (1998). Opinions of teachers and educators were canvassed about which

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[*] Paper accepted by peer review

aspects of Working Mathematically should be assessed in a paper-and-pencil test, and what the structure and format of the assessment items should be.

Working Mathematically emphasises the use of a variety of justification techniques, activities and processes involving investigating, posing and solving mathematics questions. The maim aim of the program was to design test items in which mathematics content acts as means for students to acquire and develop skills, processes and techniques, that are vital to the understanding of mathematics.

By their nature, the Working Mathematically activities tend to demand that students generate evidence of their mathematical thinking, rather than recording or choosing correct answers. The test items covered three response formats: multiple-choice, short-response and open-response. In total, there were 138 items in all nine test forms. Most of the questions were grouped into multilevel *problem fields* — groups of multilevel questions with a common initial content. An example of common content used for the construction of a group of seven interrelated questions used in Year 3 forms is shown in Figure 1.
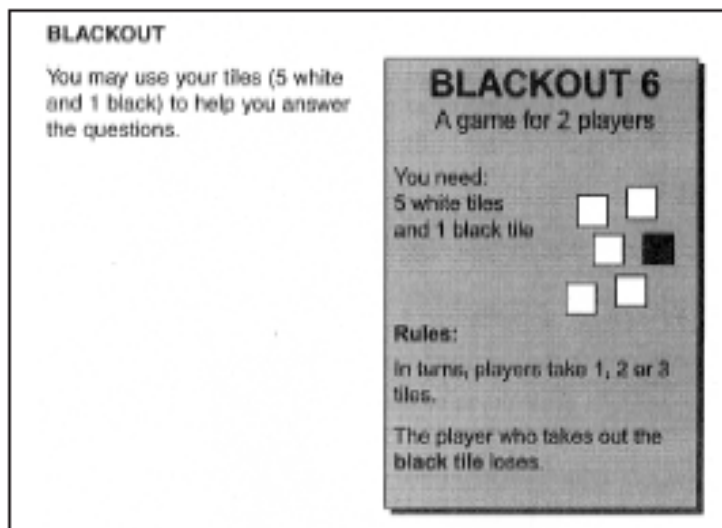

Figure 1. An example of a common content used for developing a problem field.

The problem-field questions, constructed on this common content, ranged from questions addressing students' understanding of the game rules through applying the rules of the game, to constructing similar games. Two examples of specific questions designed on Blackout are shown in Figure 2.
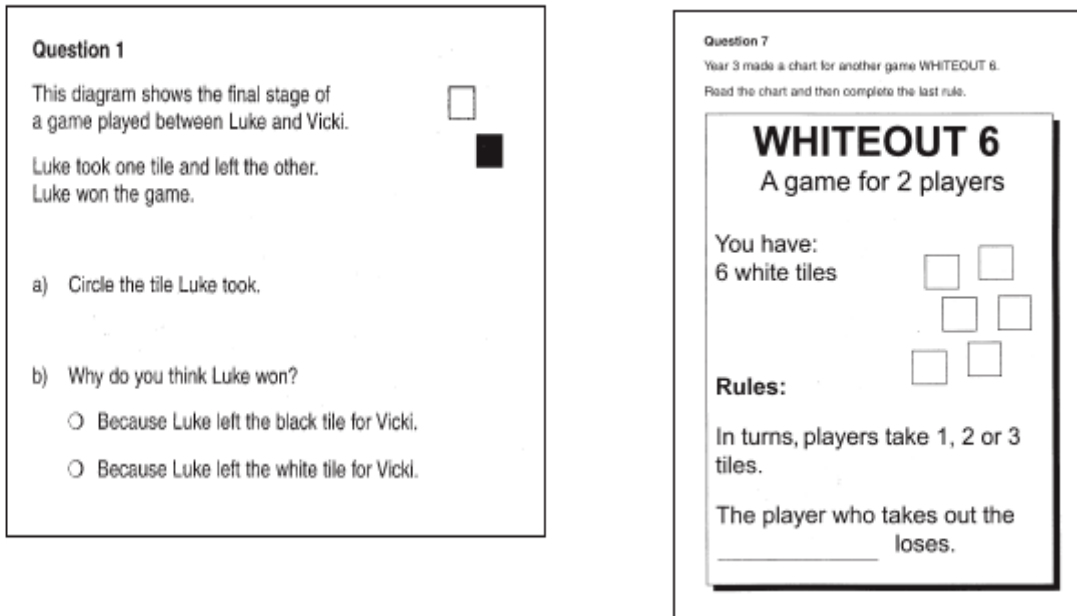
Figure 2. Examples of two questions that belong to the same problem field.

Teacher questionnaire

In order to explore the links between student achievement and the scope and nature of Working Mathematically activities used in the classrooms, a teacher questionnaire was designed and a teacher survey was conducted. Mathematics teachers, of all students selected to participate in the random sample program, were asked to complete a questionnaire: 198 primary and 376 secondary teachers did so (over 98%). The survey obtained information about teachers' background characteristics and beliefs, the scope and nature of their teaching and assessment practices, their knowledge and understanding of Working Mathematically, their professional development, and the use of Working Mathematically resources.

Sample

Samples of students were drawn at random from Years 3, 7 and 10 in government schools throughout Western Australia. Because the Years 3 and 7 test materials required a preliminary whole-class activity, the Years 3 and 7 sample selections occurred in two stages: firstly, a random selection of 75 schools; and secondly, selection of the class with the largest number of students enrolled in each of these 75 schools. The sample design for selecting Year 10 students followed the procedure used in previous MSE projects: that is, a simple random sample of students (not schools) was drawn across the State. A total of 1600 students was selected for each year sample. For all year groups, a higher proportion of Aboriginal and Torres Strait Islander students was sampled to provide better estimates of achievement. The final analysis of the data was completed, in such a way, that unbiased estimates for the total population could be produced.

Data collection and data analysis

Students' papers were marked by experienced primary and secondary teachers who were provided with extensive 2-day training. Data were analysed by using the Rasch measurement. This allowed a large sample of the population to be scaled against each

other and against the range of the assessment questions. Common questions linked the nine tests forms to the historical (WAMSE) scale. By using Rasch measurement the item difficulties and students' achievements on the WAMSE scale were estimated.

To determine whether the difference between the mean scores of the groups is a random variation or evidence of an underlying trend, a test of the statistical significance of the difference at the 0.05 level has been applied. In addition, a multilevel modeling of the data, taking into account its hierarchical structure, made it possible to answer questions on the statistical significance of differences in sub-group achievement through the effect of explanatory variables at the school level and the student level.

Results

The following teacher background characteristics and classroom activities were associated with statistically significant effects on student achievement:

Number of years of teaching experience

About 40% of primary teachers and 34% of the secondary teachers surveyed had more than 20 years of teaching practice. The number of years of teaching experience was positively associated with student achievement in Years 7 and 10. Year 3 students of teachers who had 11–15 years of teaching experience had the highest mean performance.

Mathematics as a field of study

Nearly 70% of primary teachers surveyed had studied core mathematics during their teacher training and 75% of secondary teachers had studied mathematics for at least two years during university education. Being taught by teachers with mathematics major at undergraduate level was positively associated with student achievement in Year 10.

Teacher beliefs

About 25% of primary and 17% of secondary teachers believed that at least 80% of their students were capable of achieving Working Mathematically outcomes appropriate for their Year levels. Teacher belief in student ability to Work Mathematically was positively associated with Years 7 and 10 student achievement.

It should be noted that 97% of primary and 88% of secondary teachers surveyed believed that students' ability to Work Mathematically could be improved through *explicit teaching* of problem solving. No relationship was established between teacher beliefs in *explicit teaching* of Working Mathematically and student achievement across the three year levels.

Using Working Mathematically

Nearly 98% of the teachers surveyed reported using Working Mathematically in their regular teaching. About 36% and 44% respectively of primary and secondary teachers reported that they integrated Working Mathematically into the content (*Space, Measurement, Number, Chance & Data* and *Algebra*) outcomes.

Posing mathematical questions

The proportions of teachers who reported providing opportunities for students to pose mathematical questions on a daily basis were 29% (primary) and 41% (secondary). Problem posing was positively associated with student achievement in Year 10 only.

Using problem solving

Nearly 90% of primary and 70% of secondary teachers reported providing opportunities for students to use a variety of problem-solving strategies on, at least, a weekly basis. The use of a variety of problem-solving strategies was positively associated with student achievement for Year 7 students only.

Using "What if?"-type questions

Twenty-three percent of the primary teachers reported using this activity on a daily basis, while another 49% used it once or twice a week. Sixty-four percent of the Year 10 teachers provided such opportunities on a daily or weekly basis. The use of "What if?"-type questions was positively associated with student achievement in Year 10.

Making conjectures

Nineteen percent of primary teachers reported that they prompted students to make conjectures on a daily basis, while another 46% used it once or twice a week. At the secondary level, the proportions were 15% and 43%, respectively. Prompting students to make conjectures was positively associated with Year 3 student achievement.

Using alternative ways of checking solutions

Twenty-six percent of primary teachers reported that they prompted students to use alternative ways to check solutions on a daily basis, and 53 percent did so on a weekly basis. Nearly half of the Year 10 teachers used this activity once or twice per week, while 25 percent used it each day. The use of alternative ways for checking solutions was positively associated with student achievement in Year 10.

Making sure that solutions make sense and are reasonable

About 83% of the primary teachers and 90% of the secondary teachers reported using this activity on a daily or weekly basis. Asking students to make sure that solutions are reasonable, and that they make sense, was positively associated with student achievement in Year 10.

Looking back

Nearly 54% of Year 10 teachers reported that they prompted their students to look back at questions, and check whether the assumptions they had made were reasonable. The use of the strategy "Looking back" was positively associated with Year 10 student achievement.

Partitioning a problem into sub-problems

More than 73% of Year 10 teachers reported that at least weekly they involved students in activities requiring partitioning problems into sub-problems. The use of partitioning a problem into sub-problems was positively associated with Year 10 student achievement.

Testing conjectures

Eleven percent of Year 10 teachers reported involving students in the testing of conjectures on a daily basis and about 36% reported using the activity once or twice a week. Testing-conjectures activities were positively associated with Year 10 student achievement.

Sharing strategies as a whole class

About 72% of the primary teachers and 37% of the secondary teachers reported allowing their students to share and discuss Working Mathematically strategies, as a whole class, on a daily or weekly basis. Sharing strategies as a whole class was positively associated with student achievement in Year 3.

Drawing attention to a list of problem-solving strategies

Sixty-three percent of the primary teachers and 32% of the secondary teachers reported that they drew students' attention to lists of known problem-solving strategies on a daily or weekly basis. Drawing students' attention to a list of problem-solving strategies was positively associated with Year 3 student achievement.

Telling students how to proceed when stuck

Nearly 80% of primary and secondary teachers reported that they told students, on a daily or a weekly basis, how to proceed when stuck on a problem. Telling students how to proceed when stuck was positively associated with Year 3 student achievement.

Asking students to explain and justify their reasoning

About 83% of primary and 70% of secondary teachers reported asking students, on a daily or weekly basis, to explain and justify solutions. Asking students to explain and justify their reasoning was positively associated with Year 10 student achievement.

Asking students to persist with problem solving

More than 80% of primary and about 65% of secondary teachers reported that they encouraged students, on a daily or weekly basis, to persist with problem solving. Asking students to refocus on problem solving was positively associated with student achievement in Year 10.

Using a substantial Working Mathematically task

About 40% of primary, and 22% of secondary teachers, reported that they began or ended lessons with a substantial (a task that takes a considerable amount of time) Working Mathematically task on a daily or weekly basis. The use of a substantial Working Mathematically task was positively associated with Year 7 student achievement.

Relating mathematics to the real world

Nearly 50% of primary teachers, and about 40% of secondary teachers, reported that they used activities to relate mathematics to the real world on a daily basis. Relating mathematics to the real world was positively associated with Year 3 student achievement, but it was negatively associated with Year 10 student achievement.

Assessment practices

About 70% of primary, and 54% of secondary teachers, reported feeling confident or very confident in identifying the Working Mathematically levels at which their students worked. The frequency categories with which teachers assessed Working Mathematically were: each day; once or twice a week; once or twice a month; once or twice a term; less than once a month; and never. Six specific categories of assessment were linked to the levels of student achievement: ongoing observations, checklists or notes; separate tests or tasks on Working Mathematically only; content tests and tasks with some Working Mathematically aspects; content tests and tasks with no Working Mathematically aspects; student self-assessment; and peer assessment.

Using ongoing observations, checklists or notes

Nearly 55% of primary teachers, and 40% of secondary teachers, reported using observations, checklists or notes on a daily or weekly basis to assess the levels at which their students were located. The use of ongoing observations and checklists for assessing Working Mathematically was negatively associated with Year 10 student achievement.

Using separate tests or tasks on aspects of Working Mathematically only

About 12% of primary, and 5% of secondary teachers, used separate tests or tasks on aspects of Working Mathematically only, on a daily or weekly basis. Fourteen percent of primary and 26% of secondary teachers, stated that they had never used such tasks. Using separate tests or tasks on aspects of Working Mathematically only, was associated positively with Year 3 student achievement.

Teachers' knowledge and understanding of Working Mathematically

Teachers were asked how they would describe their knowledge and understanding, at the levels appropriate for students they teach, of each of the three aspects: *Mathematical Strategies, Reason Mathematically,* and *Apply and Verify.*

Nearly 65% of primary and 38% of secondary teachers reported "good" or "very good" levels of knowledge and understanding of the nature of Working Mathematically. Teacher knowledge and understandings in the three aspects of Working Mathematically were positively associated with Years 3 and 7 student achievement.

Teacher professional development

Teacher professional development was addressed through several questions that drew on the frequency of attending specific professional development forms such as: meeting informally with colleagues, participating in action learning projects, attending a relevant professional development seminar or workshop, observing other teachers, sharing resources and collaborative planning.

More than 61% of primary, and 53% of secondary teachers, reported that they attend relevant professional development at least once every three years. Attending relevant professional development about Working Mathematically was positively associated with Years 3 and 7 student achievement.

Working Mathematically resources

Teachers were asked to name up to three of their favourite Working Mathematically resources. The resources were then classified into four categories: specific publications (content); policy documents; teachers' own prepared materials; and Internet resources. Specific publications were used most frequently, followed by teachers' own prepared materials, policy documents and Internet resources.

Teachers' preferences for six particular types of Working Mathematically resources were linked to the level of students' performance. The six categories were: sets of tasks ready for use by students; ideas for tasks but not in a form ready for use by students; guidelines for teachers on how to develop their own tasks; a range of typical student responses to tasks at each level; assessment criteria and keys; and online/Internet resources.

Nearly half of the teachers reported that they were aware of the existence of sets of tasks ready for use by students, and that they were helpful. Use of Working Mathematically tasks ready for use by students was positively associated with Year 10 student achievement. Use of guidelines on developing own assessment tasks in Working Mathematically was negatively associated with Year 10 student achievement.

Implications for teaching and learning

The study suggests that Working Mathematically is associated with a significant positive effect on student achievement at both primary and secondary levels. And, more importantly, the progression of students' achievement in Working Mathematically across Years 3, 7 and 10 implies that, as the content skills, student's Working Mathematically skills and processes develop during the years of schooling, it is vital that appropriate Working Mathematically activities to be used in the classrooms.

It should be noted that, while student achievement at primary level benefits from teacher professional development in Working Mathematically, the study suggests that at secondary level students benefit from being taught by teachers who have a background in mathematics.

The study indicates that some types of Working Mathematically activities are more appropriate, than others, for primary students. For example, Year 3 students benefit from explicit teaching of specific types of problem-solving techniques. Their learning benefits also from teachers prompting students to make conjectures and from relating mathematics concepts to the real world. Telling students how to proceed with a challenging task, and asking them to explain the solution strategy used, were also associated with a significant positive effect on student achievement.

The study suggests that Year 7 students benefit from being taught by teachers who believe in student's ability to Work Mathematically. Their learning benefits, also, if teachers use a variety of problem-solving strategies and substantial Working Mathematically tasks.

At secondary level, classroom activities such as partitioning a problem into sub-problems, posing questions, using alternative ways for checking solutions, justifying and making sense of the solutions are associated with a significant positive effect on Year 10 student achievement. The study suggests that using checklists, ongoing observations, substantial Working Mathematically tasks, or questions that relate mathematics to the real world, need to be used cautiously in the mathematics classrooms. The negative associations identified between these activities and student achievements need further research.

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ALL is happening with numeracy: A new international survey of adults' numeracy and mathematics skill levels

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An international survey of the numeracy abilities of adults was part of the Adult Literacy and Lifeskills (ALLS) survey conducted in Australia in 2006 and the start of 2007. This presentation will include information on the construct behind the numeracy component of the survey, including the complexity scheme that attempts to predict the difficulty of test items, and report on the initial data from the first wave of countries, highlighting the important role that numeracy plays in both human and social capital terms. First Australian results are due out in late 2007.

Introduction

The Adult Literacy and Life Skills survey (ALLS) took place across Australia between July 2006 and January 2007. This comparative survey has been jointly developed by Statistics Canada and the United States' National Center for Education Statistics (NCES), in cooperation with the Organisation for Economic Cooperation and Development (OECD). Its implementation in Australia is being co-funded by the Australian Bureau of Statistics (ABS) and the Department of Education, Science and Training (DEST), with support from the Department of Employment and Workplace Relations (DEWR). First results of the Australian ALLS are expected to be available in October 2007. The first wave of international data collection for ALLS has already been undertaken, and a number of national and international reports resulting from this first wave are already available. The first wave countries were: Canada, Italy, Norway, the USA, Nuevo Leon (Mexico), Switzerland and Bermuda. Countries participating in the second wave of ALLS, alongside Australia, are: Hungary, the Netherlands, New Zealand, and South Korea.

The Adult Literacy and Lifeskills (ALLS) Survey is a large-scale, comparative survey that goes beyond previous international adult literacy studies. In addition to the literacy skills measured in the previous International Adult Literacy Survey (IALS), ALLS is designed to identify and measure a broader range of skills in the adult population in each participating country. The skills directly measured are:

- prose literacy
- document literacy
- numeracy
- problem solving.

The first international comparative report from the survey was published in May 2005 (Desjardins, Murray, Clermont and Werquin, 2005).

A first for adult numeracy

In an important step forward for the field of adult numeracy, numeracy was given the opportunity to be part of the ALLS. To make it through to be part of the final survey was no simple task. It is required of all domains of the survey to first develop a conceptual framework or construct for their domain and prove that this construct is sound and can produce a range of assessment tasks that fit the construct, and that these tasks will stand thorough statistical analysis in a number of trial surveys. Only then is the 'new' domain able to take its place in the final survey. The ALLS Numeracy team succeeded in this task and the numeracy domain is now part of the final ALLS.

Why was numeracy considered for inclusion? In recent years, many countries have increasingly attended to the need to improve workplace efficiency, promote proper utilisation of public and health care services, and ensure informed civic participation. While literacy skills have been shown to contribute to adults' ability to effectively function in modern society, adults' numeracy has been a neglected, but critical, factor in this regard. In the home, workplace, or community, adults often need to manage various types of quantitative situations or interpret information that may involve numbers, measurements, probabilities, shapes, statistical information, or quantitative arguments. Numeracy skills: help adults to keep up with a rapidly changing world; underlie independent functioning and action as a parent, citizen or worker; are a gatekeeper for entrance into further education in many technical and other occupational areas; critically affect employability and career options.

Why not continue with the previous quantitative literacy scale?

One of the scales of the International Adult Literacy Survey (IALS), the Quantitative Literacy Scale, was a measurement of the respondent's ability to apply arithmetic operations to numbers embedded in diverse texts. [For an analysis of Australia's performance in the Quantitative Literacy (QL) scale, see Cumming's article 'The Quantitative Literacy Performance of Australians: Implications of Low Skill Levels' (Cummings, 1997).] While this scale produced useful data, survey developers recognised that it was limited in scope. The numeracy scale of ALLS is designed to go above and beyond the QL Scale. While there is a clear connection and relationship between numeracy and Quantitative Literacy, there are significant differences with numeracy covering a wider breadth of mathematical skills and, also, not being heavily dependent in all cases on literacy skills by having tasks embedded in text, as shown in Figure 1.
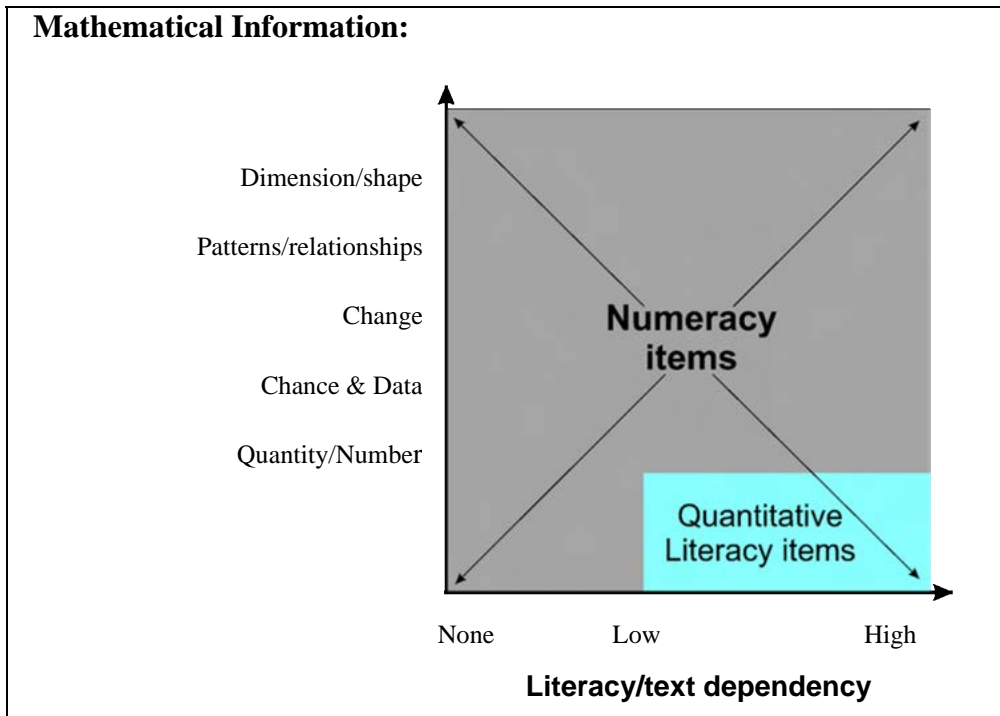
Figure 1. Numeracy versus Quantitative Literacy

Description of numeracy

The ALLS numeracy team described numeracy as a complex, multifaceted and sometimes slippery construct (Gal et al., 2003). The basic premise was that numeracy is the bridge that links mathematical knowledge, whether acquired via formal or informal learning, with functional and information-processing demands encountered in the real world. The ALLS numeracy framework (Gal et al., 2003) defined numeracy as follows: "Numeracy is the knowledge and skills required to effectively manage and respond to the mathematical demands of diverse situations."

However, since an assessment can only examine observed behavior, not internal processes or capacities, the framework uses a more detailed definition of "numerate behavior" as a means to guide the development of items for the survey: "Numerate behavior is observed when people manage a situation or solve a problem in a real context; it involves responding to information about mathematical ideas that may be represented in a range of ways; it requires the activation of a range of enabling knowledge, factors, and processes."

The ALLS description of numerate behavior distinguishes what are called five facets, each with several components. Table 1 presents the ALLS framework's elaboration of numerate behaviour and includes the five facets, each with several components.

Table 1. Numerate Behaviour and its Facets.

Numerate behavior involves:
managing a situation or solving a problem in a real context
everyday life
work
societal
further learning

by responding
identifying or locating
acting upon:
- order/sort
- count
- estimate
- compute
- measure
- model

interpreting
communicating about

to information about mathematical ideas
quantity & number
dimension & shape
pattern and relationships
data & chance
change

that is represented in a range of ways
objects & pictures
numbers & symbols
formulae
diagrams & maps
graphs
tables
texts

and requires activation of a range of enabling knowledge,
behaviors, and processes
mathematical knowledge and understanding
mathematical problem-solving skills
literacy skills
beliefs and attitudes.

Complexity factors

As mentioned earlier, there were a number of requirements expected of the numeracy construct, one of which was to develop a scheme that would be able to predict the difficulty or complexity of a numeracy assessment task. A scheme of five factors was developed that attempted to account for the difficulty of different tasks. These five factors were:

1. Complexity of mathematical information/data
2. Type of operation/skill
3. Expected number of operations
4. Plausibility of distractors (including in text)
5. Type of match/problem transparency.

These factors were used to attempt to estimate, separately and in interaction, the difficulty level of the numeracy tasks. For each of these factors a detailed description was developed against a scoring system in the range from 1 through to 3 or 5. Figure 2 (complexity flow chart) indicates how the scoring for each item worked.
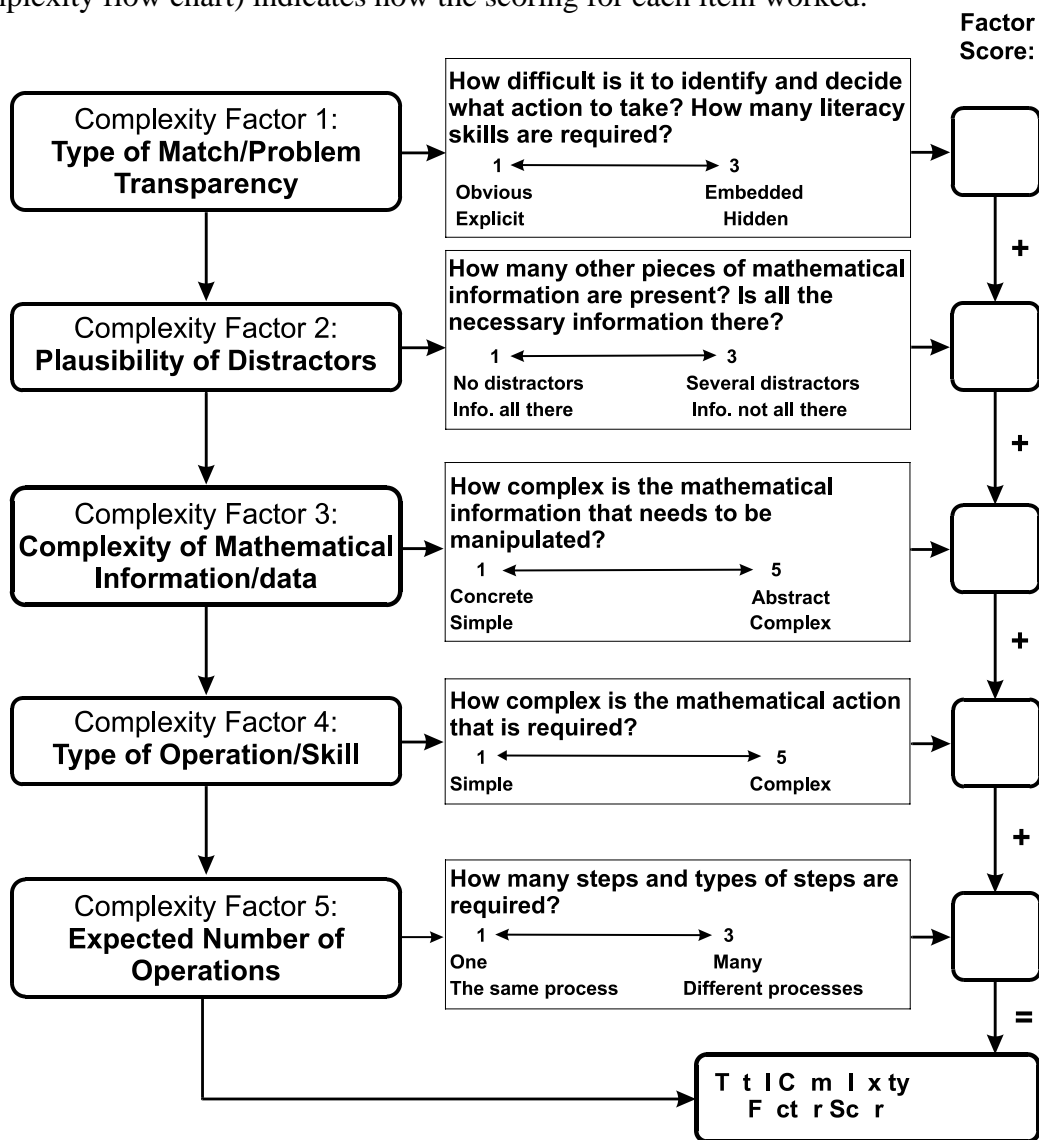
Figure 2. Complexity Flow chart

Each of these five factors was developed in detail; the draft developed is included in *Adult numeracy and its assessment in the ALL survey: A conceptual framework and pilot results* (Gal et al., 2003).

The numeracy scale and methodology

As in IALS, literacy and numeracy ability will be expressed as a score on a scale ranging from 0–500 points. The score is the point at which a person has an 80% chance of successfully performing tasks at that level. The design ensures that the results can be reported on the full scale for each domain. This methodology is unique in that an individual respondent is *not* given a score or a mark: the items themselves are scored and placed on the continuum of difficulty.

The assessment items

A pool of items was developed that attempted to satisfy the breadth of the conceptual framework and the facets described above. The items attempted to be as realistic as possible, but there are restrictions on developing items for a large scale survey that is to be delivered to 1000s of people, in all participating countries. The items are based upon simulated texts such as advertisements, newspaper articles, maps, diagrams and plans, photos, etc. A ruler and a calculator are provided to all respondents.

Based on the conceptual framework, over 100 test items were developed and tested in feasibility studies in the U.S. and the Netherlands. A pool of 80 items was then selected that included tasks at diverse levels of difficulty and that covered key facets of the conceptual framework for numeracy. For those 80 items, the theoretical factors that were predicted to account for task difficulty strongly correlated with the observed difficulty of items and, therefore, provided initial support for the content validity and the construct validity of the numeracy scale. In 2002 large pilot surveys were conducted in a range of countries and the majority of the numeracy items performed well, and the numeracy domain was accepted as a directly assessed component of the final ALLS survey. Based on the performance of the 80 items, 40 items were chosen to be administered in the final ALLS survey.

Administration of the survey

Using household survey methods, respondents are first asked a series of background questions and then presented with a screening booklet containing a small number of simple (level 1) tasks. If the respondent fails to complete two or more of these tasks correctly, the interview is concluded. If the respondent completes two or more tasks correctly, they are given a separate booklet which contains a selection of the assessment tasks. Respondents can solve the problems in any way they choose. There is no time limit and respondents are urged to try each question.

The background questionnaire

The ALLS not only tests skills directly across the four different domains, it also collects background information (almost 300 different questions), including details of participation in education and learning; educational attainment; parental education, languages spoken; labour force status and occupation; respondents' literacy and numeracy practices at work and elsewhere; social capital and well-being; use of information and communications technology; income; and other socio-demographic information. It is the linking and correlations of this background data with the

performances of respondents that allows for a wide ranging and comprehensive analysis of the abilities of adults and the different factors that impact on, and influence, performance.

Implications for Australian research, policy and practice

Despite limitations of such large scale assessments, there is valuable data within and behind ALLS that should be utilised when the results are released. The data and the results will be made available by ABS and should be of interest to:

- policy makers and governments (State, Federal and Industry bodies)
- researchers
- teachers and practitioners.

Initial messages regarding numeracy from the first wave of results

In the initial analysis and results coming out from the first wave of the 2003 ALLS countries, the data seems to be indicating that numeracy plays a more important role in economic returns for individuals than do prose or document literacy, and seems to confirm that men outperform women on the numeracy scale. Some of the outcomes and possible questions stemming from the results, that will be of interest, when the Australian results are released include:

- The overlap between respondents' abilities in literacy and numeracy was lower than expected.
- Early school leavers are much more likely to score at low levels of numeracy in all countries: e.g., in the US those without a secondary diploma are 26 times more likely to score at levels 1 and 2 than those who have completed an educational level "higher than upper secondary".
- The individual economic returns to increases in numeracy skill was larger than for the prose and document skills.
- There are indications that numeracy skills are more important than literacy at work, and skills deficit is greater for numeracy.
- Men performed better on the numeracy scale in all countries.
- From IALS research it is known that the proportion of individuals with Level 1 skills exerts a strong negative drag on growth in GDP per capita: so, one could realise quite large economic gains by investing in the bottom. IALS research also indicated that the skill levels of women seem to matter more to the growth in GDP than those of men. So, from ALLS, with its highlighted economic importance of numeracy, what does this mean about increasing the numeracy competence of women?

The initial ALLS data supports other research data from the UK that indicates the strong role numeracy plays in both human and social capital terms:

> People without numeracy skills suffered worse disadvantage in employment than those with poor literacy skills alone… Women with numeracy difficulties appeared especially vulnerable to exclusion from the clerical and sales jobs to which they aspired (Bynner & Parsons, 1997, p. 27).

> For women, while the impact of low literacy and low numeracy is substantial, low numeracy has the greatest negative effect, even when it is combined with

competent literacy… Poor numeracy skills make it difficult to function effectively in all areas of modern life, particularly for women. (Bynner & Parsons, 2005, p. 7)

The potential for researching and analysing the Australian results

Some potential questions that could be analysed include:

At the policy level

- What is the profile for the distribution of numeracy in the adult population, and what is the inter-relationship of these skills to prose literacy, document literacy and other measured skills?

- What is the numeracy distribution among males and females and how can this be explained? Are the results for adults consistent with PISA findings (and now initial ALL findings) that boys are likely to score better in mathematics/numeracy while girls score higher in reading? Is this trend consistent from country to country or within countries? Does it vary across different age groups or different educational groups?

- What is the numeracy distribution among the working age population? Is this distribution in synch with labour market sectoral demand? Where are the shortages? Where is the excess?

- Do the outcomes of IALS (and now ALL), in relation to numeracy versus literacy and the differences in performance of women versus men in conjunction with the economic returns, have implications for policies regarding the mathematical and numeracy competence and engagement of women?

- What is numeracy distribution among different age groups? Do the new cohorts entering the work force have better numeracy proficiency than the current work force, thereby raising national proficiency levels?

- We need to research more closely the association between the indicators from the BQ and Numeracy skill level — what more can we learn? For example, what are the correlations between use and familiarity with mathematics tasks in the home or workplace with numeracy competence?

- What are the population characteristics associated with persons with low numeracy? Are these the same as for literacy? Is low numeracy concentrated among some population groups?

- What are educational outcomes for those with low numeracy? What are the average numeracy scores for populations with different levels of educational attainment? Do those with low numeracy use training after initial education?

- What are the labour market outcomes for those with low numeracy? Are they concentrated in certain occupations? Industrial sectors? Are they more likely to be unemployed? Have low incomes?

At the research and educational level

- Literacy versus numeracy: is it an equal partnership at the funding level? At the delivery and curriculum level?

- Do numeracy skills wane with age, as has been shown for literacy?

- Using the theoretical constructs behind the literacy and numeracy domains, how are the cognitive processes for numeracy different from those used in literacy? What are the links or connections between literacy and numeracy?

- In terms of ethno-mathematical research, what is the relationship between formal, or school-based, mathematics and informal, "real life", mathematics learning experiences? What counts most? What do adults use the most?
- What are the implications for vocational and workplace education, training and curriculum? What are the implications for school mathematics curriculum, standards and learning and teaching?
- What is harder for (some) people and easier for others?
- What factors make items more difficult?
- What factors impact on success in numeracy?

Conclusion

It is hoped that Australia's investment in ALLS will result in valuable data and outcomes for all interested in improving the numeracy skills of the Australian population — including governments, policy and program makers, educational organisations, researchers, teachers and trainers.

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A classroom exploration of Benford's Law and some error finding tricks in accounting

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In this talk, two mathematical ideas are presented — one small and one big — that have applications in the world of finance and accounting and which can be made the basis of effective high school lesson plans.

As a warm-up task, try this:

> You are the treasurer of your local Mathematical Association and you are trying to get the known expenditure amount for the year ($11483) to balance the list of expenditure items ($12365). Whichever brand of calculator you use, you cannot escape the conclusion that the known expenditure seems to be $882 less than the total of the list of items. There must be a mistake somewhere. Is there a clue hidden in the number 882?

The smaller of the two ideas involves number theory, specifically divisibility by certain integers. To begin, the "slightly special" property of the $882 is its divisibility by 9. To an accountant this would be a clue that a transposition error might have occurred, although not necessarily.

Fun can be had in the classroom through specific examples:

> The difference between $3947 and $3497 is $450 and the difference, 450, is divisible by 9. By experiment students can check that whenever adjacent digits are transposed, the difference between the resulting number and the original is divisible by 9.

When the transposition is between non-adjacent digits separated by one digit, the difference between the original number and the corrupted version is divisible by both 9 and 11. If there are two digits separating the transposed digits, the difference will be divisible by 111 as well as 9.

The explanation is straightforward: Let a and b be digits in a numeral. The difference between, say, $ab45$ and $ba45$ is

$$1000(a - b) + 100(b - a) = 900(a - b)$$

and this is clearly divisible by 9 (and a power of 10).

Also, the difference between, say, $4a5b$ and $4b5a$ is

$$100(a - b) + (b - a) = 99(a - b)$$

and this is divisible by 9 and 11.

How then might the discrepancy of $882 be explained? It cannot be the result of a single transposition error because the difference $(a - b)$ between digits is confined to the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and none of these numbers multiplied by 9 or 90 or 99 will give

882. However, there are many ways by which the difference 882 could arise as a combination of several transposition errors.

For example: in two numerals, 2901 → 2091 and 2591 → 2519, with discrepancies of 810 and 72; or in three numerals, 1911 → 1191 with a discrepancy of 720, 54544 → 54445, with a discrepancy of 99, and 22281 → 22218 with a discrepancy of 63.

In the classroom, ideas about *implication* could be introduced at this point. Transposition errors imply divisibility by nine but divisibility by nine does not imply transposition errors. It might also be useful to note that "not divisible by nine" implies "not a transposition error."

An accountant would be happiest to find a discrepancy that could be obtained by multiplying 9, 90, 99, 900, etc. by a single digit, because this could be explained by just one transposition error. A discrepancy like my \$882 requires at least two transposition errors, which would be a less likely occurrence than a single error, and so could have other explanations.

Benford's Law

The second idea involves much more sophisticated mathematics if one is being rigorous, but aspects of it can be explored and comprehended by high school students.

Astronomer Simon Newcomb, in 1881, noticed that numbers found in statistical data or in tables of physical constants were more likely to begin with a smaller digit than a larger one. Dr Frank Benford, a physicist working for the General Electric Company, rediscovered this strange fact in 1938 and it came to be known as Benford's Law. A satisfactory explanation for Benford's Law was worked out only as recently as 1996 by Theodore P. Hill.

Forensic accountants now use Benford's Law as a tool in the detection of fraud.

As Newcomb and Benford discovered, for many naturally-occurring sets of data, the distribution of the initial digits is given by the function $P_d = \log(1 + d^{-1})$, $d \in \{1,2,3,...,9\}$. Fraudsters who have not heard of Benford's Law tend to concoct data that does not follow the logarithmic distribution and so leave themselves open to investigation.

Senior students would be able to verify that the function $f(d) = \log(1 + d^{-1})$, $d \in \{1,2,3,...,9\}$ does indeed define a probability distribution.

$$\sum_{d=1}^{9} \log(1 + d^{-1}) = \sum \log(\frac{d+1}{d}) = \log(\frac{2}{1} \cdot \frac{3}{2} \cdot ... \cdot \frac{9}{8} \cdot \frac{10}{9}) = \log 10 = 1$$

It is also true that if we let d_2 be a two-digit number, then, by the same reasoning as in the single digit case, $f(d_2) = \log(1 + d_2^{-1})$, $d_2 \in \{10,11,12,13,...,99\}$ is a probability distribution.

Indeed, if d_k is a *k*-digit string, then

$$f(d_k) = \log(1 + d_k^{-1}), d_k \in \{10^{k-1}, 10^{k-1} + 1, 10^{k-1} + 2, ..., 10^k - 1\}$$

is a more general form of the Benford distribution involving the first *k* digits of a numeral.

It is interesting to note that, to use a particular example, the sum of the probabilities of the two-digit strings beginning with, say, four is

$$\sum_{d=40}^{49} \log(1 + d^{-1}) = \sum \log(\frac{d+1}{d}) = \log(\frac{41}{40} \cdot \frac{42}{41} \cdot ... \cdot \frac{49}{48} \cdot \frac{50}{49}) = \log \frac{5}{4} \approx 0.097$$

which is the same as the single-digit probability of a 4 in the Benford distribution.

Generalising this we see that the single-digit case and the multi-digit case are essentially the same distribution. This is the "scale invariance" property that one finds mentioned in the literature.

Another generalisation of the logarithmic distribution comes from the fact that the recipe also works for numerals expressed other than in base 10, provided that the logarithm is also to the alternative base. Predictably, this phenomenon is called "base invariance."

It is also worth noting that the occurrence of a particular second digit is not independent of the first digit. For example, it is clear that

P(2nd digit is 4 given that 1st digit is 1) > P(2nd digit is 4 given that 1st digit is 9)

because

P(1st 2 digits are 14) > P(1st 2 digits are 94).

In the workshop handout there is a project based on Benford's Law that was completed by a class of Year 9 students. There is no algebra and no attempt to explain why the data that the students generate should have the Benford distribution (in truth, it does not quite have the distribution!). However, there is enough in the exercise for students to be able to get a feel for the idea, to practice some computation skills and to learn about mathematics in action in the real world.

In the project, students are asked to generate sets of twenty random numbers using their scientific calculators or with a computer and a spreadsheet program. The frequencies of the initial digits from 1 to 9 were tabulated. Most found that the counts of the initial digits were fairly evenly spread, as would be expected if the random number generators were producing numbers from a uniform distribution.

Each set of twenty random numbers was then split into two subsets of ten. The students multiplied all the numbers in one subset by all the numbers in the other subset, as in a multiplication table, to produce a set of 100 new numbers — in other words, the Cartesian product of the two subsets. Again, the students tabulated the frequencies of the initial digits.

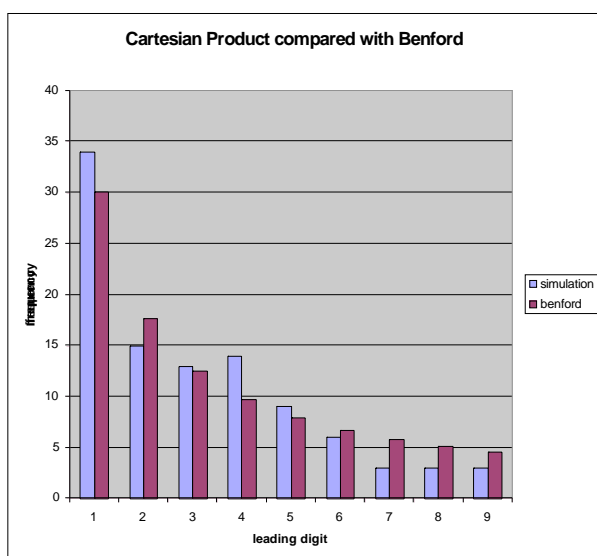Magically, the distribution of the initial digits now followed a Benford-like pattern.

Figure 1

Finally, students were asked to compare two data sets, purporting to be quarterly commercial transaction amounts, and by examining the initial digit distributions, decide whether evidence existed that one or both sets might have been fabricated. This was to be done on the basis that: "in data of this kind, the relative frequency of initial digits is likely to be highest for 1 and should decrease progressively for digits 2 to 9."

What is "data of this kind"? The exercise skates over questions like this one with the tacit excuse that the particular group of Year 9 students for whom it was developed would not have wanted to think in such detail. On the other hand, a teacher wishing to model a spirit of enquiry and of enthusiasm for mathematics could hardly avoid asking questions like the following:

- Why should certain data sets be expected to have the logarithmic (Benford) distribution over their initial digits while other data sets do not, and what processes generate data of this kind?

- Do the data sets generated in this exercise truly exhibit the Benford distribution? If not, what *is* their distribution?

- Are there other ways of generating and displaying Benford-like data in the classroom?

The second of these questions seemed easiest to tackle. It is not too difficult to simulate on a spreadsheet the process described in the classroom exercise and to graph the results in columns side-by-side with columns representing the Benford distribution. Comparing many sets of data produced by this Cartesian product process, one is forced empirically to the conclusion that the resulting distribution does not quite match the Benford distribution.

To confirm this one must derive the theoretical probabilities. That is, the distribution of the initial digits in the Cartesian product of two sets of samples from a uniform distribution on, say, the interval $(0,10^k)$.

The situation can be conceptualised graphically. Uniform $(0,10^k)$ random variables X and Y are represented on a pair of coordinate axes and level curves are drawn representing $XY = 10^{k-1}d$ and $XY = 10^{k-1}(d+1)$ for some $d \in \{1,2,3,...,9\}$. Any (x,y) pair falling between these level curves must have a product with initial digit d. But, below the level curve $XY = 10^{k-1}$ there is an infinite family of regions in which the product xy has initial digit d. The probability that the initial digit is d must be the infinite sum of the areas of these regions as a fraction of the area of the rectangle containing all allowable (x,y) pairs.

To clarify this using a particular example, suppose $k = 1$ and $d = 4$. Then random variables X and Y take values in the interval $(0,10)$. The infinite family of regions in which the product xy has initial digit 4 and $xy < 10$ could be represented as follows :

$$\{\, 4 \le xy < 5,\ 0.4 \le xy < 0.5,\ 0.04 \le xy < 0.05,\ \ldots \,\}$$

The first two of these regions are displayed in Figure 2.
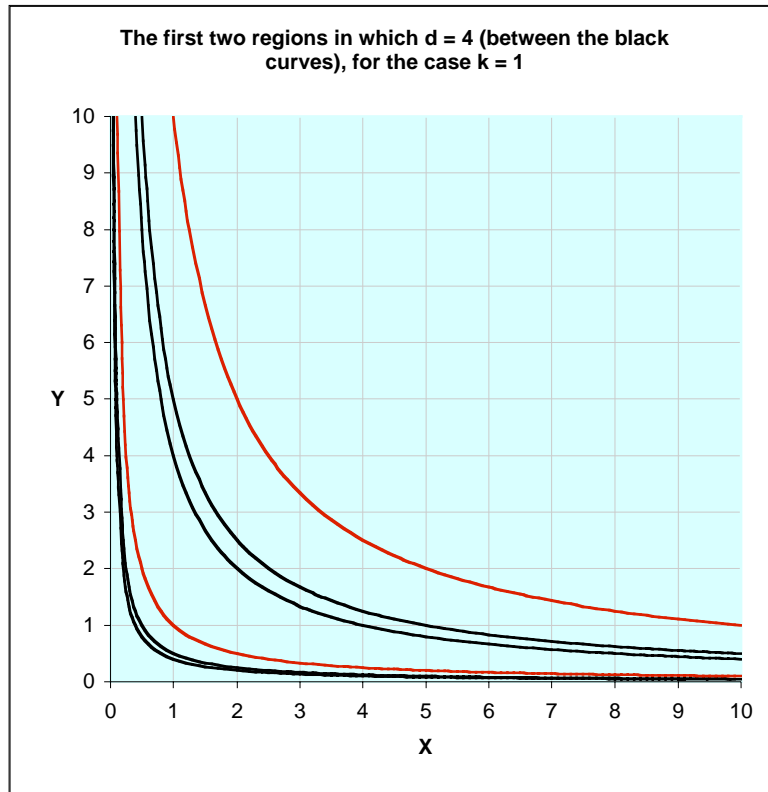
Figure 2

By carrying out some integrations, it can be shown that the area of the *j*th region in the sequence of regions in which the initial digit is *d*, is given by

$$A_j = 10^{2k-j}(d\ln d - (d+1)\ln(d+1) + 1 + j\ln 10)$$

Summing these regions over *j*, we obtain eventually

$$\sum_j A_j = 10^{2k}\left(\frac{d\ln d - (d+1)\ln(d+1) + 1}{9} + \frac{10\ln 10}{81}\right)$$

Since the permissible (x, y) pairs occur in an area of 10^{2k}, we have the probability that the initial digit is *d*:

$$P(d) = \frac{d\ln d - (d+1)\ln(d+1) + 1}{9} + \frac{10\ln 10}{81}, \quad d \in \{1, 2, ..., 9\}$$

The probabilities obtained from this distribution function agree well with the results obtained by simulation. The distribution is known as the Stigler distribution. Here it is in explicit form:

P(1) =	0.241	P(4) =	0.117	P(7) =	0.060
P(2) =	0.183	P(5) =	0.095	P(8) =	0.047
P(3) =	0.145	P(6) =	0.076	P(9) =	0.034
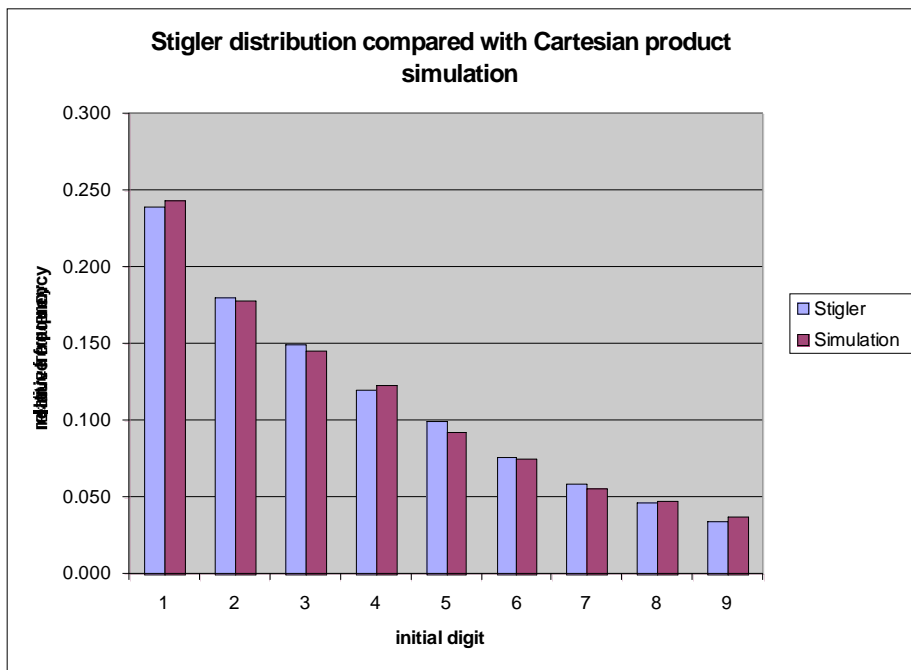
Figure 3

A partial answer to the third question on the list above, "Are their other ways of generating and displaying Benford-like data in the classroom?" presented itself while experimenting with spreadsheet simulations.

Sets of numbers can be produced, each element being the product of *three* uniformly distributed random numbers. In this process, the initial digit distributions are much closer to the logarithmic distribution — at least on the evidence of simulated data.
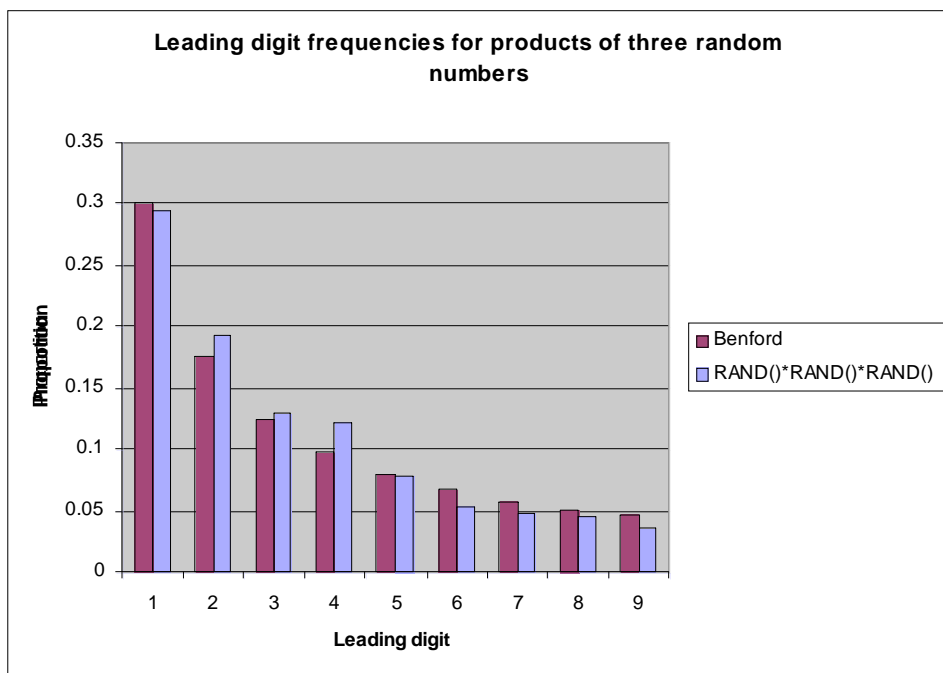
Figure 4

In a paper titled: "Benford's Law As The Property Of The Truly Random Number", A. E. Kossovsky relates a fable that illustrates another way in which logarithmic initial digit data might be generated. To paraphrase Kossovsky:

Members of an ancient society were permitted to have at most 9 gods, 8 olive trees, 7 fruits, 6 chickens, 5 sheep, 4 dogs, 3 slaves, 2 houses and 1 wife, and they had to have at least one of each. In conversations among the members of the community mentioning counts of these things, it is likely that the number 1 would be mentioned most because it can come up in relation to any of the objects. 'Nine', on the other hand would only arise in relation to one kind of object: gods, and so would be mentioned least. It is conceivable that the probabilities would be determined as in the following table:

| Object	Maximum number permitted	P(1)	P(2)	P(3)	P(4)	P(5)	P(6)	P(7)	P(8)	P(9)
Wife	1	1/1	0	0	0	0	0	0	0	0
House	2	1/2	1/2	0	0	0	0	0	0	0
Slave	3	1/3	1/3	1/3	0	0	0	0	0	0
Dog	4	1/4	1/4	1/4	1/4	0	0	0	0	0
Sheep	5	1/5	1/5	1/5	1/5	1/5	0	0	0	0
Chicken	6	1/6	1/6	1/6	1/6	1/6	1/6	0	0	0
Fruit	7	1/7	1/7	1/7	1/7	1/7	1/7	1/7	0	0
Olive	8	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8	0
God	9	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9
Average probability over 9 topics		0.314	0.203	0.148	0.111	0.083	0.061	0.042	0.026	0.012
Benford probabilities		0.301	0.176	0.125	0.097	0.078	0.067	0.058	0.051	0.046

It can be seen that the probabilities averaged over the nine topics of conversation closely resemble the Benford distribution.

This idea suggests more realistic scenarios like the following: Suppose there are forty categories of payments to be made. Each category has a randomly chosen upper bound between $100 and $10000. Within each category several payments are to be made. They are assumed to be uniformly distributed on the interval from $100 to the upper bound for the category. Such a scenario can be simulated readily by spreadsheet (Figure 5).
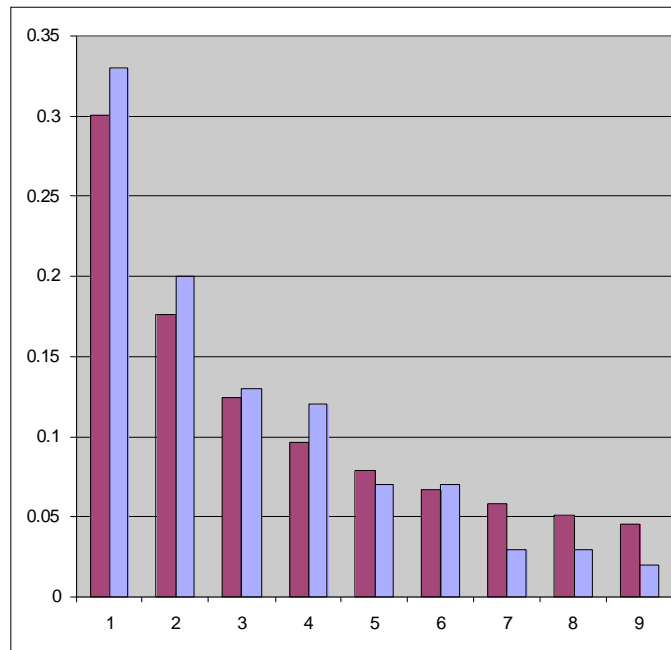
Figure 5

In this diagram, the right hand column in each pair represents a simulation, with the Benford distribution on the left. Not all trials produce as good a fit as this one but the shapes are usually similar to the Benford shape.

The scenario just described seems to accord with the explanation for the Benford distribution given by Theodore P. Hill. In his 1996 paper "A Statistical Derivation of the Significant-Digit Law", Hill introduces his main theorem, his log-limit law for significant digits, with the claim: Roughly speaking, this law says that if probability distributions are selected at random, and random samples are then taken from each of these distributions in any way so that the overall process is scale (or base) neutral, then the significant digit frequencies of the combined sample will converge to the logarithmic distribution.

The first question on the list, "Why should certain data sets be expected to have the logarithmic (Benford) distribution over their initial digits while other data sets do not, and what processes generate data of this kind?" is probably the most difficult to answer.

In essence it is this question that the paper by Hill addressed and his explanation is considered to be the first satisfactory one. In one sense the question is outside the realm of mathematics. Mathematicians have described processes that would generate data sets with initial digits following the logarithmic distribution and they have demonstrated some special properties like scale and base invariance that the distribution should have, but little can be said *a priori* about whether a given set of data should display the Benford distribution.

In the end, observation must be the best recourse. If a given data set is seen as a sample from a particular distribution, we assume it will have the characteristics of the population from which it was drawn. So, if samples of a given type of data usually display the Benford distribution, we suspect that a sample that does not have that distribution may be from a different population and therefore deserves further investigation. Thus we must take the scientific approach in these matters rather than the purely mathematical.

Appendix 1: Year 9 level 2 in-class assignment

Detecting fraud with probability

Marks: /35 There are seven sections, each worth 5 marks.
Your teacher will arrange for you to complete this assignment either in class or in a computer lab session. You will use a scientific calculator or an Excel spreadsheet to manipulate sets of random numbers.

1. Generate twenty random numbers.
On a calculator this is done using the [rnd#] button. In Excel, just copy the formula '=RAND()' into twenty cells arranged in two rows of ten as below.
Copy <u>the first three significant digits</u> of each random number into a table:

2. Look at the initial digit of each number in your table. By counting the number of occurrences of each numeral 1–9, write down fractions representing the estimated probability that each will occur as the initial digit in a random number. Use this table:

Numeral:	1	2	3	4	5	6	7	8	9
Frequency:									
Proportion:									

3. Next, multiply each of the numbers in row 1 of your table in question 1 by each of the numbers in row 2. Use the following grid to make sure each of the 100 multiplications is carried out. (You can do the multiplications quickly in Excel by copying formulas like '=C$2*$B3 into a block of cells.) Now, instead of writing the products in the following grid cells, <u>write the initial digits only</u>.

Row 1

×										
Row 2

4. Do step 2 again, this time looking at the initial digits in your table of products.

Numeral:	1	2	3	4	5	6	7	8	9
Frequency:									
Proportion:									

5. Compare the results of step 2 and step 4. Comment on any difference you observe in the ways the estimated probabilities are distributed.

_____
_____
_____

A true story
Astronomer Simon Newcomb, in 1881, noticed that numbers found in statistical data or in tables of physical constants were more likely to begin with a smaller digit than a larger one. Dr Frank Benford, a physicist working for the General Electric Company, rediscovered this strange fact in 1938 and it came to be known as Benford's Law. A satisfactory explanation for Benford's Law was worked out only as recently as 1996 by Theodore P. Hill.
Benford's Law is now used by auditors and forensic accountants to help detect fraud.

6. Analyse the distribution of initial digits in the following data sets. To do this, you should complete the frequency tables, as you did for steps 2 and 4.

Data set 1: Transactions July 1 – September 30, $'000, subject A

80	207	505	646	627	138	618	585	907	82
294	977	698	385	618	280	8	391	280	62
131	626	658	486	656	88	843	703	786	911
812	787	907	950	293	538	753	600	931	97

Initial digit frequencies

Numeral:	1	2	3	4	5	6	7	8	9
Frequency:									
Proportion:									

Data set 2: Transactions July 1 – September 30, $'000, subject B

192	201	205	232	210	261	131	95	232	273
161	168	172	194	176	218	110	79	194	478
498	510	575	521	648	325	235	575	292	329
285	298	370	186	134	329	409	427	437	493

Initial digit frequencies

Numeral:	1	2	3	4	5	6	7	8	9
Frequency:									
Proportion:									

7. Benford's law says that, in data of this kind, the relative frequency of initial digits is likely to be highest for 1 and should decrease progressively for digits 2 to 9. On this basis is there any evidence that one or more of the above data sets could be fraudulent? Explain.

_____
_____

Bibliotherapy, a powerful tool to alleviate mathematics anxiety in pre-service primary teachers: A reflective report[*]

Sue Wilson
ACU National

This reflective report synthesises the research background that has informed recent research on bibliotherapy as a tool in teacher education. Bibliotherapy provides a structure and language for educators to help pre-service teachers to reconstruct their own school experiences and reinterpret their views of themselves as learners of mathematics while studying school students' experiences. Bibliotherapy simultaneously addresses affective and cognitive domains and has the power to encourage pre-service teachers to change their image of themselves as students, and their assessment of their capacity to learn and teach mathematics. This paper also identifies potential directions for future research.

Introduction

Recent research in mathematics teacher education concluded that bibliotherapy is a process that shows potential as an innovative way of eliciting and structuring pre-service teachers' affective responses (Wilson and Thornton, 2005a, 2005b), acknowledged the importance of the dual nature (affective and cognitive) of these reflections, and provided a framework and shared language for teacher educators to analyse them (Wilson and Thornton, 2006, 2007).

Aims

This paper aims to provide a synthesis of the theoretical framework and research background that has informed preliminary research into the application of the technique of bibliotherapy to primary pre-service teachers suffering from mathematics anxiety; and reflect on the results of these studies and to indicate potential directions for future research.

Scope

This report is limited to the investigation of the use of the technique of bibliotherapy in teacher education, how it has potential to address mathematics anxiety in pre-service teachers, and to consider the associated theoretical framework and research background.

Limitations

This report does not address other areas of primary pre-service teachers' education, such as anxiety about science education, secondary pre-service teacher education, teacher in-service, or how bibliotherapy might be used with school students or their parents;

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[*] Paper accepted by peer review

although all are important issues and bibliotherapy has existing or potential applications in all these areas.

Mathematics anxiety

Dossel (1993) identified that, for many children, mathematics anxiety is a specific learning difficulty, characterised by a belief that mathematics cannot make sense and a feeling of helplessness and lack of control over one's learning. Stigler and Hiebert, (1992) suggested factors in this anxiety were inappropriate teaching practices, and a common belief that success in mathematics is predetermined by ability.

Mathematics anxiety in pre-service teachers

Mathematics anxiety is a phenomenon which is particularly significant in pre-service teacher education. Many pre-service primary or early childhood teachers have a fear of mathematics, and see themselves as unable to learn effectively (Haylock, 2001). Hembree's (1990) meta-analysis of research studies found that the level of mathematics anxiety of pre-service elementary teachers was the highest of any course major group on university campuses. Trujillo and Hadfield (1999) attempted to trace the roots of mathematics anxiety in American pre-service primary teachers. In a study of pre-service teachers, Uusimaki and Nason (2004) found that most of their mathematics anxiety could be attributed to experiences in primary school. Mathematics anxiety in pre-service teachers has been associated with low levels of confidence to teach mathematics (Bursal and Paznokas, 2006) and low mathematics teacher efficacy (Swars, Daane and Giesen, 2006).

Research has identified the factors contributing to anxiety about mathematics (Hadfield and McNeill 1994) and its impact on teaching practices. The impact of teachers' beliefs about mathematics and learning on their teaching is widely recognised (Schuck and Grootenboer, 2004; Thompson, 1992). Askew, Brown, Rhodes, Johnson and Wiliam (1997) found evidence that teachers' perceptions of mathematics, and how it is learned, were more important in promoting positive outcomes for students than different teaching methods or ways of organising classrooms.

Pre-service teacher education

A significant area of research in recent years has been the role of reflection in teacher professional learning. Mathematical autobiographies have been used as a tool to encourage reflection by pre-service teachers (Ellsworth and Buss, 2000; Sliva and Roddick, 2001). Bibliotherapy has close links with these reflective practices.

Pre-service teacher education and mathematics anxiety

Research investigating how studying subjects at university might impact upon this anxiety has focused on teaching mathematics to assist students to develop deeper knowledge (Chick, 2000), emphasising the use of manipulatives in a methods class (Vinson, 2001) or on how studying mathematics teaching strategies impact on pre-service teachers' beliefs and attitudes (Frid, 2000). Bibliotherapy offers a new and radically different approach.

Bibliotherapy

Bibliotherapy is a technique developed in psychology and library science. Hendricks (1999) traces the development of the process and its applications. Bibliotherapy aims to assist individuals to overcome negative emotions related to a real-life problem by guided reading about the situation of a third person, or a fictional animal. It is a dynamic process during which people bring their own needs and problems to the reading activity, identify with the protagonist and interpret the words according to their own experiences. Bibliotherapy can be described as: "a process of dynamic interaction between the personality of the reader and literature — interaction which may be utilised for personality assessment, adjustment and growth" (Cornett and Cornett, 1980, p. 8).

The reader is less confronted and able to experience the problem more objectively, because the situation is about a third person. Reading is followed by individual or group discussion in a non-threatening environment. From this, the reader can develop self-awareness, an enhanced self-concept and improved personal and social judgement. (Hebert and Furner, 1997).

The process of bibliotherapy involves four stages: identification (the reader identifies with the protagonist), catharsis (the reader becomes emotionally involved and releases pent-up emotions), insight (the reader becomes aware that their problems might also be addressed or solved.) and universalisation (the reader realises that the problem is shared by others).

The technique has been used to help high ability students to overcome mathematics anxiety. "Bibliotherapy is a therapeutic, discussion-generating technique which offers educators appropriate affective strategies for dealing with mathematics anxiety in *secondary* math classrooms so that students achieve success" (Hebert & Furner, 1997, p. 170).

Bibliotherapy in pre-service teacher education

Morawski (1997) suggested that bibiliotherapy can "play an instrumental role in helping both beginning and experienced teachers think about their professional practice" (p. 243). She emphasised practical applications in pre-service education in units about teaching reading and teaching students with special needs, and stressed the need for teachers to "gain an understanding of their perceptions as well as the influence that these perceptions can have on their attitudes and actions in the education setting" (p. 256)

One of the purposes of bibliotherapy ,in some pre-service education courses, has been to change attitudes towards students with issues. Marlowe and Maycock (2000) reported on the use of bibliotherapy in preparing pre-service teachers to teach students with emotional and behavioural disorders. They reported a decrease in punitiveness in pre-service teachers who read and discussed stories of how teachers dealt with such students "within the mechanics of group bibliotherapy" (p. 325). In this research, the pre-service teachers identified with the teacher, but an alternative approach is to encourage pre-service teachers to identify with the learner.

Bibliotherapy as a tool to address mathematics anxiety in pre-service teachers

The technique of bibliotherapy has been used as a tool to address mathematics anxiety in secondary students (Furner and Duffy, 2002; Hebert & Furner, 1997). In pre-service teacher education courses it has been reported to help pre-service teachers construct

knowledge and practice (Morawski, 1997) and change pre-service teachers' attitudes towards their students with behavioural difficulties (Marlowe & Maycock). Making connections between these studies gives rise to the idea that bibliotherapy has potential as a technique to address mathematics anxiety in pre-service primary teachers.

Research investigating the effect that pre-service teachers' reflections on students' difficulties in learning mathematics, including mathematics anxiety, had on their perceptions of themselves as learners of mathematics has provided evidence that the technique of bibliotherapy may enable pre-service teachers to reconstruct their own experiences, and re-evaluate themselves as learners of mathematics. In addition, their reflections may also produce a greater insight into how children's anxiety about mathematics can be minimised by teachers (Wilson & Thornton, 2005a). The research attempted to identify the way pre-service teachers envisaged themselves as learners, and doers, of mathematics and how they might reassess their capacity to learn and teach mathematics as a result of guided reflections. Themes of the role of the teacher, the cycle of fear, failure and avoidance, the pre-service teachers' perceptions of the nature of mathematics, and their self-image as a learner of mathematics, were identified. (Wilson & Thornton, 2005b). Enhancing pre-service teachers' self-image contributed to their developing the capacity to see mathematics as making connections, to seeing learning as developing deep knowledge, to seeing their role as teachers as being to provide opportunities for school students to solve rich and complex problems, and to adopting a view that all students can learn mathematics (AAMT, 2002).

Bibliotherapy was proposed as an innovative technique to assist pre-service teachers to examine their attitudes towards their own learning in mathematics. The researchers identified how a pre-service teacher's journal reflections related to the stages of bibliotherapy and suggested this technique has potential to enhance pre-service teachers' confidence for teaching primary school mathematics (Wilson & Thornton, 2006). The bibliotherapy approach used in this research differed from other approaches to addressing mathematics anxiety in pre-service teachers, in that it simultaneously addressed affective and cognitive domains and, in the process, provided pre-service teachers with the opportunity to reconfigure their identity as future teachers of mathematics. The journal entries provided evidence that students experienced the stages of the process:

- *Identification*

  The pre-service teachers' reflections showed that they identified with the character (in this case the students in the articles) and the situations in which they found themselves.

- *Catharsis*

  Through their reading of the articles, the pre-service teachers became emotionally involved and shared and released pent-up emotions.

- *Insight*

  Through their readings and discussion, the pre-service teachers became aware that their problems might also be addressed or solved.

- *Universalisation*

  Using their reflections on the readings, and sharing of their experiences, pre-service teachers were able to connect with each other and find that they were not alone in their feelings and experiences (Wilson & Thornton, 2006).

Wilson and Thornton (2007) identified a fifth stage in pre-service teachers: *projection*. Pre-service teachers' added insight into their own circumstances was

followed by a consideration of what this could mean for the future. The pre-service teachers discussed the implications of the readings for themselves, as teachers, and identified specific strategies that they might adopt in their teaching. The study presented alternate conceptions of the nature of mathematics which influenced the way the pre-service teachers envisaged themselves as potential as teachers of mathematics.

A recent pilot study by Wilson (2007), currently under review, investigated the use of the bibliotherapy technique with pre-service primary teachers in a unit where the focus was on mathematics learning, and not learning difficulties, and found that a pre-service teacher, who expressed anxiety about mathematics, exhibited some of the stages of bibliotherapy.

Discussion and conclusions

The bibliotherapy process is an innovative way of eliciting pre-service teacher reflections. One of the salient features of the process was the emotional responses of the pre-service teachers. Its power is in the way that pre-service teachers' cognitive responses are allied with their emotional response. In comparison to other reflective practices, the potential of bibliotherapy lies in the opportunity for the process to change the way pre-service teachers feel. An explicit focus on learning difficulties, through the process of bibliotherapy, may be a powerful force in addressing some of the mathematics anxiety felt by many pre-service teachers, and thus be a powerful tool in healing (Wilson & Thornton, 2006). There is evidence that this may be achieved, to some extent, even if this is not the teaching or assessment focus of the unit (Wilson, 2007).

Bibliotherapy provides a new framework for looking at pre-service teacher reflections, that has much to offer. It is an effective way of thinking about their prior experiences and looking at the reflective process. It gives teachers a framework, and language, to talk about the process. It provides educators with a shared language to talk about students' emotional responses in terms of the processes of identification, catharsis, insight, universalisation and projection (Wilson & Thornton 2006). It is, therefore, possible that, if one of the parts of the process is missing, educators can select readings to promote the entire process.

It seems reasonable to conclude that overcoming mathematical anxiety among pre-service teachers is important for them to become effective teachers of mathematics. The research suggests that it will help many pre-service primary teachers studying mathematics education to go through a carefully constructed bibliotherapy process. Bibliotherapy provides a process that others could adopt, so is very practical in its potential applications. This process may be at least as important as learning mathematics or how to plan lessons because it has potential to change the way students feel about themselves, and think about themselves, as learners and potential teachers. Ultimately, the power of the bibliotherapy technique lies in its potential to heal and enthuse students (Wilson & Thornton, 2006).

Implications and recommendations

The preliminary research indicates that bibliotherapy has potential to impact positively on outcomes for both beginning primary school teachers and their students in their mathematics classes. An immediate research focus would be the identification of reading materials that impact substantially on pre-service teachers, and some effective

strategies for incorporating this technique into teacher education units, particularly those which have a different content focus.

The potential of individual teachers to have a lasting influence, and the pre-service teacher's awareness that there are alternatives to the approaches that they experienced, were common insights developed by the pre-service teachers. These studies cannot reliably predict how these pre-service teachers will actually teach mathematics in the classroom. A direction for future studies is to research the teaching practices of these pre-service teachers. Follow-up longitudinal workplace-based studies investigating bibliotherapy ,and the change in pre-service teachers' beliefs, and their effectiveness as teachers of mathematics to young children, would provide potential further research contexts.

Future research could also investigate the application of the techniques used in the study, such as critical incident analysis and bibliotherapy through guided reading and journal writing, to investigate their potential to combat mathematics anxiety in primary and high school students. Although pre-service education is the focus of this particular paper, the technique of bibliotherapy has been used already, to an extent, with school students. It has potential to be used to alleviate mathematics anxiety in school students, particularly with the number of children's books that have mathematics as a particular focus.

There is the opportunity to transfer the process to other learning areas, and use it to address issues other than mathematics anxiety. This report does not address teacher in-service, other areas such as primary pre-service teachers' anxiety about science education, secondary pre-service teacher education or parental beliefs and actions, although it recognises that all are important areas for further research, that could be addressed by the bibliotherapy process.

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Using the history of mathematics for mentoring gifted students: Notes for teachers[*]

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The paper presents a theoretical framework, methodology and practical implications for the work with gifted students using history of mathematics. A teaching-learning model, where history of mathematics is integrated in problem-solving activities, is described. Didactical material based on the concepts of triangle geometry is given in the scope of this model. A beautiful and intriguing piece of geometry — the Lemoine point — is the focus of consideration. Its properties are investigated through appropriately designed activities for students. Different examples show the importance of history of mathematics for the development of students' mathematical thinking.

Introduction

Have your students ever asked you about the geometry of the triangle and what Archimedes could have discovered using dynamic geometry software? History of mathematics connects the content of mathematics with its past and can be a source for further reflections on the ways of teaching and learning mathematics from primary to tertiary levels. Nowadays, there is a consensus that the history of mathematics is no longer a remote or tangential issue to the mathematics curriculum and deserves a significant role. Hayes (1991) stated: "I believe that it is a grave mistake and error of strategy to attempt to teach mathematics without reference to its cultural, social, philosophical and historical background."

Undoubtedly, most teachers are aware of the importance of integrating the history of mathematics in the teaching process and many of them successfully use at least some elements in their work. But, how can this best be done in different classroom situations, with different teachers' expectations and students' beliefs? Undoubtedly, the question needs further specification. The main issue we would like to focus on is integration of history of mathematics, in particular with respect to triangle geometry , in classroom inquiry activities of gifted in mathematics students. This paper is written for teachers and has two main aims:

- to provide teachers with practical advice on how challenging material can be organised in a constructivist framework using history of mathematics;
- to attract teachers' attention to the importance of linkage between mathematical and pedagogical content and demonstrate how difficult or challenging mathematical problems can be addressed through their historical background using generalisation, visualisation and systematisation.

We will describe a fragment of a teaching-learning model, where the Lemoine point takes a central place. The Lemoine point is not the end in itself but a means to show the

huge potential in teaching challenging material. It can be presented by the teacher, and seen by students, as a chain of logical reasoning based on the geometry curriculum. We hope teachers will find this paper and the didactical material helpful, either as a direct guide in mentoring gifted students, or as an example to help produce their own materials bringing together history and mathematics.

Theoretical framework

Gray et al. (1999) pointed out that "didactical reversal — constructing a mental object from 'known' properties, instead of constructing properties from 'known' objects causes new kinds of cognitive difficulty" (p.117). We used the idea of "didactical reversal" with respect to historical context of mathematical content. We called it *didactical chronology of a concept*, i.e. we proposed students to build up a successive chain of their argumentation, which would lead them to the discovery of a certain property. In other words, we modelled the same timeline situation as it happened hundreds years ago and modified it slightly so that the final result, i.e. mathematical property, could be discovered. We considered a concept in the context of its historical-mathematical sense (Yevdokimov, 2006a):

- When was a concept posed in a certain problem for the first time?
- Who was the author, and did that author prove/solve the problem on his/her own?
- What other famous mathematicians were interested in the same problem and why?
- How long was that problem known as an unsolved one?
- and the most important issue: How it could be solved.

We called this stage *initial problem situation*. When the initial problem had been solved the next stage began — to find out more what was going on around the context of the problem:

- Which mathematical objects could be taken into further consideration?
- How could those mathematical objects be related to each other?
- Which properties could be suggested to or made known to for students? Or more specifically: Did a certain property follow from the initial problem? How could it be proved?
- Could students introduce some auxiliary elements in order to make the initial problem applicable to a range of situations? (Stoyanova, 2000);

We called this stage *advanced search over situation*. Following Brown and Walter (1990) we proposed a "situation" to mean a localised area of inquiry, within a given historical context, with features that can be taken as given or challenged and modified. We would like to note that there were different directions of students' inquiry work in this stage. To assist the teacher control a "situation" we used Mercer's idea (1995) of "the sensitive, supportive intervention of a teacher in the progress of a learner, who is actively involved in some specific task, but who is not quite able to manage the task alone". We followed Edwards' idea of *conceptual territory before proof* (1997) in conducting the process of students' inquiry activities through historical context in a classroom and took into account that students' exploration and teacher's explanation constituted the main elements that preceded students' discoveries in the scope of inquiring activities.

As advocated by Kronfellner (1996) we distinguished similar patterns in the historical development of the concepts, which could be used to identify the direction of students' inquiry activities in a classroom:

- Implicit use before explicit definition;
- Changeability of mathematical concepts;
- Understandable modification of concepts.

We chose triangle geometry to demonstrate our approach to mentoring gifted students for the following reasons:

- There are more than one hundred distinguished objects (such as points, lines, circles and conics) of a triangle, many of them related to each other (Davis, 1991). Some elements were studied in antiquity.
- Histories of some identifiable objects were recorded and written over a hundred years ago (Mackay, 1892).
- Using technology gives opportunity to model mathematical experimentation through visualisation.

Methodology

All students' activities were carried out as a part of the six months enrichment program for Year 11 students. Fifteen (15) students were selected by their school mathematics teachers to be involved in the program. All students were identified, by their teachers as gifted or, at least, as very good mathematics students. No additional confirmation of that status of each student was required. The teaching experiment was designed for teachers to provide their feedback on the proposed program and encourage them to use the similar ideas in their further work. The methodology consisted of interactions between researcher and teachers, researcher and students. These interactions included interviews with teachers and teaching episodes with students. This teaching experiment was conducted in four parts: observation part, interview part, teaching part, and analysis part. Below we provide the brief details for each part of the experiment.

Observation part

For six months 7 high school mathematics teachers observed 12 teaching episodes carried out with the group of 15 selected students. Teachers were asked to make their notes on 4 different criteria of the program: complexity for teachers, complexity for students, development of students' conceptual constructions, and importance of the historical context. The scale from 1 (strongly disagree) to 10 (fully agree) marks for each index was used. The lowest and the highest marks were removed in each assessment criterion, average values were calculated in the end. Also, we used this part to establish a collaborative relationship with the teachers and the whole class.

Interview part

During the last two months of the program each of the seven teachers was individually questioned in 30 minute interviews. The goal of the interviews was to find out more about teachers' expectations with respect to students' conceptual constructions and problem solving strategies, when history of mathematics was integrated into the teaching process and students were met with challenging problems. In particular, we asked teachers how they could identify that a problem was likely to be hard for students and how historical context could be used in the most effective way in problem solving.

Finally, we took short interviews on students' beliefs in the problem solving process, and attitudes to different forms of problems.

Teaching part

There was one 90 minute teaching episode per fortnight. During eight episodes, each of the 15 students worked individually; in the other four episodes, students worked in small groups (3–4 people in each group). All students were given similar problems, but the questions varied in difficulty, depending on their ways of thinking. We paid much attention to researcher-student interaction to keep it in the scope of the theoretical framework. The most important feature of each teaching episode was that instructions were modified continually, according to the students' performance and the ways of thinking. The need for, and nature of, the modifications were decided on the basis of their answers and explanations.

Analysis part

All interviews and teaching episodes were analysed with respect to teachers' comments and their marks for each criterion. Average values of each index were compared with overall marks of teachers' assessment. Feedback from each teacher, for the further possible inclusion of historical background in their problem solving activities with students, was received. We used teachers' protocol sheets of interviews and their comments on the teaching episodes, teachers' notes concerning students' inquiry work through historical context in a classroom and audio-files of the episodes.

Each student, or small group of students, received the first card having the following content:

- initial problem;
- figure;
- description of historical-mathematical neighbourhood for the initial problem;
- solution of the initial problem (optionally);
- questions-hints for students to encourage their work.

Work on the first cards was the first stage of each episode. Students were given 30 minutes for these activities. The first card was the same for all students. We used 12 first cards for 12 teaching episodes. The second stage of each episode was divided in two parts. Students worked for 30 minutes, making different suggestions and conjectures. For the last 30 minutes students received the second card:

- another problem;
- figure;
- description of historical-mathematical neighbourhood for the given problem;
- questions-hints for students to encourage their work.

The main research focus was to find out how students viewed the links between the two "card problems," both in terms of perceived problem properties, and linking relationships between the problems themselves, one to the other.

Also, any help, hints or ideas could be provided by the teacher verbally either in explicit or implicit form. The teacher's role was to conduct students' inquiry work on both card problems and make the transition between them in the most understandable way for every student.

Didactical implications

In this part of the paper we describe four characteristic examples, which were identified by the teachers as the most important. Also, we give short descriptions of two cards of the teaching episode, where the Lemoine point was investigated.

The following two key ideas are the fundamental ones for understanding the nature of students' inquiry activities in a classroom.

Diversity in forms

A mathematical statement (conjecture, question) can be presented in different forms.

Example A: Three forms of the same statement

1. Find all nontrivial integer solutions for the Diophantine equation $x^2+y^2=z^2$.
2. Find all points, which have rational coordinates and lie on a unit circle $x^2+y^2=1$.
3. Find all right-angled triangles, for which lengths of hypotenuses and both sides are integers (Pythagoras triplets).

It is important to note that the first form relates to analytical interpretation of the statement, while the second and third show visual interpretation of the same statement. Teachers should take into account the impact of these interpretations on students' abilities to distinguish different forms of a statement. Moreover, this example shows how different areas of mathematics are intertwined: the first form is taken from the theory of numbers, the second relates to analytical theory of the second order curves and the third form presents particular cases of the famous Pythagoras theorem in geometry.

Our previous research (Yevdokimov, 2003; 2006b) showed that gifted students, in most cases, were successful in making transition from one form of a statement to another in the problem solving process. Nevertheless, one of the teacher's main tasks remained to develop students' abilities to identify and analyse different forms correctly.

The second key idea (property) is more complex: and incorporates the first property:

Diversity in properties

A concept of a mathematical object (point, line, function, etc.) can be developed to get different properties, related or otherwise.

The teacher's role was to develop students' mathematical thinking and their inquiring abilities. Students were taught to find some properties, and their possible relations to each other, through the appropriate model of teaching environment. Additionally, we considered this idea through its historical context. See Example B.

Example B: Short history of the Lemoine point

In 1809 the French mathematician L'Huillier tried to find a point inside a triangle such that the sum of the squares of distances from this point to the sides of the triangle is the least. In 1820s Gauss was interested in the properties of this point in relation to his method of the least squares. Later German mathematician Groebe (1847) and French mathematician Aussart (1848) discovered new properties of the same point.

In 1852 Belgian mathematician Catalan was investigating the properties of a point, which was the centre of gravity of the triangle *MNL*. He proved that the distances from this point to the sides of the triangle *ABC* are proportional to the lengths of these sides (Figure 1).
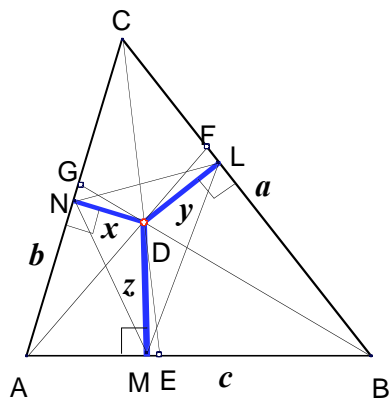
Figure 1

In 1860 the German mathematician Schlomilch proved that if lines are drawn through the mid-points of the triangle's sides and the mid-points of the corresponding heights, these three lines would intersect in the same point.

Finally in 1873 the French mathematician Lemoine discovered that L'Huillier, Groebe, Aussart, Catalan, Schlomilch and other mathematicians were investigating the same point of a triangle! Since that time it was known as the Lemoine point (Cajori, 1907).

This example shows there is a variety of didactical situations in which a certain point inside a triangle can be investigated.

Inquiry activities, in a constructivist framework, are one of the most effective forms in working with gifted students. Many beautiful results in geometry can be discovered using different kinds of generalisation, both visual and analytical. Understanding how different famous historical theorems and facts can be linked to each other through the different kinds of generalisations is important, and can be developed in a classroom. See example C.

Example C: Visual and analytic generalisations for bisectors' property

We would like to demonstrate how the same geometrical object can be a source for visual and analytic extension and generalisations. Consider the angle bisector *CD* (Figure 2a). If "angle bisector" is changed to "angle trisector" and appropriate intersection points joined, we have as result an equilateral triangle (*MKL*) known as Morley's triangle (Gambier, 1954; Figure 2b).

On the other hand, we can make an analytic generalisation with respect to the point *D* of bisector *CD*. We can use the well-known bisector's *property*: "A bisector of a triangle divides the opposite side onto the parts, which are proportional to the corresponding adjoining sides of this triangle".

For generalising we consider the case, when the parts of each side of a triangle are proportional to the squares of the corresponding adjoining sides of this triangle. The result is the Lemoine point of a triangle, which is a point of intersection of symmedians.[11] Construction is shown in Figure 2c.

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[11] A symmedian of a triangle is called a segment inside a triangle, which is symmetric to the median with respect to the corresponding bisector and going through the same vertex of a triangle.
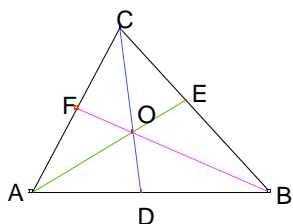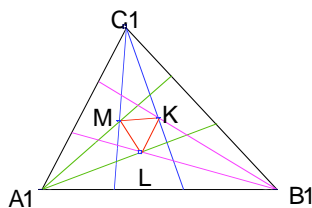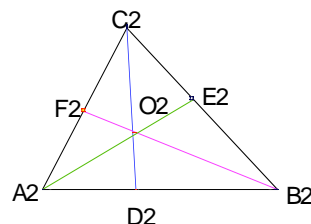
Figure 2a.   *Figure 2b.*   *Figure 2c.*

Also, we would like to show how different levels of visual generalisations can be used by teachers in their guidance of students' inquiry activities. See example D.

Example D: Levels of visual generalisations

There is a triangle *ABC* inscribed in a circle. Points A_1, B_1, C_1 are feet of the corresponding perpendiculars PA_1, PB_1, PC_1 to the sides of the triangle (Figure 3).
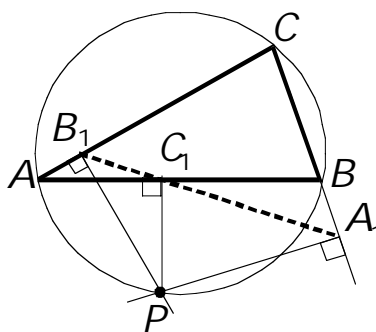

Figure 3

There are two levels of generalisation here. The first one can be easily presented visually. Any point of the circle has the same *property*: "Feet of the corresponding perpendiculars are in the same line". This line is called the Simson line.

Thus, the first generalisation relates to the circle only. The second and higher level of generalisation is hidden in the figure. However, it is very important to note that a generalisation might be possible for the points outside a circle. This is the other level of visual generalisation.

A teaching-learning model: Triangle famous points — a case of the Lemoine point

We give an example of two cards of the teaching episode in which a teaching-learning model for the Lemoine point was constructed and used in classroom activities.

The first card

Initial problem (L'Huillier): Find a point inside a triangle such that the sum of the squares of distances from this point to the sides of the triangle is the least.
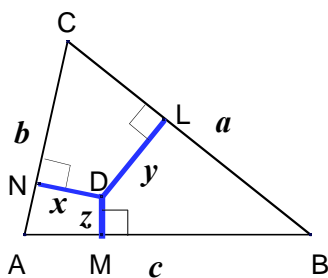

Figure 4.

Very brief description of historical-mathematical neighbourhood was given in Example B.

Questions-hints for students:

- What is the aim of the problem?
- The sum of the squares of distances from this point to the sides of the triangle should be the least. We should find the point. How can we find it?
- What should we begin with?
- "To find a point" means to specify its location with respect to at least two sides of a triangle;
- "To find a point" means to analyse an expression $x^2 + y^2 + z^2$ for a minimal value.

The second card

Problem (Lemoine): A segment *BK* is called a symmedian if *BK* is symmetrical to the median *BM* with respect to the bisector *BL* and going through the same vertex *B*. Show that all symmedians intersect in the same point.

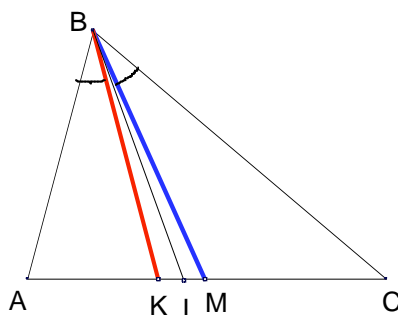A very brief description of historical-mathematical context was given in Example B.

Figure 5

Questions-hints for students:

- How does the area of the triangle depend on the expressions $x^2+y^2+z^2$ and $a^2+b^2+c^2$ from the first card, if *x, y, z* are distances from the Lemoine point to the sides of the triangle and *a, b, c* are lengths of the sides of the triangle?

- Draw the lines connecting corresponding vertices of a triangle with the Lemoine point. How do these lines divide the sides of a triangle?

Concluding remarks

Summing up the results of teachers' response on the enrichment activities we can confirm that almost all teachers approved the program and expressed interest in developing classroom inquiry work within an historical context. Teachers supported the idea that an historical perspective can help us in the work with gifted students. It was found that mentoring gifted students looks very promising and easy to follow, if we use a teaching-learning environment, which provides a logical structure and historical background for inquiry activities.

More activity development and research is needed, and more is planned.

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Working smarter, not harder

Alexander Young

Society's pressure on schools to widen the school curriculum has meant that teachers have come under increasing pressure to work even harder. Whether the poor performance of Australian students' in the 2002 TIMSS tests are, in part, related to this, is unclear. The aim of this paper is to demonstrate how teachers can work smarter, not harder to achieve considerable improvement in student outcomes.

This paper will show it is possible to process the responses of 30 students' work in less than 2 minutes using AutoMarque software. This computer software program transfers paper-based multiple choice responses to computer using a photocopier or sheet fed scanner. Within two minutes AutoMarque will:

- *reveal the quality of prior teaching by providing the results per question and the class performance per strand of learning.*
- *indicate the quality of each question.*
- *expose gaps in individual student learning which usually go unnoticed when assessing for learning.*
- *supply "weighted student results" by taking into account the difficulty of each question.*

Introduction

Most of my work over the last five years can be summed up in Gullick's comment that: "research shows the only thing proven to improve students' learning was improving teaching — not smaller classes, charter schools, vouchers, smaller learning communities or even curriculum reforms" (Gullick, p. 1). Therefore, if we are to help teachers to improve their teaching, helping them to work smarter, and not harder, should be the way ahead.

Frequent formative assessment is often both time-consuming and frustrating. This need not be the case. Assessment for learning, once set in place, can be easily and quickly achieved, resulting in improved student outcomes.

This paper is designed to demonstrate that teachers can:

- easily achieve quantified self analysis of their teaching
- save considerable time in assessment for learning
- identify previously undisclosed gaps in student learning
- quantify the quality of their questions and the overall reliability of assessments
- obtain weighted and non weighted student results.

In the current climate, where government driven summative assessment is sweeping the Western World, the education profession has responded, in many instances, by having teachers work even harder as a UK study found: "Teachers and headteachers work more intensive weeks than other comparable managers and professionals" (Pricewaterhouse Coopers, p.1).

At best, summative assessment can be seen as a post course marker. Assessment for learning, or formative assessment, is seen as the path to learning achievement. Broadfoot states "Genuine improvement in the effectiveness of learning actually requires a major rethink in the way assessment is used."

Therefore, systems which are based on a summative assessment are "fundamentally flawed in that they do not take into account the relationship between assessment and learning" (Broadfoot, p. 24)

The Australian House of Representatives Standing Committee on Education and Vocational Training (2007), in its report "Top of the Class," listed the features of high quality teaching practice. A high quality teaching practice: "integrates theoretical knowledge and professional practice across the three domains of a teacher education program; 'content' knowledge gained through a liberal education, professional knowledge, pedagogical skills and insights" and "assesses against clear delineations of purposes, roles and expectations of TEI student activity and performance" (House of Representatives Standing Committee on Education and Vocational Training, pp. 73–74).

In his book "Evidence Based Teaching," Petty cites Hattie as an international pioneer and world expert on successful learning. He quotes Hattie as subscribing to three principles:

- Achievement is enhanced to the degree that students and teachers set and communicate appropriate, specific and *challenging* goals.
- Achievement is enhanced as a function of *feedback*.
- Increases in student learning involve … *reconceptualisation* of learning (Petty p. 63).

It appears clear that if teachers can adopt Hattie's methodology their teaching effectiveness will improve. In other words, successful learning:

- sets out a series of "mile posts" (*challenging* goals) in their teaching plan
- verifies student achievement through frequent *feedback,* and
- *reconceptualises* their teaching as a result of this feedback

Revealing that which would otherwise not be known

Broadfoot asserts that "Teachers' professional understanding in relation to assessment has typically been one of the least well-developed aspects of their expertise" (p. 35). The best teaching practice is limited, especially when dealing with students whose basic grounding is missing. This is a major cause of student disenchantment, disruptive behaviour, absenteeism and early drop out from school.

One of the contributing factors to this state of affairs has been our profession's inability to identify specific gaps in student learning that were not diagnosed or addressed, early in a student's education. Consequently, the classroom teacher is faced with this dilemma: "Where do I start to try and fill in the gaps in students' grounding, when they are moved up the ladder of schooling regardless of their level of success?"

Given the results from the House of Representatives Standing Committee on Education and Vocational Training, the PriceWaterHouse Coopers UK study, and Hattie/Petty's research, educators need to have time-saving and insightful resources. These would assist them to be efficient and effective practitioners rather than just overworked and bogged down with unrealistic expectations of the general public.

Having struggled with this dilemma for a long time, I left the classroom scene and, with the benefits of computers, programmers and a son who had always said, "There has to be an easier way," I developed that "easier way." Let me tell you about it.

AutoMarque is a highly effective time saving and insightful resource to assist teachers in their assessment for learning.

Using AutoMarque, teachers are now able to have students' paper based responses to multiple choice questions processed, in less than two minutes, using their photocopier or portable sheet fed scanner connected to a computer which stores the results. By using a multi choice test of their own choice, teachers can quickly obtain an insight into the quality of their teaching effectiveness
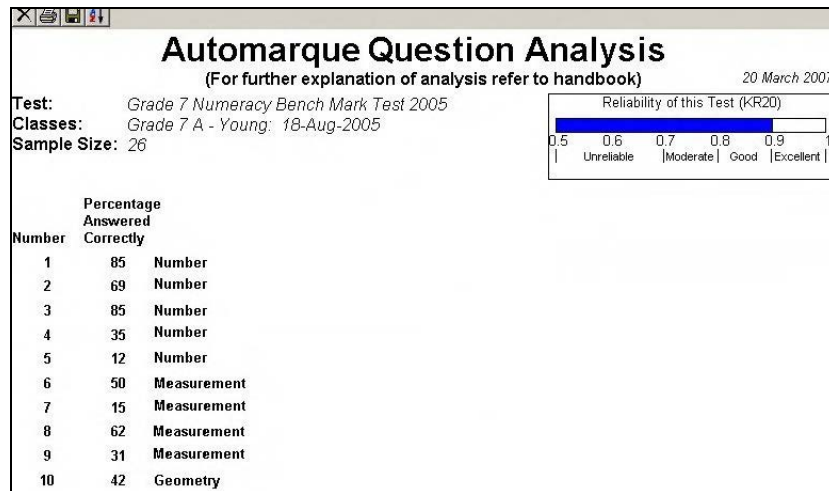
Figure 1

Figure 1 shows clearly the students' results per question and the strand of learning related to the question. An indication of the overall reliability of the test is also shown.

Reliability of the test is calculated by AutoMarque using the Kuder Richardson formula 20 (KR20) test of reliability. KR20 is the most accurate of the practical Kuder Richardson formulas for estimates of reliability. It measures consistency of responses to all the items within the test. It is the mean of all possible split half coefficients (Athanasou and Lamprianou, p. 315).

Figure 2

Figure 2, shows how to easily identify the questions that were least well performed by the class. This is achieved by clicking on the right hand icon.

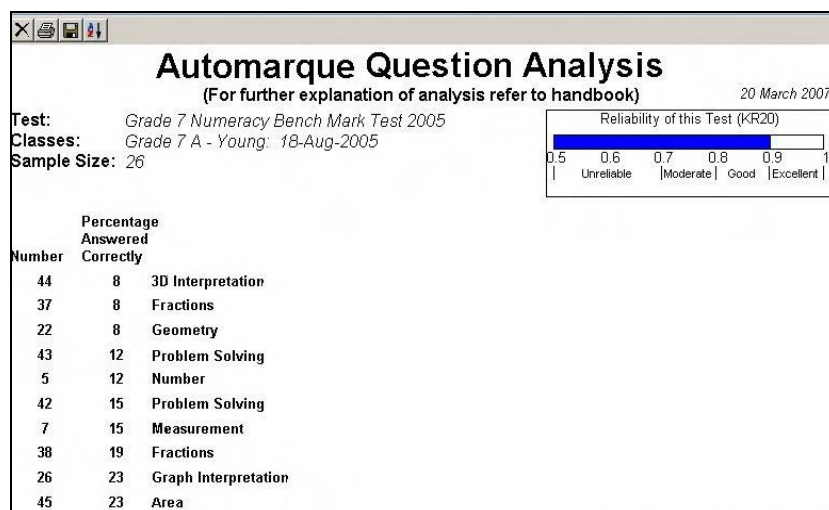When a teacher receives this information in real time, the gaps of learning are obvious, inviting remediation or as Hattie put it "*reconceptualisation* of learning" (Petty, p. 63). Such action by a teacher will help to better retain the focus of students on learning.

When compared with the traditional practice of revising the whole test, teachers are now able to define areas of weakness in learning for the whole class and concentrate their teaching resources to address these only.

By clicking on an icon, an educator can gain further insights into their overall teaching effectiveness. Figure 3, gives a summarised version of the class' success by strand. In this case, the teacher needs to reconceptualise his/her teaching in the three strands of learning on which the class performed poorly.
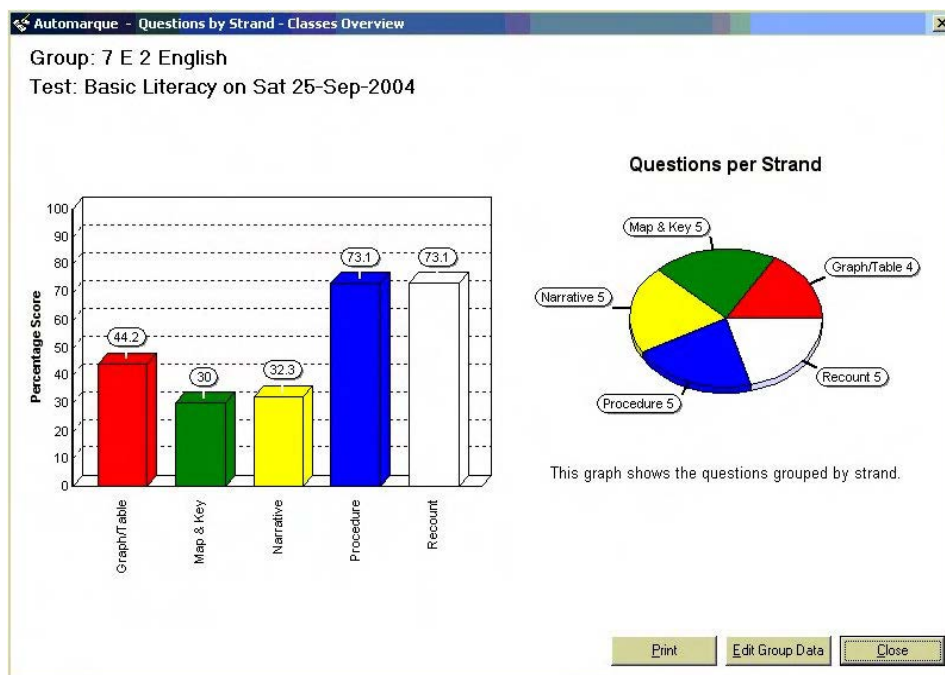

Figure 3

Using the two facilities shown in Figures 2 and 3, teachers are better informed as to their teaching effectiveness. These insights enable teachers to reflect on which teaching methods are appropriate for their students, and to help them to decide to explore alternate strategies, where necessary. As Hattie would define it, "*reconceptualise* the learning." I believe this aspect of AutoMarque will be of immense assistance to all teachers, as it provides new insights into their professional practice.

Within the teaching profession there are many highly qualified people who are gifted in their particular subject area, but who have poorly developed teaching skills. These same teachers' self awareness has potential to be improved using AutoMarque, giving them a new lease on life. These insights enable self assessment and thus reduce the need for a peer assessor in a classroom: the assessment results speak for themselves. If AutoMarque were adopted school-wide the whole school has the potential to improve student achievement. Furthermore, AutoMarque has the potential to improve teaching effectiveness across the profession.

Students at the centre

Assessment of and for learning, as detailed above, has great potential to assist teachers improve the quality of their teaching, but will not necessarily deliver insights into individual student needs. Individual student results, by strand of learning, compared with their peer group are obtained on the click of a mouse (Figure 4). This provides a powerful insight into an individual student's needs. In this instance the child's teacher was not aware that this gap existed. As Chris had out performed his peer group in all the other strands of learning, his deficiency was masked.

How many other students' needs are going un-noticed?
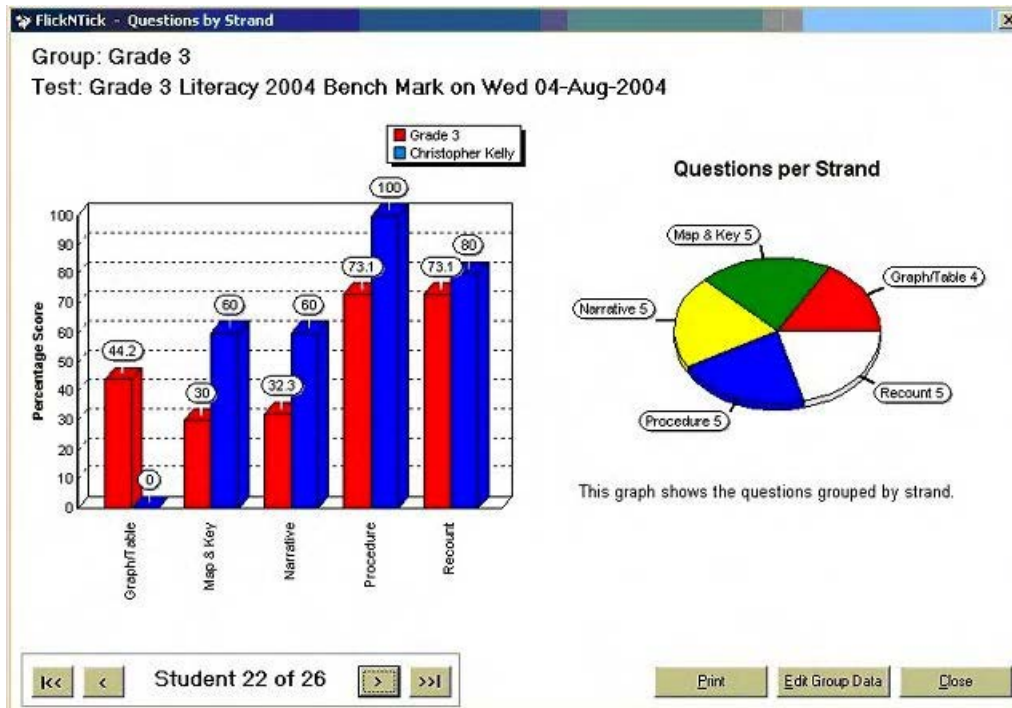
Figure 4

AutoMarque's student strand analysis result sheet is of great assistance when counselling a student, his/her parents and the teacher's supervisor, that intervention is required. As AutoMarque stores test results, a history of student achievement is readily retrievable and student progress easily, and clearly, demonstrated.

If the teacher knows the school, state or national average achievement per strand of a standardised test, the teacher can edit the group data so the individual student's results can be compared with that group.

Identification of quality questions

There are considerable resources available on the Internet, often in PDF format, for teachers to acquire. The quality of these questions can be assessed by AutoMarque. AutoMarque requires a minimum of 100 students to have completed an identical test before the question quality analysis can take place.

In Figure 5, five classes have completed an identical test. As the group using the test exceeds 100, there is an analysis of each question for 130 candidates as well as an indication of the test's reliability.

Figure 5

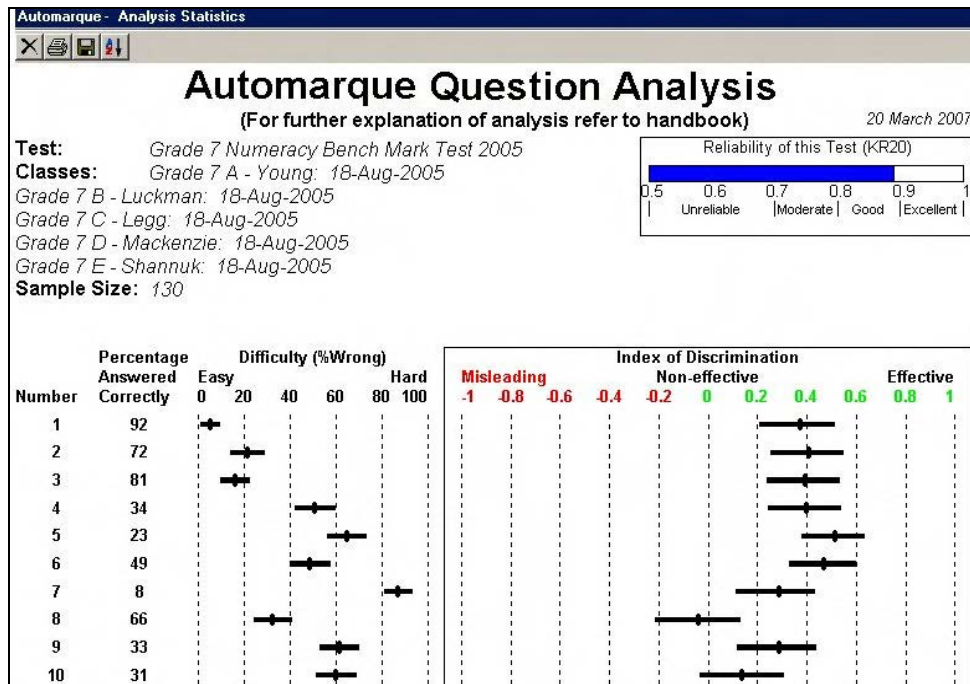AutoMarque expresses the difficulty of a question as a percentage of the students who answered incorrectly. For discrimination, the software uses a Point Biserial Coefficient of Correlation between the correctness of the response to the given question and the students' result in the test as a whole (Athanasou & Lamprianou, p. 309).

The confidence intervals displayed are indicated by the length of each line, per question, for difficulty and discrimination. The line's length is inversely proportional to the square root of the sample size.

With these features, teachers can use many sources of tests produced by others and use this facility to verify the quality of test questions. This should help raise the quality of their work and lead to consequent improvements in students' outcomes.

Most photocopiers scan a sheet per second, so school wide moderation assessments can be easily marked. After selecting an appropriate test, the teacher co-ordinator only needs to photocopy off the number of response sheets required. Once the test has been completed, the results sheets are fed through the photocopier and the results are stored in the teacher's computer. The major advantage of on-campus moderation testing is to verify that assessment of learning has occurred.

Weighted results

AutoMarque provides weighted and non weighted results. The weighted results are derived by either of two methods. The simplest method is when teachers use tests with known difficulty ratings for each question; e.g., ACER PAT assessments. The levels of difficulty are entered into the software, before marking the test. The other method is designed for tests where the level of difficulty is unknown in advance. For this to be accurate, at least 100 students need to have completed the same test. The weightings are obtained by, firstly, having AutoMarque complete Question Analysis (Figure 5) and then to click on the floppy disc icon in the left corner of the screen. The weighting is calculated by using the level of difficulty of each question.

Some of the results per student are shown (Figure 6) with weighted results shown as "W%". This column of results will change to unweighted when one "double clicks" on its title. There were 30 students in the class shown and the software automatically averaged the results.

Figure 6

Research using AutoMarque software has found that when weighted scores are used, the spread of student results usually increases.

The work of Thomson and Fleming has demonstrated that Australia has a "large tail" of intermediate and low performing students in the TIMSS 2002 tests. This "tail" appears to be larger than many nations (Figure 7). In this case it is dramatic compared with Singapore.

Figure 3.4 Proportion of Year 4 students reaching the international benchmarks in mathematics by state
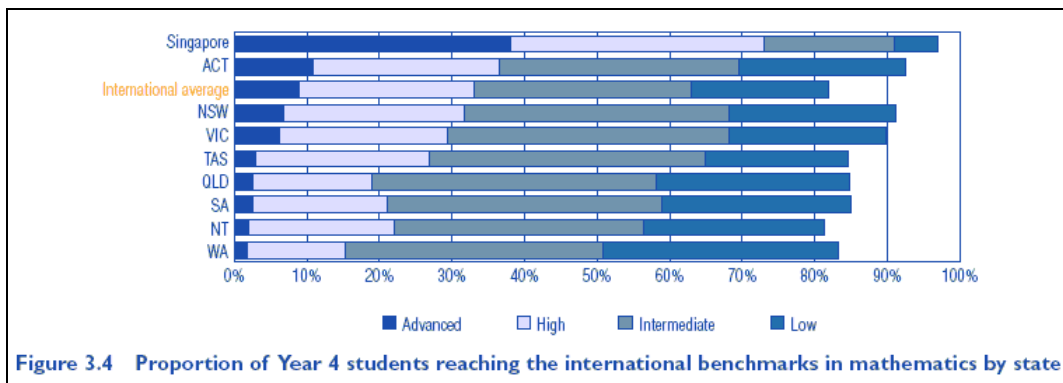
Figure 7 (from Thomson & Fleming, p. 68).

There may be many reasons for this, two being of prime importance. These are:

- society's pressure on schools to widen the curriculum has meant that teachers are required to address a variety of issues, traditionally not the role of the teacher;
- teachers have not had insightful tools available for their use to conduct assessment for learning, quickly — this appears to have resulted in large numbers of student needs not being addressed.

The potential benefit of AutoMarque's weighted results is that the less well performing students will stand out more readily, as shown in Figure 4. In this way, their

needs may be better understood and effective remediation appropriately applied. This is something that appears to be of high importance, if the results of Thomson and Fleming's work are to be taken seriously. Thompson and Fleming's work indicates how a large proportion of the student population are not reaching their full potential. After all, our only reason for being, as a profession, is to help each child reach his/her full potential.

New horizons

For those teachers who would like to be able to conduct research on student learning, AutoMarque enables them to transfer the student responses into a spreadsheet for further analysis. For those wishing to conduct Rasch Analysis, for example, it is only a matter of loading the spreadsheet generated by AutoMarque into a Rasch Analysis program. This removes the need for manual transcription. Transcription errors are also minimised, improving the precision of the results.

Also, AutoMarque can be used for conducting surveys as the results can be automatically placed in a spread sheet and therefore ideal for:

- surveying student and parent opinion;
- processing optional subject choices;
- collecting and sorting sports carnival choices;
- student research;
- teacher academic research;
- conduct of student elections.

Conclusion

This paper has demonstrated how easily, and quickly, assessment for learning can be conducted. This allows teachers to work smarter, not harder. At the same time, AutoMarque has the potential to improve student learning. The analysis by strand of learning, for each student, has been found to be particularly useful in highlighting individual student learning needs.

Most importantly, AutoMarque provides fast and insightful feedback on teaching effectiveness. It encourages teachers to reconceptualise learning in the areas which the software has shown to be poorly understood by their students. AutoMarque has the potential to improve teacher understanding of assessment and its role of assisting student learning.

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Workshops

Mathematics extension groups at Melbourne High School[*]

Ian Bull

Melbourne High School

Melbourne High School (boys only) is one of two select-intake government schools in Victoria where students achieve outstanding results across all studies. As the coordinator of the High Achieving Student Program, I have had the privilege of working with the mathematically elite in the school at Years 9 and 10 for many years. I have devised materials that are undertaken by students, chosen to participate in the program, with such questions as: "If a capacity crowd at the MCG was to be evacuated onto the playing surface of the ground, would they fit?" Given neither dimensions nor formulae is a favourite and frees up the minds of students to discussion/argument. All aspects of the program will be discussed.

Melbourne High School (boys only, Years 9 to 12) is one of two select-intake government schools in Victoria. Melbourne High School's curriculum is derived from the fact that at Year 9 students arrive after satisfying the standards of the entrance examination at a number of different, albeit a high, set of different levels. The general curriculum offering at Years 9 and 10 is designed to address transition issues and is based on the expectation that students will have the ability, and the desire, to be challenged through a broad education. It is school policy that students not be accelerated beyond their current year level; that is Year 9 and 10 students study concepts suitable to those levels, however, given the level of performance of the students at these levels, the mathematics faculty supplies additional enrichment material throughout the courses designed to apply the concepts to more complex tasks. This is delivered via the text series: Maths Dimensions 9 and 10 (Pearson) where an enrichment section follows each chapter.

In addition, the mathematics program targets the top 10% of the students in Years 9 and 10 to undertake further tasks of a more challenging nature. These "high-achieving" students are presented with a program of non-routine tasks. The groups are called the Year 9 and Year 10 maths extension groups, respectively.

The mathematics extension groups

As a lover of mathematics, along with the language of logic, the opportunity to work with students of a high calibre is a fantastic opportunity for advancement to all involved. To work with these students, where they argue and fight over the finer points of mathematical thought, is a special experience.

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[*] Paper accepted by peer review

Picture a room full of students from Years 9 or 10: assembled separately once a fortnight, working – thinking – arguing – seemingly fighting (intellectually) and there you have it. The students engage in a mathematical jousting match either teaming up with others or working alone to tackle the current fortnightly challenging problem(s). For these boys mathematical conundrums are the vehicle for discussion, exploration and learning. A difference of mathematical opinion is an occasion for open debate — a free for all — a time to convince other participants of the validity of the logical conclusions of their solution. This is the ultimate meeting of mathematical minds, where all are challenged to reach their potential.

Structuring the mathematics extension programs

Operating the Year 9 and 10 programs requires the selection of appropriate students, to find a time that suits all participants, to provide interesting material and, finally, to have the energy and belief to oversee such a program. Let me examine each dimension separately.

Selecting the students

Student selection into the program needs to be fair and open to all students in the year level. It must not be subject to any form of favouritism. A method of selecting students to participate in the program is done by:

- undertaking a series of tests — IQ, reference to entrance exam data, other externally derived tests;
- teacher recommendation — this is a very accurate indication but needs to happen after a number of weeks in to the semester. The mathematically elite students show themselves to their teachers in the classroom environment — it is the most efficient way to find them.

Finding a time for the program

In general, school programs are under continual pressure, where the time required for additional programs such as drug education, pastoral care, work education, work experience, sex education, driver education, and so on, impinge on the time that mathematics occupies in the curriculum. Time for a program such as this cannot easily be found within the normal teaching program. At Melbourne High School the only time to run this program is a negotiated fortnightly lunchtime, as the timetable operates on a ten day cycle, which suits all selected students.

Having students commit to the program

Students can start off well committing to the program, however, this initial enthusiasm can ebb away. Students need to commit to the full program and this is ensured by the completion of a contract whereby the supervising teacher and each student, and their parents, sign an agreement detailing aspects of content, assessment requirements and the date or timing of the program. This process formalises the program and overcomes problems later when students withdraw from it, yet claim credit for the part that they have completed. My time is valuable and I require a full commitment from the participating students to complete the program and show their work in progress for each task. This is an attempt to overcome plagiarism of others' work where students show their initial response to the problem.

Sample contract

Melbourne High School High Achieving Students Program, 200

STUDENT NAME: Thomas Ng-Santomartino   FORM: 10H

SUBJECT/COURSE AREA: Mathematics

SUPERVISING STAFF MEMBER: Mr Bull

The content to be covered

Thomas has been selected into the Year 10 Maths Extension Group wh
once a fortnight at lunchtime during Terms 2 and 3. Eight challenging questions will be
distributed during these sessions covering a wide variety of unusual and non-routine
Maths topics.

Assessment method

To obtain a satisfactory grading:

• A complete response to the eight extension questions needs to be submitted including
   a reflection statement for each task.
• All work is to be submitted in a bound workbook with an index showing the
   placement of all questions undertaken.
• Parents/guardians need to sign the beginning and end of each task in the workbook to
   show that they have seen the work undertaken by the student.
• The supervising staff member the student and their parent/guardian need to provide
   comments on the completion and reflection of their work both before and at the
   completion of their work and sign the following relevant sections.

Date for submission of work

All work needs to be submitted at the end of Term 3.

BEFORE WORK ON THE UNIT OF WORK

STUDENT STATEMENT:

I am pleased to be selected into this group of the best maths students in Year 10 and will
work through all the work that is expected.

                                   Signature:

PARENT STATEMENT:

We will track the progress of Thomas as he works through the tasks given to him.

                                   Signature:

AT THE CONCLUSION OF THE EXTENSION UNIT:

STAFF COMMENT:

Thomas Ng-Santomartino *completed all the questions required on time showing a full*
range of sophisticated methods of solution. Well done, you are a talented student and
your enthusiastic approach shows that you enjoyed the challenge.

STUDENT COMMENT:

The work was at times challenging but I enjoyed working through each question.

PARENT COMMENT:

Thomas Ng-Santomartino *worked through this work with us looking over it. We are*
pleased to see the school providing him with work of a more difficult nature and have
looked through his bookwork.

Material

The material presented to the students is designed to parallel the concepts, and the levels, covered by the students in their mathematical studies. With the program operating during terms two and three, the first section deals with topics covered during the first semester, whilst the second section deals with the topics studied in the second semester. The tasks have been, and are being, developed over a period of time and have been sourced from a variety of materials: mathematics competitions, enrichment materials, problem solving books and interesting questions taken from mathematics texts, and so on. A small list is included at the end of this paper.

Formulating tasks

Mathematical tasks are around us all the time. The more traditional ones are sourced from texts and competitions and altered to suit the level of the students in the program. More interesting tasks are the ones that we stumble on in everyday life. An example of this type of task was conceived as I sat at the MCG, before a football match, listening to the emergency evacuation announcement.

This is a favourite task which is presented to the students at the start of the program. It would be unlikely that all the stands at the MCG would be affected in this way, but the task is unusual, and accessible, being based on measurement concepts. No information is given — the students need to estimate all measures — the capacity of the ground, the dimensions of the ground and the formula for the area of an ellipse. This is where team work in groups pays dividends.

The students glue this task into a bound workbook and are then given twenty minutes to make preliminary explorations of the task working either in groups or on their own. The workbook is then ruled off and the students are able to complete work on the project in their own time, before it is finally sighted and signed off at the start of the next fortnightly session.

Task 1

If there was a disaster at the Melbourne Cricket Ground on Grand Final Day and the people were asked to leave all their possessions and stand on the ground, would they fit?
Draw a diagram and estimate the dimensions involved and justify your answer.

Mathematics: Essential for Learning, Essential for Life

Solution: Task 1

The initial estimate required is that of the dimensions of the ground. The fifty metre line marked at each end of the ground could be used as a guide.

The Melbourne Cricket Ground is in an elliptical shape as shown.
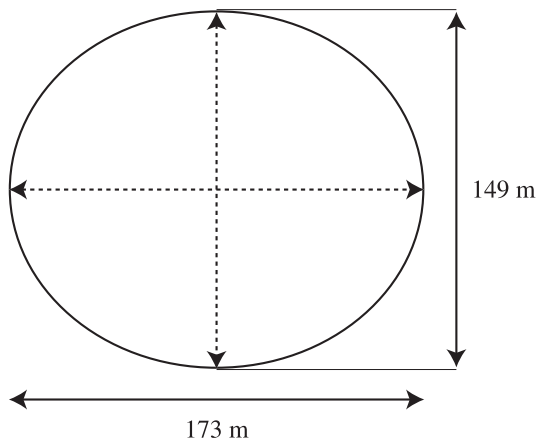
The area of an ellipse is as follows:
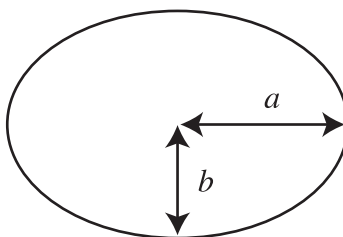
$$\text{Area} = \pi ab$$

The accurate area of this ellipse is: Area $= \pi \times 74.5 \times 86.5 \approx 20\,245$ m^2

The next question then is to estimate the number of people that could fit into one square metre. It would perhaps be reasonable to suggest that 4 people could fit into one square metre or that each person would take up 0.25 m^2 of space.
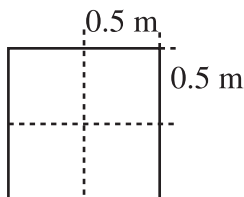
This would certainly be true if some of the people were children, although the issue of children keeping still, whilst packed tight on the ground with the emergency going on around them, could be debated.

Theoretically then, $4 \times 20\,245 = 80\,980$ people.

Mathematical arguments — such as to the reasonableness of this estimate or to how people could be assembled on the surface to enable such packing — are invited and would need to be made in order to justify the solution; it is up to the students to justify their solution.

Case study — formulation of an extension task, Task 2

Often in our teaching of classes at a higher level interesting questions are raised which can be adapted to tasks of lower levels.

This task grew out of a question asked by a Year 11 student whilst working through the topic of sequences and series in a General Specialist Mathematics class from the text *General Maths Dimensions — An Advanced Course* (Bull & Nolan). Often questions and tasks examined at a higher level can be repackaged into a task suitable for use in the Maths Extension Program.

Task 2

An arithmetic sequence is a set of numbers which continues in a pattern by adding a constant difference to generate the next term. The terms of an arithmetic sequence with a first term of a and a common difference of d is:
{*a, a + d, a + 2d, a + 3d,…*).
The symbols for the first term is t_1, t_2, t_3 with t_n being the nth term in the sequence.

(a) Show that the nth term of an arithmetic sequence is given by $t_n = a + (n-1)d$, hence find an expression for (i) t_{n+1} (ii) $t_{\frac{a}{d}}$ (iii) $t_{-\frac{a}{d}}$.

The sum of n terms of an arithmetic sequence is given the symbol S_n.

The general rule for the sum of n terms is: $S_n = \dfrac{n}{2}\left(2a + (n-1)d\right)$.

(b) The sum of an arithmetic sequence is $2n^2$. Using the methods (i) of generating the sum of terms in the sequence and (ii) using algebra find the values of a and d hence find the first three terms of the sequence.

Solution: Task 2

(a) $t_1 = a$, $t_2 = a + d$, $t_3 = a + 2d$, $t_4 = a + 3d$,… so by observation the coefficient of d is one less than the term number, i.e., n. Hence

(i) $t_{n+1} = a + \left(n + 1 - 1\right)d = a + nd$

(ii) $t_{\frac{a}{d}} = a + \left(\dfrac{a}{d} - 1\right)d = a + a - d = 2a + d$

(iii) $t_{-\frac{a}{d}} = a + \left(-\dfrac{a}{d} - 1\right)d = a - a - d = -d$

(b) This produces an interesting mathematical scenario.
(i) $S_1 = t_1 = 2(1)^2 = 2$
$S_2 = t_1 + t_2 = 2(2)^2 = 8$, $\therefore t_2 = 8 - t_1 = 8 - 2 = 6$, so the sequence is (2, 6)
$S_3 = t_1 + t_2 + t_3 = 2(3)^2 = 18$, $\therefore t_3 = 18 - (t_1 - t_2) = 18 - 8 = 10$, so the sequence is (2, 6, 10)
Hence $a = 2$ and $d = 4$.

(ii) $S_n = \dfrac{n}{2}\left(2a + (n-1)d\right) = an + \dfrac{dn^2}{2} - \dfrac{dn}{2} = 2n^2$
Equating coefficients:
n^2: $\dfrac{d}{2} = 2$, $\therefore d = 4$

n^1: $a - \dfrac{d}{2} = 0$, for $d = 4$, $a - 2 = 0 \therefore a = 2$

The sequence using the first term of a with a common difference d of 4: (2, 6, 10)

In this task for part (b) a Year 11 student approached the question from a formula approach — because for him the solution of mathematics problems was based about the application of a suitable formula.

Reaching the step: $an + \dfrac{dn^2}{2} - \dfrac{dn}{2} = 2n^2$, with one equation and three unknowns, this working step stopped this student in his tracks. I showed him the Part (b) (i) approach which required him to think of the information in a different way — using a problem solving approach and then wondered why his algebraic approach had met a dead end. I then saw the algebraic line in terms of powers of n and achieved a solution by equating the coefficients of n as shown in the solution above.

This example was presented to the class, where they were led through thinking processes highlighting where, if a problem can be appreciated from various views, multiple solution processes can be applied. This led to a discussion of the power of mathematics as a problem solving tool — an algebraic or "apply the formula" approach is a disappointing simple method which most students depend on to solve the type of questions that we often manufacture.

Often in the traditional, mainstream, classroom with time restrictions and so on, mathematics is reduced to the use of formulae. The strength of this Extension Program is to present students with a series of non-routine, thought providing, experiences which are intriguing and require the students to be adaptable thinkers, according to the problem situation.

Reference materials

These are best collected over time from a variety of sources, both local and overseas. A short list might be:
Mathematics competitions such as
- Australian Mathematics Competition: AMT
- University of NSW: ICAS
- Maths Challenge: AMT
- Maths enrichment stages: AMT
- IBM – Melbourne University Maths competition

Useful reference materials
- Set of books: Can you solve these: Tarquin publications
- Edwards, King O'Halloran. All the best from the Australian Maths Competition. Australian Mathematics Trust.
- Soifer. Mathematics as Problem Solving.
- Canadian Mathematics Competition. Problems, Problems, Problems. University of Waterloo.
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Out of the square, into reality: Bringing mathematics and science into the real world[*]

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The project "Out of the Square and into Reality" is a dynamic and collaborative partnership between the Queensland University of Technology, five Brisbane schools (West Moreton Anglican College, St Peters Lutheran College, Kelvin Grove State College, Marsden State High School and Moreton Bay College), private industry and a state government agency. The project has been selected as a participant in the Australian School Innovation in Science, Technology and Mathematics (ASISTM) project, which is funded by the Commonwealth Government through the Department of Education, Science and Technology.

Introduction

The project's creation and its collaborative nature are an attempt to arrest the significant problem facing secondary schools, universities and industry of attracting and retaining high quality and appropriately prepared individuals (Hylton & Otoupal, 2005; Orlich, Thomson & Zollars, 2005; Zarske, Kotys-Schwartz, Sullivan & Yowell, 2005). The problem begins in secondary school, with a declining number of students, particularly girls, studying senior mathematics and science subjects (Felder & Brent, 2005; Goodrum, Hackling & Rennie, 2000; Orlich et al., 2005). The underlying issues for this predicament include: deteriorating school student interest; negative student perceptions towards gender difference; perceptions of students who study mathematics and science as being "nerdy"; and competition from a new wave of business subjects and historically "female-friendly" humanity subjects (Hylton & Otoupal, 2005; Jolly, Goos & Smith, 2005). This is further exacerbated by students lacking information about the diverse and important roles undertaken in the world by engineers, mathematicians and scientists. With their teachers, parents and peers as the only available information sources, secondary students have scant appreciation for these professions (Millican, Richards & Mann, 2005). Consequently, this has a flow-on effect for tertiary institutions, which compete with other faculties for a dwindling pool of appropriately prepared students who can make a successful transition to tertiary study. Furthermore, the problem is compounded by a declining youth interest in the professions (Goodrum et al., 2000; Zarske et al., 2005). Finally, industry competes for the resulting shortage of capable and qualified individuals, thereby inheriting the problem (Hylton & Otopul, 2005). Coupled with the current global resource and industrial booms, have increased the demand for these professionals, the significance of the problem becomes quite clear.

These complex and interwoven problems facing secondary schools, universities and industry, form the basis of the project's aims to:

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[*] Paper accepted by peer review

- enhance student interest and engagement in mathematics, science and engineering;
- increase the real-world practicality and enhance the pedagogy and perception of senior mathematics, science and engineering subjects in secondary schools;
- increase secondary student participation and success in studying senior mathematics, science and engineering subjects;
- develop the ability of students to be innovative and "think outside the square";
- assist students to gain an appreciation of the important role that engineers, mathematicians and scientists play in today's and tomorrow's world;
- facilitate the opportunity for teachers to work collaboratively with and learn from real-world academics and professionals;
- foster quality teaching through improved approaches and techniques, whilst attracting and revitalising quality teachers;
- increase the number of quality and appropriately prepared secondary students making the successful transition to tertiary study.

Project's theoretical basis

Current pedagogical theory and practices have informed and inspired the project's development. The theoretical basis for the project's teaching and learning is constructivism. Constructivism asserts that students assimilate new information to build on their existing knowledge and understanding, thereby constructing knowledge. Accordingly, this knowledge is not passively received from the teacher (Al-Weher, 2004; Cobern, 1995; Lesh & Doerr, 2003). Furthermore, each individual's process of learning is different, as each individual builds their own understanding, based on their own interpretation of the situation, along with their pre-existing knowledge and understanding (Watts, 1999). Consequently, this view of learning requires the roles of students and teachers to change. Students need to actively participate in and direct their learning process, so that they are able to construct new and improved knowledge through problem solving, discussions, and the design and implementation of projects (Al-Weher, 2004; Cobern, 1995). Conversely, teachers need to move away from their traditional position as "transmitters of information" and the conventional application of a "linear sequenced approach" to instruction. Instead, teachers should move to a facilitating role, arranging suitable conditions to enable students to be involved and to direct their learning process (Hand & Treagust, 1994).

Constructivist theory has significantly influenced the design aspects of our project. This is evident in the combination of student-directed, inter-disciplinary, hands-on activity kits with learning from and communicating with real-world professionals. The kits will give students control of their learning process, and require them to assimilate existing and new knowledge, through both problem-solving and extension activities. Students will work together in small peer-directed, collaborative groups, instead of through the traditional independent and individual approach. The potential benefits of such a move include the powerful effects of students giving and receiving help and explanations, to and from their peers (Webb, Troper & Fall, 1995; Yackel, Cobb, Wheatley & Merkel, 1991). Also, this group approach encourages the important learning processes of: clarifying and recognising information in new ways; recognising and filling gaps in understanding; developing new and improved perspectives; constructing more elaborate conceptualisations; strengthening the connections between

new and previous information; and improving problem solving skills (Webb et al., 1995; Yackel et al., 1991). In contrast, teachers and participating real-world professionals, together, assume the role of facilitators. Their roles are to create a non-threatening learning environment, in which students are in control of their learning and are able to take risks (Al-Weher, 2004).

Project components

The development and teaching team aspires to instil an understanding in students and to promote an appreciation of engineering, science and mathematics as both a rewarding career and a means to benefit society. Combining the talents of engineering, science and educational professionals in cross-curriculum design will allow students to experience these subjects in ways that are meaningful to them in their daily lives. The activities are designed to introduce curriculum concepts in a fun and engaging manner, and demonstrate the student's potential to understand and enjoy mathematics, engineering and science.

The initial phase of the project will be the development of the hands-on activity based kits mapped to educational curricula. They will comprise hands-on activities that simulate the work of a real-world professional, and will require students to make connections between the activities and what they have learnt in the classroom to actively construct new and improved knowledge. In addition, each kit will include extension exercises, assignment and examination questions, created in line with current senior syllabi and common curriculum elements, to extend student learning opportunities. The activity kits will emphasise authentic experiences, encourage creative problem solving skills, and prepare students for advanced education. Kit design and implementation will be developed along similar strategies as Rhoads et al. (2005) who reports on a National Science Foundation initiative, describing authentic learning experiences that engage students in personal construction of new knowledge; result in students conducting disciplined inquiry; and have value beyond the classroom. Therefore, students apply their knowledge to their everyday lives. Hylton and Otoupal (2005) support these strategies by using an approach called "Mathematics with Real World Correlation", and established that when student paradigms are connected to science and mathematics concepts through authentic exposure, student learning becomes meaningful in a way that it otherwise would not. Their study showed a substantial increase in understanding engineering concepts (46%) as well as an improvement in the comprehension of mathematics concepts (21%).

A Steering Group comprising a school representative, relevant Teacher Associates (university, industry and government agency) and a Critical Friend (supplied by ASISTM) will create the kits, to ensure that they meet the identified needs and outcomes of students and teachers and fit into the curriculum. The Steering Group will also allow the workload to be shared among the cluster. The kits will be progressively created across the 18-month period, with at least three developed initially. This process will enable the Steering Group to reflect on the process and use this to improve this process for the remaining kits. The development (and trial) of the kits, complemented by the involvement of real-world professionals will improve student interest, learning outcomes, and ability to think outside the square.

The project coordinators (the authors of this paper) act as a medium between schools and non-school organisations and administer the project. Initially, the schools will work closely with the coordinators and non-school organisations to develop and create activity kits that meet identified teacher and student needs. The schools will integrate

the kits within the curriculum, and will evaluate their impact, in consultation with the coordinators. The main role of the non-school partners is to enrich the kits during the trial phase through guest lectures, mentoring, university staff and student-led seminars, and laboratory and test site visits.

As a result, partnerships will be created between the schools, enabling teachers and students to work directly with each other. For example, students from the various schools will form affiliations based on shared interests, and can then contact practising professionals from one of the non-school organisations to foster their interests. Partnerships will also be created between the schools and non-school organisations, enhancing student prospects of a successful transition to university study and future career opportunities. This could include scholarships, work experience and placements where small groups of students undertake university subjects as part of the new Senior Certificate.

Stakeholder benefits

The project is orientated towards achieving a number of sustainable benefits for each of its key stakeholders, including Queensland secondary students, teachers and schools, QUT, and the engineering professional body. A majority of these benefits have been noted in the related corpus of literature, including Brown (2005), Crouch and Haines (2004), Davies and Heath (2004), Fabijan and Pope (2005), Orlich et al. (2005), Zarske et al. (2005). In most cases, however, the benefits are specifically focused at particular stakeholders and/or areas of need. In contrast, this project will work towards achieving several identified benefits in one package focusing on each of the key stakeholders. The particular benefit for each of the identified key stakeholders follows.

Benefits for secondary students

The intended benefits for all students, who use the project resources, are derived from the enhanced learning outcomes in the specified subjects using a constructivist perspective teaching methodology. The benefits are also derived from the use of activity kits related to real-world disciplines to stimulate their learning. In addition, students will have an increased capacity for innovation, by learning in an environment that requires them to connect classroom learning with its real-world application. Finally, by taking on the role of scientists and engineers, and communicating with, and learning from, both tertiary educators and engineering professionals, students will appreciate engineering and science as potential careers. They will also understand the important role undertaken by engineers and scientists in the world today and into the future. Consequently, it is foreseen that increased student engagement and interest in studying mathematics, science and engineering technology in their senior years of schooling will stimulate their progress into future study and careers in these areas.

Benefits for secondary teachers

Benefits to secondary teachers are fashioned from fostering a culture of innovation within the teaching fraternity. This will stem from the opportunity for secondary teachers to work with, and learn from, tertiary educators and practising professionals in developing, implementing and reviewing the project and its effects on student learning. Consequently, obstacles between teachers from different schools and education sectors will be removed, allowing them to work together in an environment focused on improving student learning and achievement across their schools. This will lead to a

dynamic professional network between these schools, assisting teachers to gain improved teaching approaches and techniques, and new resources to positively influence their students, including students outside of the project. This network also aims to revitalise, retain and attract quality teachers to this dynamic environment within the cluster schools.

Benefits for secondary schools

The benefits for secondary schools are derived from the benefits that it provides for its teachers and students. Students who perform at a higher cognitive level in mathematics and science will improve overall academic results for the school, thereby improving school marketability and attracting new students. Furthermore, for schools that can revitalise, retain and attract quality teachers, this brings with it the associated benefits of improved instructional practices and quality learning resources. This ultimately results in improved student learning outcomes and results. Finally, project involvement gives schools an opportunity to leverage the collaborative relationship between other schools, a leading tertiary institution and industry, in the interests of many other benefits. These benefits do not necessarily need to be confined to the areas of engineering, science and mathematics. They can apply to, and be used in, any school area to improve any existing programs.

Project progression

As outlined earlier, the initial phase of "Out of the Square and into Reality" will be the development of the hands-on activity based kits. The first three kits are an Ergonomics kit, Materials and Alloys kit, Rocket kit and finally an Engineering kit. The Ergonomics kit will be used across Mathematics A and Technology Studies subjects. Here, students will integrate several learning exercises across survey design, data collection and interpretation (statistics), drawing scale diagrams, constructing a scale model and, finally, evaluation of an ergonomic cardboard chair. The Materials and Alloys kits will be used across Mathematics B and C, Physics and Engineering Technology and focused on recording the cooling curves of various alloys and materials and then modelling these curves with mathematical models. The Rocket kit will focus on simulating flight data. This data will be recorded and automatically logged for further manipulation and used in Mathematics and Physics subjects. The Engineering Kit will focus on utilising the mathematical concepts of complex numbers, vectors and calculus (integration and derivatives) in applicable fields of engineering. At different stages within the project's development, various activity kits will be implemented.

Once developed, the kits will be implemented within the cluster schools' curriculum. The Teacher Associates, Queensland University of Technology fourth year students, who have expertise in each kit's discipline, will facilitate and manage this process. Their role will be to enhance student learning experiences by: engaging with students in discussion about the relevance of the activity to its real world context and future study and career options; assisting with the hands-on aspects of the activity; and acting as role models and mentors for students. The role of the classroom teacher during the university students' visit is to facilitate a safe learning environment and assist students, where needed.

Successes to date

The project has been extremely successful to date, due to the hard and dedicated work of Steering Group. The successes to date for the Out of the Square into Reality project include:

- successfully attaining a BP Education Grant for the development of the project's interactive website, www.realitysquare.com.au. This website has been successfully launched and is focused on disseminating the project's ideas, resources and professional development opportunities throughout Australia and the world. It also facilitates the opportunity for teachers to safely share their resources online, through the development of a password projected "Teacher's Access Area";
- significantly increasing the number of secondary schools utilising the project's activity kits;
- increasing the scope of existing kits to be used in other subject disciplines, such as Business Education;
- forging links with the Queensland Association of Mathematics Teachers and the Science Teachers Association of Queensland to develop professional development "Showcase Days". These professional development days will focus on teachers working with and learning from real-world professionals and tertiary students and educators, in gaining a greater appreciation and understanding of contextual based teaching and the development of resources to be utilised in participants' schools;
- disseminating information about the project through conferences, such as the Australasian Association for Engineering Educators Conference in Auckland and the Queensland Association of Mathematics Teachers May Day Conference;
- forging new partnerships with the Engineering Link Group, Central Queensland University, James Cook University and secondary schools in Townsville, Cairns and Rockhampton, to extend the real-world and interdisciplinary ideas of the project into middle school mathematics and science. The aim of these new partnerships is to increase the number of students taking senior mathematics and science in secondary schools.

Conclusion

It is imperative that secondary students be not only be well informed about the career options available to them within engineering, but that they also see the links between engineering and the secondary school subjects of mathematics and science. The promotion of engineering as a career in schools needs to make the connection between what students are learning in theory, and what happens in real life in the provision of engineered products, services and infrastructure. We need to demonstrate, in interesting and exciting ways, the value and importance of the work of engineers to people's every day lives, and to the environment. We need to engage students, their teachers and career counsellors, in activities that are part of or linked to the work of the engineering team. In addition, the programs should integrate with existing elements within the school curriculum, provide appropriate resources for teaching staff and assist in their professional development, and foster links with partner organisations.

It is envisaged that there will be a follow-up article to report on the progress and success of our project through the development and implementation phase of its life. In this article we plan to focus on the actual outcomes achieved by the project's key

stakeholders (secondary students, teachers and schools, tertiary students and institutions and the professional body. These outcomes will be identified through the analysed results of student and teacher surveys, which will accompany the activity kits during implementation in participating schools.

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Acknowledgements

MathBusters (apologies to MythBusters)[*]

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This paper looks at some of the urban myths surrounding mathematics education. Several of these myths are held up to scrutiny and submitted to the rigours of comparison with classroom practice and research findings. Most myths are "busted", but not all of them.

The popular television program *MythBusters* is a show where truth is separated from urban legend and folklore. The cast and crew use science to test the veracity of popular beliefs. This idea prompted me to think about mathematical myths and use modern-day science (mathematics education research) to show what is real and what is fiction. Using thorough investigation and a bit of "trial and error" I will seek to examine the truth or otherwise of these common mathematical myths.

1. Maths is boring.
2. Calculator use "rots children's brains".
3. You never use the maths you learn in school.
4. Some people just don't have a maths brain
5. Maths is hard.

I will take this folklore and examine each idea then decide whether the myths are CONFIRMED or BUSTED. Taking one myth at a time, I will look at some classroom activities and research findings that test the myth, then strategies to try to address each myth will be offered to the reader.

Myth 1. Maths is boring

For many "people in the street", mathematics lessons probably were boring. In the past the format of mathematics lessons was fairly predicable: the teacher demonstrated some new skill or procedure — often on a blackboard — then the students were required to complete similar exercises from the textbook. These practice exercises were corrected and another set was given as homework. Few practical applications were used in class and the relevance of the lessons was often not clear to students. It must be said, though, that the same adults who think maths is boring seldom stop to consider the mathematics involved in everyday life when they go shopping, fill the petrol tank, back into a parking space, withdraw money from an electronic teller and so on.

The good news is that times have changed and mathematics lessons are now varied and interesting (or they can be!) However, if the students you inherit come to school with a slightly jaded view of mathematics lessons, I have some ways to encourage them to think again.

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Intriguing "bits and pieces" often help to create the variety and novelty that prevents mathematics becoming boring. Here are some ideas under the categories of games, tricks and curiosities. A few examples will illustrate how they can be used.

Games

There is always time for a good mathematics game in the classroom to stimulate concept development, to contextualise mathematics learning and to practise skills. The following games are just two examples.

Ten snap

This is a game that is a variation of the old Snap card game.

AIM:

   To make a pair of cards that total ten.

YOU NEED:

   2 players

   A pack of cards — Aces are worth one and Jacks, Queens and Kings are worth zero.

TO PLAY:

   Cards are shared between the two players.

   Each player places a card, numbers up.

   The players continue to place one card in turn on the pile and when two consecutive cards add to 10 they snap.

   The player who snaps correctly wins the pile of cards.

   The winner is the player with more cards.

Of course this game can be adapted to any target total. However, as it stands it is a good context to practise some fundamental number bonds.

The second example is a strategy game dealing with factors and multiples. As it is played children become very aware of the counters that are "powerful" and begin to consider why that is so. It is a game that can be played again and again.

Cross the grid

AIM:

   The winner is the first player to cross one counter to the opposite side of the grid.

YOU NEED:

   2 players

   A set of counters numbered 1 to 7/8 for each player (7/8 is either 7 or 8).

TO PLAY:

   Players begin at opposite ends of the board.

   Counters begin the game off the board.

   One counter is advanced one row on each turn.

   The counter can only be placed on a gameboard square that is a multiple of itself. For example, 3 can only be placed on 3, 6, 9, … 78.

   Choose a counter to place on the gameboard.

   Take turns to advance your counter one row at a time.

You may play any of your counters on the board before all the counters have "begun".

You may "capture" your opponent's counters if you can advance your counter onto a square that their counter already occupies.

Tricks

There is something intriguing and challenging about mathematical tricks. The *3 Dice trick* is one of my particular favourites because it certainly promotes an investigation of the properties of dice and is simple enough for every child to use to trick an adult. The mathematical thinking it stimulates can be taken beyond the specific case of 3 dice to 4 and beyond and it leads children to generalise a rule for doing the trick. A version of the trick (from Swan, 1993, p. 43) appears below.

Thrice dice

You will need 3 dice and a pen and paper.

Ask a volunteer from the audience to roll the dice, while you stand facing away from the audience.

Choose a second volunteer to add up (silently) the number of the *five hidden faces* of the dice as they stack them up.

When they have added up the numbers ask them to write the total down on the paper provided and show it to the audience. Ask the volunteers to tear up the paper. Look at the stack of dice and note the top face of the stack. Announce the total to the class after some show of consternation (and some simple mental addition).

To determine the total, subtract the number shown on the top face from 21.

Curiosities

Finger multiplication

Under the heading of curiosities, here are some ways to enliven the topic of multiplication. The first is the use of hands (not the forbidden thing of the past) but the curious short cut to "times tables"! *Finger Multiplication* is certainly the easiest way that most children know to calculate multiples of nine. The trick is easy.

Lay both hands palm up. Number the fingers from left to right 1–10. Now to calculate 4 × 9 bend down the fourth finger. The fingers to the left of the bent finger are the tens and the fingers to the right of the bent finger are the ones. So 4 × 9 = 36.

Curious?

Why does it work?

30 6
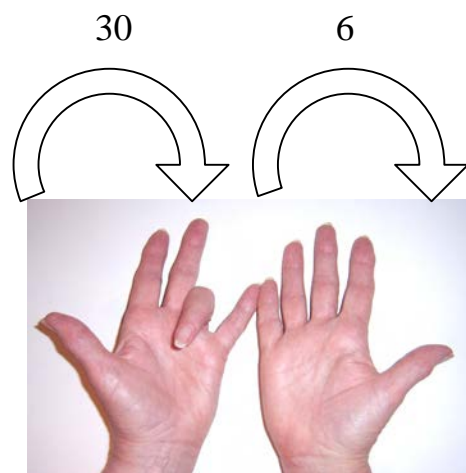
Weird and quirky things that happen with numbers sometimes motivate and intrigue children. For example (from Swan, 1994):

Double up

Enter a 3-digit number on your calculator.
Multiply by 7.
Multiply by 11.
Multiply by 13.
What has happened? Discuss with your neighbour.
Try using other 3-digit numbers.
Try using numbers where the digits are all the same; e.g., 222, 555, 888.
Try using numbers with trailing zeros; e.g., 100, 400, 700.
Write about what you notice.
Does the order in which you multiply matter?
Try multiplying by 13, then 11 and then 7.
What happens?
Try using a different order.
Try to explain why it happens.

The lovely little investigation posed in *Double up* touches on some very powerful mathematical ideas. These include the commutative property of multiplication (the order of the factors is irrelevant to the product) and the place value effect of multiplying by powers of ten. The best aspect of the task is that it has children puzzling over what is going on with the numbers.

Strategies that will stop mathematics being boring:

- Get the level of challenge right.
- Allow for some student choice.
- Use intriguing and sometimes quirky content to help create variety.
- Pose problems and initiate investigations that capture children's imagination.

Myth 2. Calculator use "rots children's brains"

Although calculators have been recommended as a tool for the teaching and learning of mathematics for decades, there has been the view in the community that they make children dependent on them and stop children thinking for themselves.

Research findings

The consistent pattern in research findings for more than two decades has been the demonstration that calculators do no harm to children's acquisition of computational skills (Groves 1994). Britain's *Calculator Aware Number* project found that children using calculators developed a wide range of strategies for carrying out calculations and, in general, reached a high level of numeracy for their age. In Australia, the *Calculators in Primary Mathematics Project* reported that; the calculator provides

> a rich mathematical environment for children to explore and promotes the development of number sense by removing previous restrictions on the types of numbers children use, by exposing children to written symbols which they can easily record and manipulate, and by facilitating sharing and discussion (Stacey & Groves, 1998).

The project investigated the effects of the introduction of calculators on the learning and teaching of primary mathematics. Classroom observation revealed that children were dealing with larger numbers than would normally be expected, as well as, in many cases, negative numbers and, to a lesser extent, decimals. After long-term experience of calculators children performed better than children without such experience on a range of computational and estimation tasks and some "real world" problems. They had a better knowledge of number, particularly place value, decimals and negative numbers. They were more able to choose appropriate calculating devices, and were better able to interpret their answers when using calculators. No detrimental effects of calculator use were identified. In fact results confirmed the anecdotal evidence from project classrooms and "supported the assertion that the presence of calculators provides a learning environment to promote number sense" (Groves, 1994).

As well as having effects on children's learning, calculators influence teaching approaches that could also be said to promote number sense in young learners (Beeby & Cheeseman, 1991). They stimulated some teachers to use more problem solving, to integrate mathematics with other areas and to use more open-ended discovery learning.

Kilpatrick (2001) in a study synthesising the research literature in America said, "it is clear that the question is no longer whether calculators should be used, but how."

Australian research used classroom observation to document the major ways teachers and children used calculators. Illustrations of these uses can be seen in the videotape *Young Children Using Calculators* (Groves & Cheeseman, 1993). Calculators were used as an object of investigation, to generate patterns, to stimulate discussion, as a recording device, to count, to solve problems, to calculate, and to stimulate mental computation.

For young children calculators are manipulatives that foster experimentation and investigation with number. They are a toy and a tool that stimulates curiosity and supports rich mathematical environment for conjecture and discovery (Groves & Stacey, 1998; Huinker, 2002).

Some ideas for investigation

- Which would take you longer: to count from 1 to 100 by 1s or to count from 10 000 to 1 000 000 by 10 000s on your calculator? Why?
- Can you find $\sqrt{2}$ without using the square root key? How?
- How many ways can you find to constantly add a number to generate a display of 1 on your calculator?
- How does the percent button work on the calculator? Can you teach someone?
- Find the counting pattern with the longest repeating sequence in the units column.
- What is the largest number (and smallest number) you can enter on the calculator?

Strategies that will eradicate the myth that calculators harm children's number sense:

- Use calculators as a teaching and a learning tool.
- Know the research evidence to be sure of your facts.
- Be prepared to address parents' concerns.
- Choose tasks carefully to demonstrate the potential of the calculator to stimulate conceptual development.
- Support calculator use with an increased emphasis on mental computation and strong teaching tactics for understanding the number system, especially decimals.

Myth 3. You never use the maths you learn in school

These days more and more of the mathematics that children meet in classrooms connects to their lives. When I get the opportunity to visit mathematics lessons, I try to write accounts of the problems I see the children tackle. Two examples spring to mind. The story of "A Bag of Apples" first appeared in *Australian Primary Mathematics Classroom* (Cheeseman & Ward, 2001) and was subsequently published in the Netherlands in *Volgens Bartjens* (2004/2005). This lesson challenged children to pose questions about the bag of apples that they could use mathematics to answer. The main focus became the question, "How long is an apple peel?" The second lesson concerned the design and testing of "Hermit Crab Trays" and is described in full in *Prime Number* (Cheeseman & Bergfeld, 2006). In each of these lessons problems were solved. These problems connected the children's interests with the teacher's mathematical agenda.

Strategies that will help to "bust" the myth that school mathematics is not useful:

- Connect the mathematics in school to the lives of your students.
- Use contexts that show mathematics "in action"; e.g., scoring diving competitions.
- Engage students in investigations and problem solving.
- Invite parents in to school to talk about their use of mathematics in the workplace.

Myth 4. Some people just don't have a maths brain

Sometimes you hear a parent (more often a mother) say, "I don't really have a maths brain; I was never any good at maths at school either," as an excuse for their child's lack of progress (particularly daughters' lack of progress). This idea that you either can or cannot do mathematics is patently false. Everyone can learn mathematics.

Part of the book *How People Learn* was devoted to how children learn. The authors reflected on new conceptions of the learning process and the development of competent performance. Children were described as dynamic learners where, "in particular domains, such as biological and physical causality, number, and language, infants and young children have strong dispositions to learn rapidly and readily" (Bransgrove, Brown and Cocking, 1999, p. xiv). These authors also stated that children lack knowledge and experience but not reasoning ability. They went on to say that children are both problem solvers and problem generators who seek novel challenges. It is clear that children from early infancy have the disposition and ability to learn mathematics. So, rather than offering up empty excuses, adults need to support children's curiosity and persistence, and structure their learning experiences.

Strategies that will help to counteract the view that some people cannot learn mathematics:

- Actively dispute this myth as everyone can learn mathematics.
- Be conscious of choosing tasks that appeal to both boys and girls.
- Support and arouse children's mathematical curiosity.
- Structure learning experiences where children experience success.

Myth 5. Maths is hard

This myth is one that may not be able to be "busted" but that might have to be "confirmed". Doing mathematics is challenging and sometimes really difficult but, with perseverance and persistence, it can be wonderfully rewarding to meet the challenge.

- Keep the challenge of the task almost within reach to that its achievement will be an exciting reward (hard but not too hard!).
- Develop the attitude within the class that nothing is accomplished without a struggle; i.e., it is supposed to be hard, it is good to "think hard" and figure things out — that is what gives children such satisfaction.

In conclusion

Popular myths abound in mathematics education. Here I have examined a few and most of them have been "busted"!

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Reflective teaching + connective teaching = effective teaching[*]

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Reflective teachers consider reasons for their students failing to like and learn mathematics. Connective teachers see links between different aspects of mathematics, between mathematics and other learning areas, and between mathematics and the "real world". They also engender such "link-making" in the thinking of their students. Two aspects of mathematical learning of particular importance are the use of "real" problems, as opposed to contrived problems, and the role of language in the understanding of mathematics. Recent research suggests that teachers can motivate students towards learning mathematics by using strategies to help them connect their mathematical learning more effectively.

Considering the problem

Reflective teaching

Have we turned students off mathematics? Have we "demotivated" them about mathematics and even contributed to the development of a phobia of mathematics? Do our students see a real purpose in learning mathematics? If we think that "Yes" might be an appropriate answer for any of those three questions, then we need to look for causes. Perhaps one factor might be that we have made mathematical learning more difficult than it needs to be and have ignored some obvious links that could make mathematics more interesting, understandable and applicable for our students.

Burns (1998) suggested that one of the reasons for students developing a dislike and even a fear of mathematics is that it has often been taught in isolation from any real context and that there is no substitute for the experience of learning about concepts in situations in which they are actually applied. Burns used several analogies to illustrate her point, including learning how to effectively play games like Monopoly through actually playing them, and learning to play a musical instrument through playing music, as opposed to endless theory lessons. Learning our language is the same in that "we acquired the skill from being immersed in the real purpose of using language to communicate" (Burns, 1998, p. 67).

Burns added that "skills should be learned in the context of problems and situations and should not exist isolated from the problems and situations that give them their purpose" (1998, p. 69). In this, she is suggesting that mathematics is best learned by immersing children in activities, explorations and experiments, and letting them learn in the context of thinking, reasoning and solving problems.

Parkin and Hayes (2006) raised a number of issues related to literacy in their consideration of why students, particularly those from indigenous or ethnic

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backgrounds, experienced difficulty with solving worded problems. They viewed the linguistic demands of the mathematics curriculum as being one of three aspects: general language demands, the language of textbooks, and the language of worded problems. They also noted that "it is important that we can articulate what may need to be explicitly taught if students are to access the language in maths" (2006, p. 23). While the three part distinction is clear, it could be more effective and engaging if the language was taught through the experience of doing the problem, along the lines of the "immersion" that Burns suggested. Some examples of how this might occur are described later in this paper.

Parkin and Hayes also noted that one of the first issues for students to deal with when solving worded problems is "the need to suspend disbelief" (2006, p. 26) as the context of the problem may be foreign to their experience. This can lead to a feeling of futility and questioning as to whether there is any real point in working out a problem that bears no relation to them. Indeed, a common reaction may be, "Who cares?" While Parkin and Hayes described the situation with respect to indigenous and ethnic students, teachers who reflect on their experience with most classes would most likely recognise the same reaction to solving worded problems.

The language of worded problems was also identified by Carnellor (2004) in considering difficulties faced by students with learning difficulties. Indeed, "problems for these students can also become more prevalent and understanding more complicated when we consider that very often the grammatical structure of mathematical word problems takes on a much more complex form than the students would be likely to encounter in oral language or reading activities" (Carnellor, 2004, p. 87). Carnellor advocated the use of "normal language" in the context of everyday mathematical experiences as being better, as well as the use of a constructivist approach where students are engaged through skilful questioning and meaningful discussion. Also, learning is more effective when "mathematical language is embedded in the development of many everyday verbal communication skills" (Carnellor, 2004, p. 88) and when fostered in natural settings.

This was supported by Trafton (1999) who claimed that teachers, in their attempts to enhance mathematical learning, often took the reality out of potentially rich learning situations by simplifying them too much. He argued that children prefer to work with problems that are "messy" and involve numbers that do not, for instance, divide evenly with no remainder. Trafton claimed that learning can be highly effective if teachers pose a problem and "let children work with others to solve it in ways that make sense to them; they let children take ownership of it" (Trafton, 1999, p. 10). This seems to imply that it is important to use the "natural language" of the context in which the problem occurs, rather than couch it in complex mathematical terms.

Connective teaching

So, what do "connective teachers" do and how can they make learning more enjoyable and effective? Askew, Brown, Rhodes, Wiliam and Johnson (1997) found that the most effective teachers of numeracy, in terms of gains made by their students, were those teachers whom they described as having a "connectionist" orientation. They identified a number of features of the beliefs and practices of such teachers including the following:

- emphasis on linking different aspects of the mathematics curriculum;
- encouragement of students to draw on their own mathematical understanding to solve realistic problems;

- belief in the ability of most pupils to learn mathematics given appropriate teaching;
- teaching mathematics is based on dialogue between teacher and pupil;
- encouragement of students to initiate their own ideas, design and plan their own work, and justify their reasoning to others (adapted from Askew et al., 1997).

These points appear to support assertions made by Parkin and Hayes as well as Carnellor and Trafton, and are summarised in the following comment made by one of the teachers involved in the research of Askew et al. (1997): "If mathematics was to be meaningful and be seen to have relevance to everyday life, it had to be presented as far as possible in a realistic context and not just be a book based activity" (Askew et al., 1997, p. 44).

Recent research (Hurst, 2006) suggests that the use of specific strategies can enhance student motivation about mathematics and their capacity to connect their mathematical learning to a range of contexts. For example, the *Mathematical Search* is a task where students are asked to identify the mathematical ideas in a given context, describe what those ideas tell them about the context, and pose questions that use the mathematical ideas. *Finding the Maths* is a similar task where students have to find the context themselves. Both tasks were found to have enhanced student thinking in that students displayed a greater degree of contextual and critical thinking after using the tasks on several occasions. As well, direct interviewing about mathematical ideas embedded in a range of contexts proved to stimulate student thinking in similar ways.

Another task that enabled students to connect their mathematical knowledge to other areas was the use of concept mapping. The following example was developed on a teacher's whiteboard during a debriefing lesson following a *Finding the Maths* task:
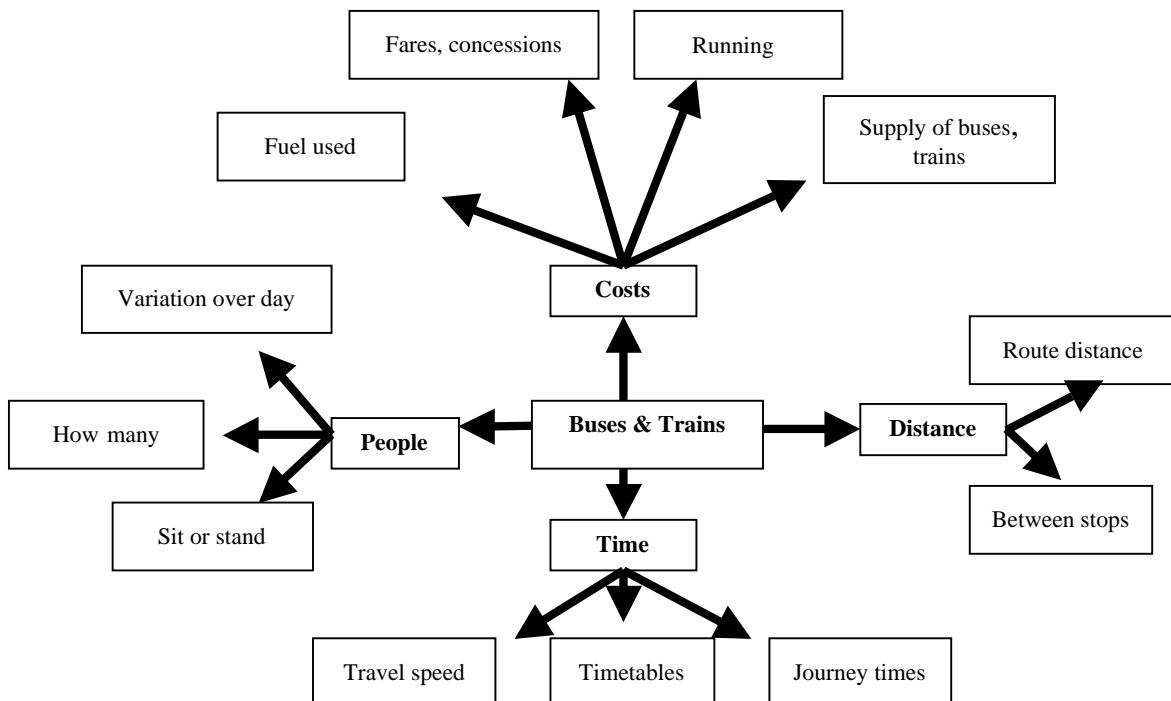

Figure 1. Concept mapping: the mathematics of buses and trains.

In all of the above tasks, the level of language was what was naturally used by students. By ensuring this, one potential obstacle, that of the complex mathematical language referred to by Parkin and Hayes (2006) and Carnellor (2004), was avoided.

Considering some solutions

Effective teaching

The following three anecdotes provide examples of effective numeracy teaching as they embody the principles outlined in the previous two sections of this paper. In particular, they highlight the use of appropriate language in natural and realistic settings relevant to the interest and experiences of the students concerned. As such, the activities described allow for learning to occur in the context in which the mathematics is embedded, as opposed to contriving an artificial problem for children to solve. They also encourage the connection of different aspects of mathematics with one another.

The Year 6 camp

Bickmore-Brand (1998) described the efforts of one Year 6 teacher to make the learning of her students more realistic and authentic. Following negotiation between the teacher and students, it was decided to investigate the most cost effective way of staging their annual camp. The agenda for the task was completely open and direction was provided by posing questions like, "What are some of the things that will need to be paid for?" and, "What are the costs of travelling to the camp and back, as well as travelling about while we're there?" The task was not one that could be dealt with in one or two lessons and the students were highly engaged over a period of six or seven weeks. Some of the specific activities that formed part of the overall task included map reading, finding fuel efficiency of various vehicles, contacting public and private transport companies, and reading timetables, "all the time collating and comparing information, building a big picture in which small decisions along the way could make a significant difference" (Bickmore-Brand, 1998, p. 6).

A key feature of this task was that various aspects of language and mathematics were inexorably connected and the language used emerged naturally as students investigated the problem. Another aspect of mathematical learning that was highlighted for students was the myriad places in which mathematics is found in everyday situations. For instance, they explored ideas from petrol tank capacity to comparative petrol costs and averages, from food preferences to catering costs, and from the use of timetables to scheduling educational outings and events. None of these were directed by the teacher but arose naturally from discussion about the task. Perhaps the best aspect of the task, according to Bickmore-Brand was that "children learn[ed] by trying to make sense out of the world they find themselves in" (1998, p. 5). The Year 6 Camp task provided a refreshing alternative to the normal mathematical "problem" which is often characterised by "incredibly complex rules of language" (Bickmore-Brand, 1998, p. 5).

A pool for the school

During 2006, Paul taught a class of Year 4 and 5 students in a small independent school that has a strong swimming culture, having won its inter-school swimming carnival for the past eight years (Hurst & Hurrell, 2007). Paul discussed ways to ensure that this success continued. Students quickly came up with the idea of building their own school pool so Paul posed the question, "If we could have a swimming pool built at the school, what kind of information would we need to know? The students were easily motivated

by the thought of continuing their winning form and, consequently, with very little prompting, came up with a list of questions.
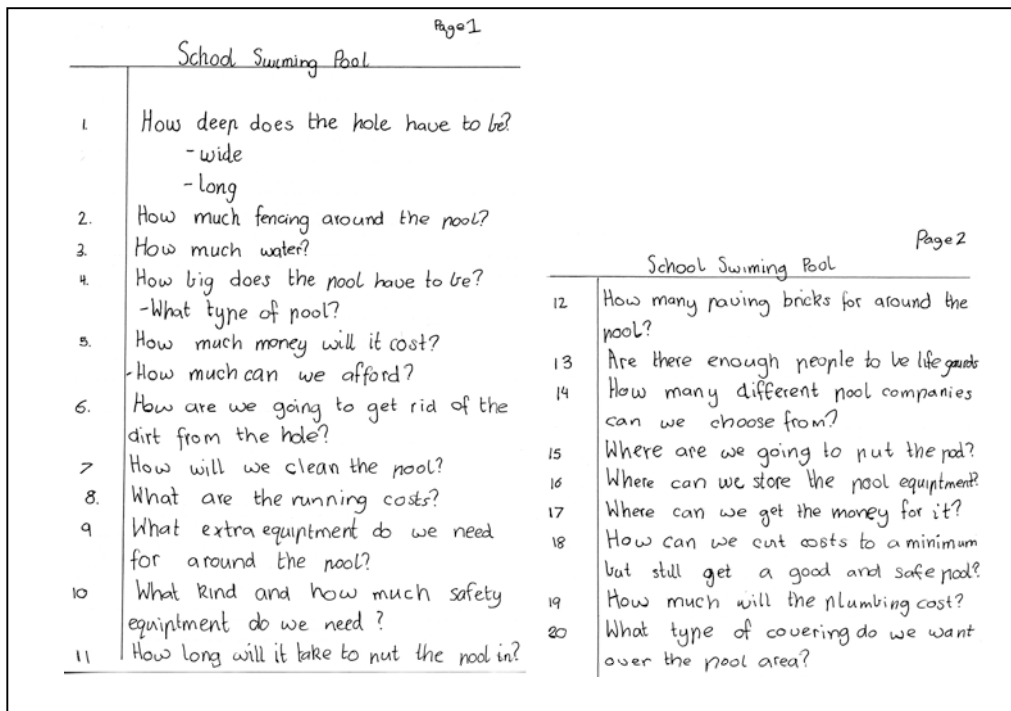

Figure 2. Questions posed by children about building a swimming pool.

The task quickly gathered momentum and the students became excited and very involved in the various logistics of building a pool. As with the previous task, the language emerged naturally from the task, as did the mathematical ideas that needed to be explored. The initial session where questions were posed was followed by a "site walk" to explore possible places for a pool site. Pros and cons of various sites were discussed during the site walk; e.g., noise levels for neighbours of school or for other classrooms, other uses of school grounds, and the effect on the school environment through having to remove trees.

Some students did a search for pool companies using the Yellow Pages and discovered three types of pools (concrete, liner, and fibreglass). They discussed the location of pool companies (distance from the school), and the effect this would have on transportation costs. Others began by pacing out the approximate area to gain an estimation of the amount of room that the pool would occupy. They started with the edge of the school oval.

As students became more engaged with the task, other aspects needing consideration came to light and the richness of the task was becoming increasingly obvious to Paul. For example, students discussed how to cut costs and volunteered the help of parents who own businesses that could assist at various stages of construction! They even discussed the order in which the project could proceed and started scheduling all the various trades, leaving room on the schedule in case some trades went overtime. The task started out as an exploratory activity but such was the authenticity of the task to the children, that they had convinced many of their parents that the school should indeed build its own pool.

A virtual maths trail

During 2006, Marion taught a Year Seven enrichment class at an independent school for girls. Most students in her class experienced difficulties in learning mathematics and/or had poor perceptions of their mathematical ability (Hurst & Hurrell, 2007). The task that Marion used with her students was the construction of a virtual maths trail. This entailed taking the girls to a local shopping centre where they photographed aspects of the centre that had some mathematical ideas embedded in them.

Whilst at the centre, the girls were asked to be thinking about what mathematical questions they might ask as part of their trail and the information they would need to provide so that the problem could be solved. They would also have to consider the language they would use. The questions in the maths trail were to be suited to two different year levels within the primary school and the girls chose Year Three and Year Six. The virtual maths trail was constructed using the Microsoft PowerPoint program.

A 180g bag of lollies cost $5.95 what would 1kg of lollies cost?
How many 180g bags would you need to get to have at least 1 kg of lollies?
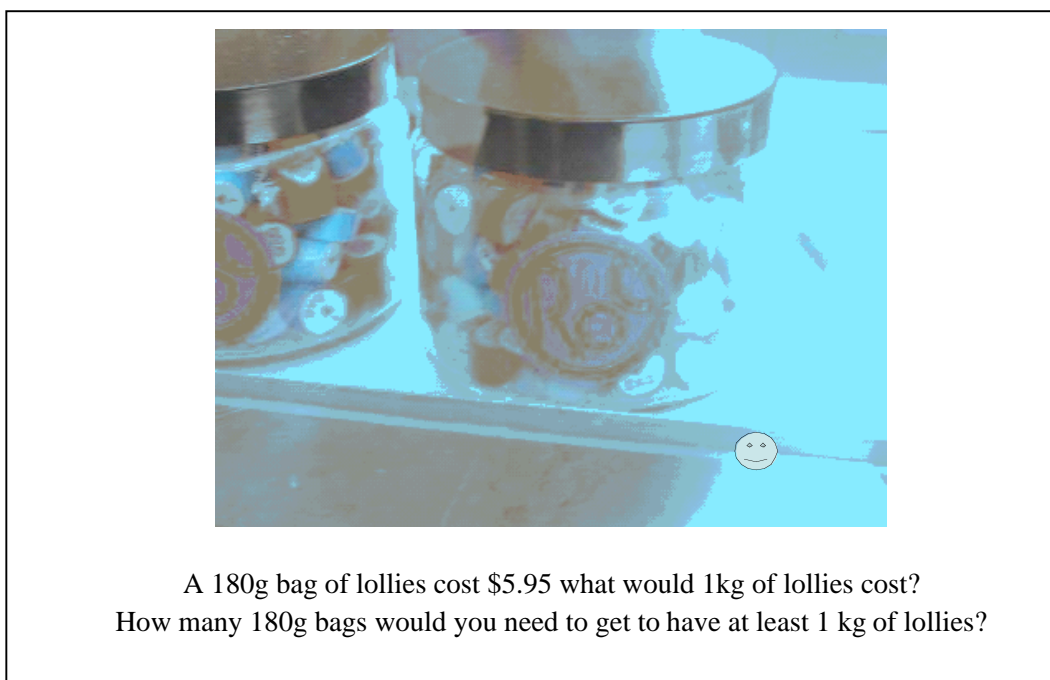
Figure 3. Example of a PowerPoint slide from the virtual maths trail.

Responses to the task were very positive and included the following:
The Year 7 students:

- enjoyed teaching the younger students;
- were keen to make another attempt when they had remedied the problems with the first testing of their maths trail;
- enjoyed the hands-on aspect of the task;
- saw that mathematics involves much more than classroom activities and came to realise the importance of being able to use mathematics in real life.

Marion felt that:

- the task empowered girls who would probably have had little success in working mathematically in the past, and that the mode of engagement suited them;
- it was a good opportunity to tackle the whole question of drafting work and that working through the questions would result in changes being made;

- the exercise proved to have positive spin offs in so much as it allowed students to work in different parts of the school;
- the cross-school liaison between students was matched by liaison between staff.

As with the previous two tasks, the virtual maths trail was highly engaging for students not considered to be strong in mathematical ability. The language and mathematical concepts emerged during the course of constructing the trail and were meaningful for the girls as they were embedded in an authentic context.

Conclusion

Researchers such as Haylock (1991) have mentioned the importance of using "real" and "authentic" contexts to enhance student learning and this seems to be a significant part of the issue of motivating students towards learning about and applying mathematics. However, it is also important that teachers are constantly reflective about what their students are thinking and doing so that they are in a better position to provide the sort of learning experiences that are relevant to the learning and use of mathematics in the modern world. Specifically, teachers should consider whether or not they are providing learning situations that genuinely allow students to connect their mathematical learning to a range of applied situations. In particular, it is important to consider if we are putting barriers in the way of our students in the form of unnecessarily complicated and cumbersome language as opposed to allowing them to develop mathematical ideas in the context of their own everyday language.

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Teaching algebra using instructional games[*]

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The teaching of algebraic understandings through instructional games provides a rich program with students and teachers gaining expertise on many levels. This paper will present a framework and examples to show how students work from practice and consolidation of skills through to a deep understanding of the concepts and how teachers, through planning within the framework and the interactions with students during game play, gain insight into where the students are in their learning, and have the opportunity to move them forward when they are ready. The teaching of important interpersonal skills will also be discussed.

Instructional games

There are many definitions of instructional games used in mathematics. Gough (2000) defines a mathematics game as "something you play to learn or practise mathematics." Ainley (1990) refines this by distinguishing the effective game as being born out of, and inherently relying on, mathematical ideas and understanding in order to run. There are many examples of using instructional games for numeration concepts (Booker et al. 2004; Booker 2004; Booker 2000; Ortiz 2003; Gough 2000; Onslow 1990; Pearn & Merrifield 2004; Randel et al. 1992; Rowe 2001; Schmidt & Ellen 1995).

The research

The research is overwhelmingly in favour of using instructional games in the mathematics classroom. Randel et al. (1992) reviewed the literature and they present a powerful case for changing the mode of instruction in the mathematics class. They found that the use of games in teaching mathematics is superior to traditional instruction in terms of achievement, retention and motivation for learning. For students using instructional games in the classroom, the literature shows an increase in achievement, improved retention of information, heightened motivation to learn, greater participation, increased opportunities to learn, more complete engagement in the task, greater interest, improved confidence and more risk taking/problem solving (Booker et al., 2004; Booker, 2004a; Booker, 2000; Ortiz, 2003; Gough, 2000; Onslow, 1990; Pearn & Merrifield, 2004; Randel et al., 1992; Rowe, 2001; Schmidt & Ellen, 1995). Pearn and Merrifield (2004) based their effective intervention program on the use of games. They wanted the students to be actively engaged in the program and found that games provided this. The benefits of positive cooperative social interactions are also well documented (Booker, 2000; Ortiz, 2003; Rowe, 2001).

The positives for teachers focus on the benefits to students. Booker (2000) also mentions increased opportunities for teachers to communicate with students, one on

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one, while they play and use varied assessment opportunities via observation of student interactions during the game.

> Within a game, children are not endeavouring to provide an answer or reason which they think will match a teacher's expectations, but focus on those that make sense to them (Booker et al., 2004, p. 27).

There are management issues for teachers highlighted in the literature. The management of the competitive spirit needs close attention as it could negate the social benefits and the timing of games in the lesson sequence needs careful planning (Ainley 1990). Rowe (2001) cites several studies and discusses some of the disadvantages a teacher encounters in using instructional games. In Particular, the increased workload in preparation and management of resources, the lack of writing mathematics and the increased noise levels in the classroom were highlighted. During her study, in her own classroom, she noticed the increased noise levels, the lengthy preparation and the unexpectedly difficult management of the cards for the game. She states that the sessions when the games were played were "hard work" (pg 14) in terms of resource management. Booker (2000) acknowledges the perceived need to implement strategies for the management of the resources and spends a chapter giving suggestions that lead to a productive environment.

The program

The philosophy underpinning the planning for each lesson, and for the program as a whole, was derived from the teachings and two books by George Booker: *Teaching Primary Mathematics* (Booker et al., 2004) and *The Maths Game* (Booker, 2000). A brief summary of: algebraic concepts are abstract representations, is considered here. The summary takes the form of materials / patterns, words or symbols as illustrated in Figure 1. An understanding of an algebraic concept involves linking these three forms.
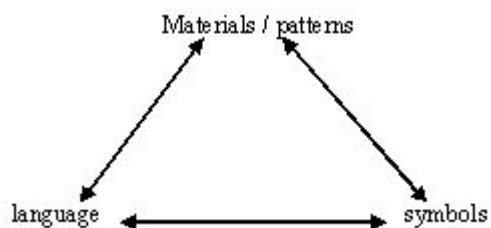
Figure 1. Representations of an algebraic concept.

Understanding the relationships shown in Figure 1 is essential to being able to teach algebraic understandings effectively, as the links must be explicitly taught. It is this relationship which gives meaning to algebra. Generally, the materials and language are investigated and understood before the symbols are introduced. The materials were counters and cups as described in *A Concrete Approach to Algebra* (Quinlan, Low, Sawyer, White & Llewellyn, 1989) and *Have-a-go! Maths* (Quinlan, 2004b). "Materials are fundamental to learning mathematics in all forms and at almost all levels" (Booker et al., 2004).

The program went over six weeks with three 70 minute lessons per week. The primary sources of activity were self designed instructional games, *A Concrete*

Approach to Algebra (Quinlan, Low, Sawyer, White & Llewellyn, 1989) and *Access to Algebra Book 2* (Lowe, Johnston, Kissane & Willis, 1993).

Specific games

Several instructional games were used in the teaching of the program. In this paper, three will be discussed in detail. They demonstrate the philosophy of the course, the style of teaching, and some of the different game genre that can be used in the classroom. The games are best used as an integral part of the lesson. The games allow the teacher to differentiate the learning for students in their class. Students can move on from where they are in their understanding by playing the game they are ready for. Teachers can have several different games being played at the same time within their classroom, and work with each group as the need arises. The small group is less threatening to students and they are more likely to risk-take and ask questions of each other.

Throughout all the game playing, links are made between symbols and language and between materials and language. The students are required to read the "question" out loud and verbalise their response. The renaming aspect involved in all the games discussed here encourages students to compare various possible responses. The particular algebraic language of variable, constant, collecting like terms, simplifying, expanding brackets are explicitly introduced during the teaching, and reinforced in the game playing.

Dolphin Magic

This game explicitly teaches the link between materials and symbols. Also, the concept of the difference between a variable and a constant is reinforced and the concept of renaming is taught through the game play. The sectors of the spinner have cup pictures representing cups that have y counters in them and large dots representing counters outside the cups. The spinner has six sectors each with a different combination of cups with y counters in them and counters outside. The track has multiple symbolic representations of each combination. The player has to move to the closest position on the track with a symbolic representation of the sector spun on the spinner. The players are given the concrete materials to engage with. An example of a sector of the spinner is shown in Figure 2. This group of materials can be written in symbols as $3y + 4$ or as $y + y + y + 4$ or as $y + 1 + y + 1 + y + 1 + 1$ or as $2y + y + 4$ and each of these "names" can be found on the track. In this ways students learn that there are several ways to symbolically represent this group of materials and that each way is equivalent to the other ways: that is, they learn renaming. After this, the concept of simplifying was taught.

Figure 2. An example of a sector of the spinner.

Renaming using brackets (i.e., the distributive property) is a natural inclusion at this point in the students' learning. Students are very comfortable, from the outset, with

using this symbolic representation and it is included on the track of this game. Figure 3 demonstrates an example of this.
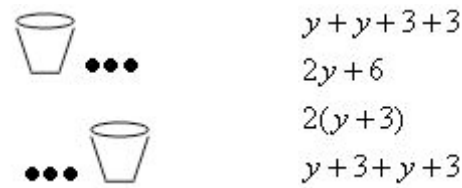

$$y + y + 3 + 3$$
$$2y + 6$$
$$2(y + 3)$$
$$y + 3 + y + 3$$

Figure 3. Materials on a spinner sector and the corresponding possible symbolic representations.

The game genre is a track. Four pictures of dolphins form the track and players move counters from one dolphin to the next from the bottom left, upwards then across, then down to the bottom right. The game board is shown in Figure 4.
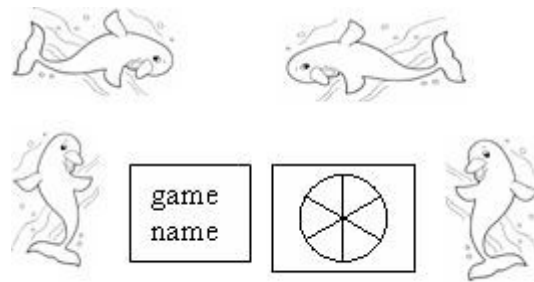

Figure 4. Dolphin Magic game board.

Each dolphin is broken up into segments big enough to fit the various symbolic "names" for each group of materials on the sectors on the spinner, and big enough to place a counter onto.

After playing this game, students can create a table showing the materials for each spinner sector with the corresponding possible symbolic representations. In this way, students can create a written record of their understandings gained from playing the game.

The interpersonal teaching that can arise from using instructional games is highlighted in *Dolphin Magic*. The finish spot is below the tail of the fourth dolphin and the players need to determine, as a group, the way to reach that spot. Many a debate has occurred as a result and it is an excellent way to embed the teaching of group dynamics skills into the mathematics program.

Hit the Spot

As in *Dolphin Magic*, this game explicitly teaches the link between materials and symbols. Again, the concept of the difference between a variable and a constant is reinforced, and the concept of renaming is taught through the game play. *Hit the Spot* specifically introduces the joining of two different variables — x and y — and constants. There are two spinners. The sectors of the spinners have cup pictures representing cups that have y counters in them, envelope pictures representing envelopes that have x counters in them, and large dots representing counters outside the cups. Each spinner has six different combinations of cups with y counters in them, envelopes with x counters in them, and counters outside. Players spin both spinners and

join the two sectors spun to form one symbolic representation. The players are given the concrete materials to engage with. Figure 5 shows an example from the game. The use of the addition symbol for joining variables with variables and variables with constants needs to be made explicit to the students, and the use of the materials helps to show this. This physical demonstration of addition helps the students to form it as a separate concept from multiplication. This is essential for success in further algebra, particularly with the distributive property.

$$x+y+4+x+y$$
$$2y+4+2x$$
$$2(x+y+2)$$
$$y+1+y+1+1+1+x+x$$

Figure 5. Materials on two spinners and the corresponding possible symbolic representations

The game genre is a grid board. The game is for two players, or pairs of players. The board has forty squares each being large enough to hold a counter. The possible symbolic representations from the spinners are in the squares. The player places a counter on a square with a symbolic representation of the joining of the sectors spun on the spinners. The first player, or pair of players, to have four connected spaces on the board is the winner. For this grid board, there are four squares as free choice or "wild" squares. They have a picture of a butterfly on them instead of algebraic symbols and can be accessed by the players, at their discretion.

In *Hit the Spot* there is also the opportunity to teach interpersonal skills, with group discussion occurring over game etiquette. Players are given the suggested definition of "four connected spaces" on the instruction sheet yet there is room to enhance this if the group desires. This genre has the added dimension of strategic placement of counters, as a feature, and this can lead the players to discuss if removing an already placed counter for strategic reasons is allowed, and to decide on the criteria for accessing the "wild" squares.

Renaming with algebra — expanding brackets

This game is a rich exploration of the distributive property — the algebraic renaming process of "expanding brackets." The game explicitly teaches the link between pattern and symbols. Spinners are used to generate random expressions, with brackets written with algebraic symbols, and students are required to expand the brackets. The teacher has several different game boards available to issue to students. Figure 6 shows three possible game boards. The boards are increasing in difficulty and the teacher can choose which board to give, dependent on where the students are in their learning. Each board has three blank spaces to be filled by each of the three identical spinners on the spinner board. One of the spinners is shown in Figure 7. The spinners each have four integers and two letters. Thus, there are 147 different possible expressions per game board.

$$\boxed{} \; \left(\; \boxed{} \; - \; \boxed{} \; x \; \right)$$

Figure 6. Three possible game boards.
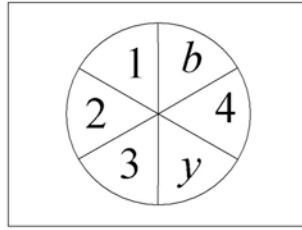
Figure 7. A spinner.

The first game board can be used as a whole class investigation, to establish the pattern for the process of expanding. The randomness of the spinners creates the "need to know" in the students, when an expression such as $b(4 + y)$ is generated. At this point, the cups and counters model, described previously, is not useful. Prior to now, in the program, students have used the materials with other games and activities to replace the materials with an understanding of the pattern and, thus, the multiplication involved in the distributive law. An example of this is demonstrated in Figures 8 and 9 using the expression $4(y + 3x)$.

Figure 8. Expansion represented in an array using materials.

$$y + 3x$$
$$y + 3x$$
$$y + 3x \quad \Longrightarrow \quad 4 \times y + 4 \times 3x$$
$$y + 3x \qquad \qquad = 4y + 12x$$

Figure 9. Expansion using symbols as an array and then as a multiplication.

When an expression such as $b(4 + y)$ is presented, students are required to apply that pattern, in a symbolic way, to rename $b \times 4 + b \times y$ and then use the symbolic conventions to simplify to $4b + by$. As the students progress through the game boards, they will be required to extend their understanding of the distributive law and the symbol conventions of algebra by expressions such as $y(b + yx)$ and $b(bn - y)$. These expressions require a thorough knowledge of the distributive law and symbolic

convention knowledge such as $y \times y$ is written as y^2 not yy and $y \times b$ is generally written as by not yb.

The game genre is a concept game, in that it is used to introduce and explore a new understanding. To create the competition element inherent in games, it uses a generic grid board shown in table 1. Each square is blank and large enough for a counter to be placed on it. Similar interpersonal issues can arise as in *Hit the Spot*.

When a player gets a question correct, the player places a marker on the grid game board anywhere they choose. The first player or pair of players to have 4 connected spaces on the board is the winner.					

Table 1. Generic grid board.

Conclusion

Using instructional games in the teaching of algebraic understandings, is a rewarding experience for both the teacher and the students. The students gain insight at many levels, from reinforcement of concepts, through to a thorough understanding of complex concepts together with interpersonal skills. The teacher is given flexibility in the management of the classroom and is readily able to meet the learning needs of the students. Groups work from where they are at in their learning. Planning the program within the framework discussed in this paper, so as to explicitly teach the links between symbols and materials, symbols and language, and materials and language, ensures the teaching is about understanding as opposed to procedural knowledge. The management of the physical resource of the games, in terms of time taken in preparation, can be made easier through the use of computer generated game boards. The classroom environment is different to that of a traditional worksheet or textbook lesson, as the students willingly engage with the activities, discuss their mathematics and negotiate processes which are meaningful to them. At the same time, the teacher gains insight into the students' thinking that informs the planning of future teaching and learning pathways.

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Mathematics and food: Essentials for life

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This paper focuses on the interpretation of tables as an essential skill needed in our everyday lives. There are two main kinds of tables: firstly, the tables accessed for specific information such as those on food or medicine packets and transport timetables and, secondly, tables of data such as those on environmental and social issues. This paper will highlight the mathematical concepts and skills which can be developed in the mathematics curriculum and that are needed for interpreting both types of tables. It will also describe how a Five Step Framework *can be used in primary, secondary and tertiary mathematics education to help students develop their skills in the interpretation of tables of data.*

Introduction

Advances in technology and communications have increased the extent to which people worldwide are exposed to quantitative information in text, graphs and tables. Over the last twenty years, more attention has been given to the development of the numeracy and statistical literacy of citizens (Gal, 2003).

There is a wide range of issues (e.g., pollution, terrorism, traffic accidents, logging, nuclear power) in which informed decisions need to be made at national or local levels and for which some understanding of mathematical concepts and skills is essential. The context chosen for this paper concerns food and health but the choice of context is quite arbitrary and the ideas presented are relevant to other contexts of contemporary relevance.

There is growing public concern about health issues such as obesity, lack of exercise, high alcohol consumption, smoking, diabetes and cardiovascular diseases. In countries such as Australia and the USA there is concern about the increase in the percentage of overweight and obese children and the associated risk of diabetes. The media give us an almost daily update from nutritionists, environmental health experts, fast food companies, school principals and parents on the social, economic and environmental effects with respect to the perceived impact of the consumption of various kinds of food.

To make informed decisions in this area of food health, the general public need to be able to interpret the quantitative information they encounter in text, graphs or tables published in magazines and newspapers, on the television, in films and on the Internet. School and university students also use sources such as textbooks, health reports, environmental health reports and academic journals.

Much of the information people encounter is in graphical form. The literature provides a rich source of information concerning considerable research on the advantages of graphical representations (e.g., Tufte, 1983; Wainer, 1992) and on ways that these graphical displays can be misleading (e.g., Dewdney, 1993). It is common

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practice in schools and universities for students to examine graphs and look for such aspects as incomplete, inaccurate or missing scales. As well, it has been suggested by Wainer (1992) that "there is ample evidence that the ability to understand graphically presented material is hard-wired in" (p.18).

On the other hand, research into the interpretation and construction of tables is much more limited and probably reflects the limited emphasis given to the developing and applying skills of interpreting and constructing tables in the education system (Koschat, 2005). However, Mosenthal and Kirsh (1998) have made significant contributions to the field through their work in the 1980's on three interrelated constructs Prose/Document/Quantitative Literacy, which informed the large-scale NAAL study in the USA (OECD, 2003).

Two kinds of tables

There are two main kinds of tables. Firstly, the kind consulted for specific information, referred to by Mosenthal and Kirsch (1998, p. 639) as *reading-to-do*. Examples of these are found on medicine, food and health supplement packages and on bus, train and flight timetables that require some understanding of travel on specific days such as Sundays, and crossing of time zones.

Secondly, there is the kind of table that includes collected, produced and collated data that can be used to develop an understanding of the associated context. This is the kind of table referred to by Mosenthal and Kirsch (1998, p. 639) as *reading-to-comprehend*. Examples of this kind of table include those with data on unemployment, traffic statistics, poverty, disease, and educational achievement that would be found in newspapers, journals and reports. It is unlikely that people consult this kind of table simply for a particular value; they are more likely to want to make comparisons, to look for trends over time, and to draw conclusions.

Being able to interpret both kinds of tables is important. The interpretation of *reading-to-do* tables receives some attention at the school level but the *reading-to-comprehend* much less so (Kemp & Kissane, 1990). The interpretation of both kinds of tables will be considered in this paper in the context of food and health, and the use of a *Five Step Framework,* to help students build up strategies for interpreting *reading-to-comprehend* tables, will be described in some detail. The importance of the careful construction of both kinds of tables is closely associated with the interpretation of tables, but addressing this is beyond the scope of this paper. The work of Mosenthal and Kirsch with their IKIRSCH/PMOSE readability analysis (Mosenthal & Kirsch, 1998) and associated papers is useful in this area.

Tables for specific information

In this section the context of food has been chosen to illustrate the processes needed to read tables for specific information. In the current climate of concern about underweight, overweight and obese young people there is a need for information about what is contained in the foods that they are consuming, and what they should be eating for a healthy life. There are many books and resources on the Internet that can be consulted to find out about such things as suitable kilojoule intake, proportions of fat, sugar and salt, servings of meat, fish, vegetables and grains, many of which present the information in tabular form.

A specific example of the relationship between food consumption and health concerns the incidence of Diabetes type 2 in young people, which has been increasing

over the last decade. For these young people and their families a sound understanding of dietary requirements in essential. It is generally recommended that for people with Diabetes type 2, low glycaemic index (low GI) foods are preferred. A general rule of thumb suggests that the foods should contain not more than 5 g of total fat per 100g of food, and not more than10g of sugar per 100 g of food (or it can be higher in sugar if a low GI as these contain fructose as well as glucose). Some people also recommend the salt intake to be less than 2300g per day.

Reading the tables on food packets

Where do people go for the information they seek? Packaging in the supermarket provides a wealth of information on the labels in table form that needs to be interpreted. The tables are often quite complex with a number of different kinds of units such as millilitres (ml), grams (g), milligrams (mg), micrograms (µg), kilojoules (kJ) and, sometimes, with all of these in the same table. Packets usually provide information per serving and per 100g; as serving sizes can differ, the best place for comparison across packets is in the Average Quantity Per 100g column. Sometimes the figures in the serving size column and the 100g column show inconsistencies, probably due to rounding, which may be confusing to the general public.

For example, three tables from *Premium* crispbread packets: *Original*, *98% Fat Free* and *98% Fat Free High Fibre* are reproduced in Figure 1.

Figure 1. Labels from three Premium Crispbread packets (Kraft Foods).

The packets say the contents of the *Original* packet weigh 250g, the *98% Fat Free* weigh 205g and the 98% *Fat Free High Fibre* weigh 240g. The packets all advertised 2 STAY CRISP packs. In addition, the *Original* is LIGHT AND CRISP, the *98% Fat Free* LOW IN SUGAR and the 98% *Fat Free High Fibre* HIGH FIBRE. In the supermarket, where these packets were purchased, each packet cost $2.29. They have been chosen to illustrate the kinds of mathematics needed to decide whether or not to buy a product, with health issues in mind. There could be a number of health reasons for looking carefully at the labels: for example, checking content of particular foods for people allergic to foodstuffs such as nuts or honey, for restricting overall calorific intake or for catering for special requirements like those related to Diabetes type 2.

For the purposes of this paper the following scenario is proposed:

> A shopper looking to buy a packet of crispbread suitable for a person with Diabetes type 2 observes that for this brand there are three versions: Original, 98% Fat Free and 98% Fat Free High Fibre. Which one (if any) would you choose and why?

Firstly, the consumer needs to compare each packet with the dietary requirements for Diabetes type 2 and, then, make comparisons between the packets. This involves looking at the headings, the lists of ingredients and the columns that indicate serving size and information per 100g. When comparing three products, the consumer needs to make multiple comparisons, the number depending on the nature and complexity of the information they seek. In addition, any comparisons are probably made in the supermarket itself in practice, at least at first.

Under NUTRITIONAL INFORMATION the first observation concerns the number of servings per package and the serving size. The *Original* has 8 servings of 30g (approx 5 biscuits) which multiplies to 240g (although the packet says 250g), the *98% Fat Free* has 7 servings of 30g (approx 5 biscuits) which multiplies to 210g (although the packet says 205g) and the *98% Fat Free High Fibre* has 7 servings of 30g (approx 5 biscuits) which multiplies to 210g (although the packet says 240g). There are some inconsistencies here in weight and cost per biscuit!

To investigate the weights, the biscuits were taken out of the packaging and weighed. For the three packets purchased the actual weights were: *Original:* 266g which is 6% more than the packet states, *98% Fat Free:* 228g which is 11% more than the packet states, and *98% Fat Free High Fibre:* 273g which is 14% more than the packet states. This sample of one set of three packets has highlighted the variation and could lead to some investigation of variation in the average weights of the biscuits in the packets. People will no doubt interpret the average as the mean but some discussion about the ways that the term average is used in everyday life would be useful, in this context. It is helpful to have the number of servings per package but the manufacturers often omit the number of items per serve, choosing instead to indicate only the weight of the serving size, without the number of pieces.

For the scenario above, the essential information concerns the total fat content and the sugar content per 100g, both of which need to be located in the table, amidst the other information provided. In this instance the sodium content will just be noted.

Premium Original (LIGHT AND CRISP)

The total fat content of 16.1g is above the 5g per 100g requirement and so is too high, but the sugar at 2.4g per 100 g falls into the guidelines of less than 10g per 100g. The product is not suitable for the Diabetes type 2 diet (sodium 829mg per 100g).

Premium 98% Fat Free (LOW IN SUGAR)

The total fat content of 1.6g per 100g confirms the claim on the packet of being 98% fat free. The sugar content is less than 1g per 100g and the product is acceptable for the Diabetes type 2 diet (sodium 505mg per 100g).

Premium 98% Fat Free High Fibre (HIGH FIBRE)

The total fat content of 1.8g per 100g confirms the claim on the packet of being 98% fat free. The sugar content is 3.5g per 100g and , thus, the product is acceptable for the Diabetes type 2 diet. However, it may be noted that this is an increase of 1.1g of sugar per 100g on the *Original* version (sodium 473mg per 100g).

None of the packets claims to be "low-salt" and so there is no need to conform to the requirement of not more than 120mg of sodium per 100g; nevertheless, it can be seen that the sodium content decreases from *Original* (829mg) to *98% Fat Free* (505mg) and to *98% Fat Free High Fibre* (473mg).

Which product should we choose? This everyday decision requires a good deal of skill at extracting and interpreting information from the tables. None of the products is

labelled low GI. The *Premium 98% Fat Free* and the *Premium 98% Fat Free High Fibre* products offer viable options. The *High Fibre* version has an advantage of the added fibre of 10.2g per 100g but, ultimately, the choice may rest with the preferred flavour after trying both products.

Misleading information

There is a very large range of information presented in the media concerning the constituents of fast foods. An example is the article in the *The West Australian* on 27 December 2006, "Hamburgers 'better for kids than some cereal'" (p. 3), which gave a pictorial, tabular analysis of the claims of the cereal manufacturers and the actual situation. It is difficult to read and only gives partial information about the products. The only tick went to Uncle Toby's Oats. Other articles such as "Fat and salt out of the box as Japanese meal put to the test." published in *The West Australian* on 10 May 2007 illustrate the need to look at information provided at takeaway outlets. Many of them provide information on the packaging. The article suggests that the Japanese takeaway might not be the "healthy low fat alternative option you have been led to believe", "you would be doing yourself a favour by choosing a greasy doner kebab instead" (p.3). The Japanese box has twice the total fat of a doner kebab and 50% more salt.

There are many sites on the Internet giving advice about nutrition and food. For example, current one gives advice about *Ten Foods You Should Never Eat (2007)*, showing that the results of not reading the label properly can be substantial:

> Judging by the label, **Pepperidge Farm Original Flaky Crust Roasted Chicken Pot Pie** has 510 calories and 9 grams of saturated fat. But look again. Those numbers are for *half* a pie. Eat the entire pie, as most people probably do, and you're talking more than 1,000 calories and 18 grams of saturated fat. Then add the 13 grams of hidden trans fat (from the partially hydrogenated vegetable shortening) in each pie and you're up to 31 grams of artery-clogging fat — that's far more than a day's allotment.

Interpreting the *table-to-read* kind of table

This section has highlighted the importance of being able to competently understand the quantitative information provided in tables on food packets and in fast food outlets. It has also indicated how some consumers need to be aware of the kinds of units used and their relative sizes. It has given examples of the kinds of comparisons that can be necessary to ensure that consumers are, indeed, aware of the food content in relation to their requirements. To make these comparisons within and across the tables requires mathematical skills of a higher order, which need to be developed at school. In the scenario chosen for this paper the tables are fairly straightforward but there are many that are not. Students can start to develop these skills in primary school, while becoming aware of healthy lifestyles.

Tables of data

In this section the focus is on reading and interpreting tables of data; that is *tables-to-know*. These are different from those in the previous section in that the numbers in the table may be derived from a variety of sources such as census data, health surveys, attitude surveys and opinion polls. In this case, the reader needs to be able to decide what the data are about; to gauge the reliability of the data; to understand what the numbers represent; to look at changes over time and differences within and between

categories, and to consider implications of the information in the light of prior knowledge. Experience has shown that many students skim over these kinds of tables, relying on the possibly biased interpretations of the authors of the accompanying text. Concern about the lack of engagement of first year students studying to be primary teachers led to an empirical study to investigate the effectiveness of an intervention workshop, using a *Five Step Framework* (Kemp, 2006) (see Table 1) and a follow up study with first year science students (Kemp & Bradley, 2006).

Table 1. Five Step Framework.

Step 1: Getting started

Look at the title, axes, headings, legend, footnotes and source to find out the context and expected reliability of the data.

Step 2: WHAT do the numbers mean?

Make sure you know what all the numbers (percentages, '000s, etc.) represent. Look for the largest and smallest values in one or more categories or years to get an idea of the range of the data.

Step 3: HOW do they change or differ?

Look at the differences in the values of the data in a single data set, a row, column or part of a graph. Repeat this for other data sets. This may involve changes over time, or comparisons within categories, such as male and female, at any given time.

Step 4: WHERE are the differences?

What are the relationships in the table that connect crime, ages and gender? Use your ideas from Step 3 to help you make comparisons between columns or rows in a table or parts of a graph to look for similarities and differences.

Step 5: WHY do they change?

Look for reasons for the relationships in the data you have found by considering societal, environmental and economic factors. Think about sudden or unexpected changes in terms of state, national and international policies or major events.

The *Five Step Framework* was designed to help students develop strategies to interpret tables of data. Both the framework, and the measuring instrument, were based on the SOLO taxonomy (Biggs & Collis, 1982) and the measuring instrument, which rated the responses in terms of the SOLO levels (unistructural, multistructural, relational and extended abstract) was refined using Rasch analysis (Rasch, 1960). Using dependent t-tests it was found that there was a significant improvement ($p<0.001$) in performance on a table interpretation task as a result of completing the workshop using the framework with the students.

The interactive workshop was designed to use this framework as a basis for developing skills to interpret tables by applying it to a specific table, providing students with a set of questions relating to the table, and encouraging them to think critically about what the data tell them.

In the original workshop students examined the table *Proportion of selected risk factors by quintile of relative socio-economic disadvantage of area, 1995* concerned with health risks by quintile from the Australian Bureau of Statistics *Yearbook 1999*. Students examined the figures for each quintile and risks for the average wage earners. They discussed, and thought critically about, the relative occurrence of people at risk and the possible social, economic and environmental influences in society.

Subsequent workshops, that have not been in the study, have used different tables depending on the context of the students' studies. They have included ones on environmental concerns, on poverty, on crime and punishment and so on. They are always conducted with the students in groups and incorporating interpretation of the data together with critical discussion about the issues. An example of such a workshop on changing food consumption, which has been used both with upper primary school and university students, is included here. The level of sophistication of the questions can be adjusted according to the level of sophistication of the students. Table 2 is based on Table 16.19 from the Australian Bureau of Statistics *Yearbook 2003* and Table 3 includes some of questions from the workshop concerned with changing Australian food consumption over the 20th century.

Table 2. Apparent Per Capita Consumption of Foodstuffs

Commodity	Units	1939	1949	1959	1969	1979	1989	1999
Beef	Kg	63.6	49.5	56.2	40.0	64.8	40.0	36.3
Mutton	Kg	27.2	20.5	23.1	18.8	3.6	7.3	5.4
Lamb	Kg	6.8	11.4	13.3	20.5	14.4	14.9	11.3
Bacon/Ham	Kg	4.6	5.3	3.2	3.6	6.0	6.9	8.6
Chicken	Kg	NA	NA	NA	8.3	17.1	24.1	29.4
Milk & products	L	106.4	138.7	128.7	128.2	100.5	101.7	103.2
Cheese	Kg	2.0	2.5	2.6	3.5	5.3	8.8	10.7
Margarine	Kg	2.2	2.8	NA	4.9	8.5	9.0	6.6
Butter	Kg	14.9	11.2	12.3	9.8	5.1	3.2	2.8
Tea	Kg	3.1	2.9	2.7	2.3	1.7	1.2	0.8
Coffee	Kg	0.3	0.5	0.6	1.2	1.6	2.0	2.2
Beer	L	53.2	76.8	99.7	113.5	133.2	113.1	94.4
Wine	L	2.7	5.9	5.0	8.2	14.7	20.2	19.5

Australian Bureau of Statistics *Year Book 2003*. The Australian Bureau of Statistics has calculated the average amounts of food eaten per person over about the last 70 years. They calculate the food production in Australia, subtract exports, add the imports and divide the total by the number of people living in Australia for that year.

This paper does not present an extensive discussion of student responses to this table and the use of the Framework in Table 3; only a few observations can be made here. Typically, students do not notice, at first, whether the numbers are in percentages or '000s and so on. In this table, students need to realise that the values in the table do not show the total food that an actual person eats but, rather, the average of the available food after adjustment for imports; in doing this, they begin to recognise the importance of reading footnotes and headings.

Students have difficulty in deciding how big a change needs to be, if it is to be significant; whether, for example, there is much change in mutton consumption from 1939 to 1969, or whether the fluctuations are normal variation. The comparisons across two rows are quite complex and need to be built up to slowly by making comparisons of two years in two rows initially, then expanding further. The younger students need prompting to consider what has happened over the years, while mature age students remember the phasing out of free school milk, for example.

Table 3. Use of the Five Step Framework for Interpreting Table 2

Step 1: Getting started

Q1: What is the table about?

Q2: What is meant by NA?

Q3: What is meant by per capita?

Step 2: WHAT do the numbers mean?

Q1: What does the number 27.2 in the first column of the table for mutton represent?

Q2: What does the 103.2 in the last column of the table represent?

Step 3: HOW do they change?

Q1: For which year was the consumption of mutton the least? _____the most? _____

How does the consumption of beef change between 1939 and 1999?

Q2: For which year was the consumption of lamb the least? _____the most? _____

How does the consumption of bacon and ham change between 1939 and 1999?

Q3: For which year was the consumption of tea the least? _____the most? _____

How does the consumption of tea change between 1939 and 1999?

Q4: For which year was the consumption of coffee the least? _____the most? _____

How does the consumption of coffee change between 1938 and 1999?

Q5: For which year was the consumption of beer the least? _____the most? _____

How does the consumption of beer change between 1939 and 1999?

Q6: For which year was the consumption of wine the least? _____the most? _____

How does the consumption of wine change between 1939 and 1999?

Step 4: WHERE are the differences?

Q1: Compare the consumption of mutton with lamb from 1939 to 1999

Q2: Compare the consumption of tea with coffee from 1939 to 1999

Q3: Compare the consumption of beer with wine from 1939 to 1999

Step 5: WHY do they change?

Q1: Overall how have the diets of Australians changed?

Q2: What kinds of influences in society and the environment might explain the changes?

The experience of students, generally, is that the support provided by the Framework for tables of this kind is both necessary and helpful. Once students become accustomed to considering the statistics in terms of external factors, they usually become interested in the research necessary to make connections. Hopefully, they carry these skills to other areas of study.

Interpreting the *table-to-comprehend* kind of table

When effectively interpreting these kinds of tables, students can use the kinds of steps in the framework to help them to consider the kind of sampling used in the collection of the data, to consider the variation in the data and to think about these in context of the table. Students need to develop critical thinking skills to analyse the data in terms of social, environmental and economic factors. This context encourages students to examine the influences which have impacted on our changing diets over the last century. A logical step next is to make comparisons across countries using international data. For younger students the interpretation of the numbers needs more scaffolding. Some of this is on the current Australian Bureau of Statistics website in the *Statsercises*.

Implications for the mathematics learning area

The tables that students encounter, such as those presented in this paper, are related to our everyday lives. Starting from a context, students can see why they need the mathematical and statistical skills necessary to make sense of the tables. Watson (2006) has examined the links connecting those components of schools chance and data curricula, in accordance with the *National Statement on Mathematics for Australian Schools* (AEC,1991), and concludes that in terms of mathematical skills students need "an understanding of rates, percents and part-whole relationships generally" and for statistical skills "an understanding and calculation of averages, basic numerical probabilities and probabilities of compound independent events" as well as an understanding of basic terms (p.249).

The development of the mathematical concepts and skills necessary to competently interpret both kinds of tables should be done in contexts relevant to the students. The ability to interpret tables of data can be developed using the *Five Step Framework* which can be used to encourage students to gain a deep understanding of the data and to think critically about the associated social, environmental and economic issues. Ideally, students will transfer this approach to other areas in the curriculum, resulting in developing critically thinking people who can make informed decisions in essential area of their lives.

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Spreadsheets, graphics calculators and mathematics education[*]

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Over recent years, spreadsheets have been increasingly recognised as having a role in teaching and learning mathematics, but have continued to suffer the limitation of requiring access to computers. This paper summarises some of the potential benefits of spreadsheet use and briefly assesses some of the available research about it in schools. A spreadsheet application for a calculator is described and interpreted in terms of its capacity to handle the essential features of spreadsheets. The comparative merits of calculator and computer spreadsheets are discussed. Some ways in which calculator spreadhseets might be used for teaching and learning mathematics are outlined.

Introduction

While originally developed for the business world, spreadsheets continue to be used for purposes of mathematics education in secondary school. While this can be explained partly because of their widespread availability on personal computers in schools, homes and businesses, it is also a consequence of their particular suite of capabilities, which lend themselves to some kinds of mathematical work. Access to a spreadsheet has remained dependent on access to a computer, until fairly recently. In this paper, we first identify the features of spreadsheets likely to be of benefit to teaching, learning and doing mathematics, and then consider the prospects of a spreadsheet on a graphics calculator having significance for school mathematics.

Features of spreadsheets

This section describes briefly the key features of spreadsheets from a mathematical perspective, particularly with inexperienced readers in mind. Several useful links to augment this information considerably are provided on the web (Kissane, 2007). Some of these are exceptionally useful for inexperienced spreadsheet users.

Essential features

A spreadsheet comprises a matrix of cells that can contain either numerical or textual information. Data in the cells can either be entered directly by a user or can be calculated by the spreadsheet itself, with appropriate instructions from the user stored in the cell with a *formula*. The use of formulas allows some numerical values in a spreadsheet to depend on values elsewhere in the spreadsheet, effectively allowing a number of calculations throughout the spreadsheet to be completed simultaneously. Although other spreadsheets exist, Microsoft's *Excel* is in such widespread use on

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[*] Paper accepted by peer review

personal computers that it is almost universally relevant, and will be used here as the archetypal spreadsheet, without further direct reference. Other spreadsheets (such as those included in *ClarisWorks* and *AppleWorks*) are very similar at the level of basic operations: differences appear only at the more sophisticated end, which is not of particular interest in this paper.

Spreadsheet formulas generally look a little different from typical algebraic formulas, although they have the same function. A spreadsheet formula uses a cell address (typically a letter that references a column and a number referencing a row, such as B3) to refer to the number in a cell, rather than the single letter characteristic of algebraic representation (such as x). Spreadsheet formulas are recognisable in spreadsheets as they start with an equal sign and refer to some other cells. Thus the formula, =A1+3, stored in cell A2 would calculate three more than the value in cell A1 and store it in cell A2. In conventional algebraic representation, if the value in cell A1 was represented by x, this formula would calculate the value $x + 3$ and store this in cell A2.

As well as direct arithmetical relationships, modern spreadsheets include a number (in fact, a large number) of mathematical, statistical and other functions that can be used in formulas and abbreviated ways of referring to sets of arguments of the functions. As an elementary example, the formula, =AVERAGE(A1:A20), calculates the average of the values in the twenty cells, A1, A2, A3, …, A20. If this formula were stored into cell A22, the spreadsheet would display the mean of the column of values in A1 to A20. If any of the values in the column were changed, the mean would be changed accordingly (when the spreadsheet is set to recalculate automatically). Formulas provide a means of constructing mathematical relationships of various kinds, and thus generating them can be thought of as a kind of programming.

A powerful feature of spreadsheets concerns *filling* a formula, which allows for equivalent formulae to be constructed automatically by the software, rather than having to be entered manually. To illustrate this, the average formula above could be readily filled across into row 22 of columns B to D; the effect of this would be to automatically generate the formulae to find the means of the first 20 values of columns B, C & D. Although the formulas could be entered separately in each case, this facility allows an efficient means of doing the same thing.

These two capabilities (*formulae* and *filling*) together allow a spreadsheet to be readily used to study *recursive* situations in mathematics. The most obvious examples are those concerned with recursively defined sequences, which can be represented easily in a spreadsheet. The recurrence relation can be defined in terms of a formula (a very simple case is the one described at the start of this section, using the formula, =A1+1), and then filling this formula down a column . This process effectively repeats the same relationship in successive cells. For this case, the filling down process will define an arithmetic sequence with common difference of 3.

While the spreadsheet itself comprises an array of cells with text, formulae or numbers in them, spreadsheets generally have mechansims for representing the information in more helpful ways, especially graphical ways. These essentially allow the user to define a graphical representation (such as a scatter plot or a histogram) of selected data from the spreadsheet itself.

Advanced features

The features described above as essential have been built into spreadsheets for many years. Later versions of spreadsheets have included extra features that make them more

powerful, easier to use or otherwise more versatile in representing and handling numerical information. Space precludes an extensive description here, although it is important to acknowledge that advanced features are unlikely to be available on graphics calculators, in the near future.

Some advanced features arise from the capacity of a computer to use a range of colours on powerful visual displays. Thus, conditional formatting features allow different cells to be coloured differently, depending on their content which, in turn, allows for impressive visual displays of mathematical objects (such as the Sieve of Eratosthenes, Pascal's triangle or a Sierpinski Gasket).

Other advanced features concern the range and sophistication of mathematical and statistical functions routinely available. Thus, successive versions of *Excel* tend to contain yet more mathematical functions, to meet the wide range of needs of users for calculations of various kinds.

Other features are concerned with designing user-friendly interfaces, with easy ways for users to input values into a cell (e.g., to vary the values of a parameter in a relationship). Two methods used to good effect recently by Staples & Smith (2005) and Drier (2001) involve *scroll bars* and *spinners*. Such features are very useful for designing stand-alone spreadsheets for students to use to experiment with mathematical ideas.

Spreadsheets in mathematics education

In the first issue of a new electronic journal devoted to the use of spreadsheets in education, Baker & Sugden (2003) surveyed the literature on the topic over 25 years, and concluded that " there is no longer a need to question the potential for spreadsheets to enhance the quality and experience of learning that is offered to students." (p.32) While some of the areas of educational use concern matters that are not directly related to mathematics, and others concern mathematical ideas not of concern to school mathematics, they highlight the considerable potentials of spreadsheets in some key areas. These areas include the elementary study of algebra, aspects of financial mathematics, statistics, numerical analysis and combinatorics (e.g., Abramovich & Pieper (1996)), among others.

Similarly, in reviewing published research in the area, Jones (2005) identified a number of aspects of algebra which researchers have claimed are supported by work with spreadsheets, and also recognised issues related to teaching and learning statistics. An important early study in this area was that of Sutherland & Rojano (1993), concerned with the development of early algebraic understanding using spreadsheets, in both the UK and Mexico. The spreadsheet offers an environment in which early ideas can be developed regarding functions and equations and ways of respresenting these.

One of the major attractions of spreadsheets, it seems, is that they are becoming available for use in a very wide range of settings, not only the mathematics classroom. Two important settings are those of the home and the workplace, in both of which personal computers are becoming much more common, and virtually ubiquitous in workplace office settings. These days, the bundled software for new computers usually includes a spreadsheet application, and this is certainly true for office computer systems. An application that is likely to be available so widely certainly deserves to be considered as a potential tool for teaching and learning mathematics.

Ruthven & Hennessy (2002) studied teacher accounts of their practices with ICT in a small number of English secondary schools and noted that spreadsheets were available in all of the schools studied; further, they found that in many cases students are more

likely now, than previously, to encounter spreadsheets in settings outside mathematics. This means that some of the lost time getting started is dimishing. Such matters, of course, depend on the practices and facilities at particular schools, to some extent.

In contrast, D'Souza and Wood (2003) studied Year 11 General Mathematics students in NSW, with a focus on the Financial Mathematics topic of the course. They found that in many schools, there were significant practical access issues related to the adequacy of computer resources and support and professional development associated with new curricula, which together reduced the success with which spreadsheets were used. In concluding their study, the researchers identified "… practical difficulties of the implementation and reasons for students' resistance towards working with computers: too few working computers, computers not working properly, the time taken to learn the software, lack of computer confidence, etc." (p. 293). The authors note that these problems are not insurmountable but, nonetheless, will impact on the success of innovations of the kind described.

Baker and Sugden (2003, p. 24) suggest that a major advantage of spreadsheets for school mathematics is that they might save time for many activities, thus creating new opportunities for exploratory work of various kinds. Ruthven and Hennessy's (2002) paper describes, in some detail, the kinds of pedagogical approaches which technology (not restricted or unique to spreadsheets) seems to support, at least in teachers' reported practices. In circumstances more favourable than those referred to by some of the students in the sample of D'Souza and Wood, spreadsheets appear to offer promise.

Of additional interest is the extent to which software tools are permitted for student use in formal high-stakes assessment. Generally speaking, computers are not permitted in formal examinations in Australia, although there are opportunities for schools to permit their use for in-school assessments, which are significant for several school systems and states.

Increasingly, textbook authors seem to assume some level of access to spreadsheets, especially in mathematics courses related to finance. An early example of textbook use was Lowe et al. (1994), who used spreadsheets as one environment in which algebraic relationships can be represented and equations solved. Recent textbooks published for the NSW General Mathematics course have included some work on spreadsheets, supported in part by formal requirements for students to complete a Computing Skills test, which includes work on spreadsheets, by the end of Year 10 in that state. In recent years, commercial publishers have included CD-ROMs with textbooks for secondary school, and these frequently contain examples of spreadsheets for students to use.

The development of some advanced features, described above, has permitted the production of spreadsheets designed as stand-alone objects to support mathematical exploration by students, demonstration by the teacher, or both of these activities. Good examples, using sliders and spinners in *Excel*, covering a wide range of senior secondary school mathematics topics have been provided by Staples and Smith (2005). Similarly, Drier (2001) has described a number of interactive *Excel* spreadsheets developed for use by teacher education students, some of whom have begun to use them in secondary schools as well. At a more sophisticated level, publications like that of Beare (1994) have focussed on extended examples of mathematical modelling, with significant support from pre-prepared spreadsheets designed for the purpose.

Modern spreadsheet software, such as *Excel*, now includes significant statistical capabilities, not only for descriptive statistics, but also for various inferential purposes, such as hypothesis testing. While these capabilities fall some way short of those of professional data analysis packages, they provide substantial data analysis opportunities for early undergraduate students of statistics, as well as those who work in the business

world, such as MBA students. The advantages of access in computer laboratories and workplaces (without additional software costs beyond those already invested), together with home use advantages, have increased the popularity of this use of spreadsheets in recent years. As noted by Nash & Quon (1996), who seemed to be not very enthusiastic about this use, there are significant criticisms of this practice, especially related to graphical capabilities and some issues of data handling. However, many Australian universities now use *Excel* routinely in the early years at least, because of the significant access advantages.

A graphics calculator version of a spreadsheet

Graphics calculators are clearly much less powerful than computers armed with sophisticated software, which is hardly surprising in view of the very great differences in costs of the two platforms for connecting technology and mathematics. What they lose in capabilities, they gain in access, as noted by many authors recently. For example, Ruthven & Hennessy (2002) reported the common practice in many schools of using hand-held machines "making it relatively easy for teachers to give students access to such technology in the normal classroom setting; sometimes also with projection facilities" (p. 58). To date, graphics calculators have not usually included spreadsheet applications routinely, although exceptions to this have been the very high end devices such as the Casio ClassPad 300 and the Texas Instruments TI-89. No doubt, one reason for this has been related to screen size and memory limitations. Another reason has possibly been related to the limited extra capabilities offered by a spreadsheet over and above what a modern graphics calculator provides.

The recent Casio fx-9860G AU calculator includes a spreadsheet as one of the standard applications, and so is here used to interrogate the potential value of hand-held spreadsheets in mathematics education. One part of the significance of ready access to technology is the flow-on effects of permission to use such technologies in formal examination settings; as noted above, this is rarely possible with computer spreadsheets.

As calculators have a much smaller screen than computers, fewer cells are visible on a calculator than on a computer, so that it is necessary to scroll more often to see the contents of cells. However, the same general cell characteristics are evident in both environments: cells contain text (for labelling), numbers (as data) or formulae (to define some relationships between cells). The syntax for formulae (e.g., starting with an equals sign and referring to cells by row and column) is similar to that for *Excel*, making it convenient for those already familiar with the similarity. There are filling operations on a calculator, as for computer spreadsheets, as well as a range of mathematical functions, similar to those normally available on a graphics calculator, which generally accommodate the needs of secondary school students.[12]

There are fewer statistical operations available in this calculator version of the spreadsheet than in computer spreadsheets. However, all the necessary statistical calculations, including those concerned with hypothesis testing, can be carried out as usual in the statistical area of the calculator; this is facilitated by the calculator commands to import and export data between the spreadsheet and the statistics or matrix areas of the calculator.

Calculator spreadsheets can be saved, edited and retrieved, as well as transmitted to other calculators or exchanged via the Internet. Of course, each might be constructed

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[12]   The operational details of the Casio fx-9860G AU spreadsheet are not fully described in this paper, but will be included in an associated workshop and are described in detail in Kissane and Kemp (2006).

instead by students themselves to explore a mathematical situation, rather than be prepared in advance by the teacher.

In summary, the key capabilities of spreadsheets described above are available via a hand-held calculator, while the advanced capabilities characteristic of recent software is not available in this environment.

Some examples

Clearly, space precludes an extensive catalogue of examples in this paper. Instead, three examples are offered to illustrate the kinds of work made possible by a spreadsheet on a calculator, and to consider this in light of the existing calculator capabilities. Each is briefly described here as an example of a prepared spreadsheet, to be used by students or used for demonstration purposes via a suitable projection device.

Chance

A spreadsheet can be used to provide experience of chance processes in an efficient way. For example, Figure 1 shows a spreadsheet called *Toss a Die* that simulates the tossing of an eight-sided die 200 times in Column A. The average result of the first 20 tosses is calculated and shown in cell B2, while the average of all 200 tosses is shown in cell C2.Every time the recalculate key is pressed, the entire spreadsheet is simulated again, with fresh averages displayed. Although rather slower than a computer, the calculator is still fairly fast, taking about two seconds to regenerate the entire spreadsheet each time. By inspecting the results of generating data in this way, students are able to see that there is a consistent pattern of results for the average of 200 tosses (usually close to 4.5), while the average for the first 20 tosses is less consistent.
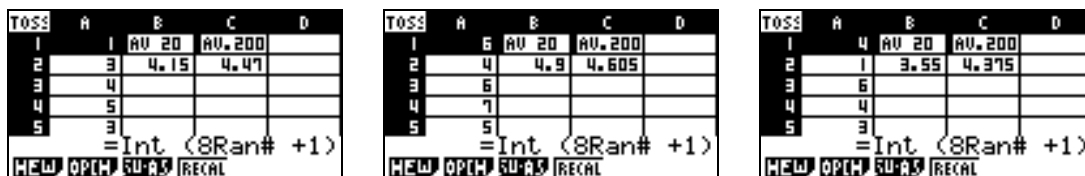
Figure 1. Successive results of tossing an eight-sided die.

A spreadsheet like this can help students get a feel for the nature of chance processes: while results are never certain, they are more predictable with larger amounts of random data available. Furthermore, issues related to sampling and sample sizes can be generated by comparing the stability of the averages of smaller samples.

In addition, changing the formula used in column A to generate the data allows for patterns connecting the die used with the averages obtained, to be inferred and checked. Changing from an 8-sided die with all sides equally likely to occur to a regular *n*-sided die requires only a change from 8 to *n* in the formula.

Simulations of this kind are also available on the graphics calculator through the STAT mode, but it is not possible to obtain immediately the means of the simulated die tosses. For each set of tosses, the mean would have to be separately retrieved, and it would not be possible to consider partial results, as was done in this case with the first twenty tosses. So, the spreadsheet offers opportunities not otherwise available, and seems to augment the existing capabilities well.

Sequences and series

Spreadsheets are good tools for examining sequences and series numerically, and a relatively simple spreadsheet provides a mechanism for doing so. To illustrate this, two separate examples are provided here, one with an explicit definition of terms of the sequence and one with a recursive definition.

The middle screen of Figure 2 shows successive terms of the harmonic sequence, given by the reciprocals of the counting numbers and its associated series in a calculator spreadsheet called *Series*. The first screen illustrates the way in which the graphics calculator spreadsheet is designed to permit easy evaluation of a sequence in column B, when an explicit definition of terms is available. The corresponding series is obtained with standard spreadsheet formulae filled down column C, as suggested by the middle screen.

Figure 2. Exploration of the harmonic sequence and series on a calculator spreadsheet.

The third screen of Figure 2 illustrates how data from the spreadsheet can be readily graphed, using essentially the same mechanisms as those that apply for statistical data. In this case, the graph has been traced to explore the (lack of) convergence of the series.

Figure 3 illustrates the use of a spreadsheet called *Fibonacci* to explore a recursive sequence, in this case the Fibonacci sequence. The first screen shows how the two-step recursion is defined in a formula, which is filled down column B as far as desired. Column C is calculated as the ratio of successive terms of the sequence in Column B. These data can be graphed to show the (rapid) convergence of the ratios to the golden ratio, as shown in the middle screen.

Figure 3. Exploring the Fibonacci sequence.

This spreadsheet allows for easy variations on the theme to be explored in a *What-if* kind of way. For example, the spreadsheet begins with 1 in each of cells B1 and B2, which are needed to define the Fibonacci sequence. The third screens shows that the change of these two cells to B1 = 3 and B2 = 4 results in the rest of Column B changing immediately. Inspection of the graph (not shown here, for space reasons) reveals the remarkable and quite unexpected result that the ratios once again seem to converge to the golden ratio.

In this case, the calculator already permits exploration of sequences through the RECUR mode. However, the spreadsheet introduces a capacity to quickly and efficiently explore the sequences and series; indeed, as noted by Baker and Sugden (2003), the spreadsheet saves time.

Finance

A common office use of spreadsheets involves financial planning of some kind. As noted above, D'Souza and Wood (2003) studied the use of computer spreadsheets in the Financial Mathematics component of General Mathematics in NSW. There are many ways in which elementary financial mathematics ideas can be represented on a spreadsheet for student analysis.

One example is spreadsheet *Reducint*, shown in Figure 4. This spreadsheet has been constructed to explore the concept of reducible interest, which it is necessary for students to understand as consumers, not only as mathematics students, since most practical uses of interest are of this kind. (Credit card statements and housing mortgage repayments are two examples of everyday significance to most Australian families.)

Figure 4. Exploring reducible interest.

The first screen in Figure 4 shows how the spreadsheet has been set up to calculate monthly balances for a $5000 loan at 12% reducible interest and a monthly payment of $170. (The END balance after the final payment is displayed, to avoid needing to scroll to the bottom). The second screen shows immediately the effects of a variation in interest rate to 16%, while the third screen shows the effect of increasing the monthly payment to $180.

Although calculations of these kinds can also be made in the calculator's finance (TVM) module, the spreadsheet allows for all the reducing balances to be seen simultaneously and dynamically, in contrast to the less efficient TVM procedure.

Conclusions

Spreadsheets have acquired a place in the repertoire of technologies of significance for learning, teaching and doing mathematics over the thirty years since their invention in the early days of the personal computer. Part of their significance lies in their widespread availability inside and outside school environments. While recent embellishments of the basic concepts have improved their range of contributions to school mathematics, the essential features are valuable for constructing objects that allow students to explore some aspects of mathematics. The inclusion of these essential features on graphics calculators seems to extend the range of influence of the spreadsheet as a useful device for mathematics education in secondary schools, and is deserving of attention to exploit it appropriately.

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Mathematics websites

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The world-wide-web provides teachers and mathematics educators with access to an extensive selection of papers, references, activities and information. These include materials and activities for topics in the curriculum, mathematical background and content for teachers and students, pedagogical approaches, technology related resources, assessment, mathematics education research, teacher and student sites and authority and department websites. This paper gives a range of illustrative examples and outlines how teachers might use such material to assist and inform their mathematics teaching their professional learning.

Introduction

The terms "internet" (a networking infrastructure that is a global network connecting computers), "world-wide web" (a system of internet servers that provides an information sharing model), "website" (a particular location or site on the world-wide web), "URL" (a *u*niform *r*esource *l*ocator which specifies the global address of resources and documents on the world-wide web) and others, including the more recent "wiki" (a browser interface that enables an individual to edit material placed on a collaborative website) and "blog" (from *web log* is a web page that acts as a publicly accessible journal of an individual) are part of the evolving language associated with our use of a central aspect of information and communication technology (see, for example, HREF1; HREF2).

There are many mathematics and mathematics education related websites that contain useful and interesting material. Websites are typically referenced in terms of their urls, the organisation or person to whom the website belongs, the date of access and sometimes a brief summary description of the nature and/or purpose of the website. More broadly, websites can be characterised in terms of their nature (encyclopaedic, interactive, tutorial, diagnostic), scope (level, area, topic) purpose (describing discipline content, curriculum structures, pedagogical approaches) and intended audience (students, teachers, parents, educators). The ephemeral nature of web-based material makes careful and ongoing scrutiny a critical and essential aspect of its use.

Over time, teachers will likely research, bookmark and systematically review various websites as they find them to be of assistance to their teaching, student learning and assessment. In conjunction with colleagues, systematic cataloguing (for example, by reference to: level, standards, topic, use of technology) and periodic updating of websites can be used to develop a good collection of resources. A selection of relevant websites can be incorporated into course outlines, planners, curriculum documentation, and the like, as applicable (these will, of course, need to be carefully checked for relevance, suitability and currency). This process could be part of occasional faculty staff activity, perhaps following on from an initial detailed survey. A naïve search on

almost any topic, for example, "Fibonacci sequences" using a search engine will yield a diversity of interesting materials (see, for example HREF3).

Types of websites

There are a range of different types of websites that teachers might access, both in terms of their daily work and also for planning purposes. These include:

Curriculum and Assessment Authority/Board/Council and/or Department of Education websites

Teachers would be familiar with such sites in terms of information about curriculum and assessment and related resources, as well as a source of data about systems. For example, the Victorian and Curriculum Assessment Authority (VCAA) website (HREF4). includes the preparatory – Year 10 mathematics curriculum document, progression points and assessment maps, the state-wide tests for Years 3, 5, 7 and 9 as well as Victorian Certificate of Education mathematics courses, past examination papers and assessors reports, and various data about schools and enrolments. This site contains links to Department of Education implementation materials and similar websites for the other Australian systems. Teachers will find that there is a wealth of material from across the states and territories system websites that can be used in implementing curriculum and related assessment. The recently developed Nationally Consistent Curriculum Outcomes (NCCO) *Statements of Learning for Mathematics* and their *Professional Elaborations* can be accessed from the MCEETYA website (HREF5). The College Board US website contains some excellent materials for senior secondary mathematics courses as part of its Advanced Placement (AP) Calculus and Statistics Programs (HREF6).

Professional and subject associations

State, national and international professional and subject associations have websites with various information, forums and materials of interest to teachers. For example, the Australian Association of Mathematics Teachers (AAMT) website contains information about resources (HREF7) while the Mathematical Association of Victoria (MAV) website (HREF8) outlines professional learning related to school based (coursework) assessment. The National Council of Teachers in Mathematics (NCTM) is well known internationally, and its website has a good collection of research and issues articles (HREF9).

Technology companies

Calculator company websites provide some good information and support about their products and how to use them (Casio HREF10; Hewlett-Packard HREF11; Sharp HREF12; Texas-Instruments HREF13). Common information on these company websites includes product specifications, where to buy them, guide books, software and drivers to download, programs and applications, activities, forums and chat groups and a frequently asked question section. Some companies also provide free online tutorials and/or videos via the net to help customers learn how to use their product. The ability to express interest for professional learning workshops, or actually book into them via the web, is also becoming more frequent.

Some companies such as Texas Instruments (HREF14) have a large education site based in their country of origin, as well as additional regional websites with more

tailored information for other parts of the world; when looking for information and materials related to their products it is a good idea to look on both websites. New operating systems, applications and tutorials are frequently found on the main website, whereas market specific activities may be more likely found on the local and/or regional website. For example, Figure 1 shows the type of information that can be accessed from the Texas-Instruments Asia Pacific regional website.

Figure 1

As well as company websites there are a range of related third party websites. These websites can offer great support for reviews, additional programs and applications, a place to ask advice and troubleshooting support, and are well worth exploring to keep up to date with what is being developed and what students might be accessing. It is important to be aware that any programs and applications accessed from these sites have not been checked or endorsed by the actually company and can, at times, have flaws and errors in them, such as giving an incorrect answer under certain conditions. There is also the risk of accessing the occasional program or application that may corrupt the operating system of a calculator and could jeopardise the warranty .

The following are some third party sites: Casio (HREF15; HREF16), Hewlett-Packard (HREF17; HREF18), Sharp (HREF19), and Texas-Instruments (HREF20; HREF21). If one is looking for something specific, a Google search frequently produces a useful list of helpful sites from around the world.

Other institutions and organisations

In recent years international assessment programs, such as TIMMS (HREF22) and PISA (HREF23), have provided systems with data on the performance of students in a broader context, what is perhaps not so well known is that they also provide a wealth of related materials and research findings that are of direct use in classroom teaching and learning. Organisations such as the Australian Bureau of Statistics (HREF24) have provided both data sets for statistical analysis and statistics related curriculum materials on the education section of their website, of which the most recent is the *CensusAtSchool* project. The website of the School of Mathematics and Statistics at University of St Andrews in Scotland (HREF25) is an excellent source of historical material on mathematicians and mathematical developments.

Individual and third party websites (wikis, bulletin boards and blogs)

The *Wikipedia* website contains a range of mathematics articles, which often provide useful teacher reference material: for example, the discussion on the golden ratio φ (HREF26) is comprehensive and includes references to the literature and other relevant websites. However, as material on this site, while moderated, is available for "open-editing" it is important that teachers cross-reference to other sources as well. For an insight into the views of (some) students on various aspects of senior secondary mathematics education around Australia, the *BoredofStudies* website (HREF27) is an interesting student community Bulletin Board : for example, topics include within-state and territory discussions on texts and the like, or the inter-state and territory conversation on topics, techniques and questions from mathematics courses of a similar nature, such as the advanced mathematics courses.

There are many individual websites that provide material of interest to teachers and or students, such as the *Library of Virtual Manipulatives* (HREF28). While this site does not replace students working with the actual tangible items themselves, it offers an alternative by using a medium many students like to work with. The site offers activities for the students to work through for each manipulative as well as a parent/teacher section to help explain the purpose and ideas behind the activities with the manipulative, and Frank Tapson's *Dictionary of Units* (HREF29) which provides a comprehensive dictionary of measurement units, including historical references and links to related websites. For those interested in geometry, *Paul Yiu's* website (HREF30), from the Department of Mathematics at Florida Atlantic University, gives a rigorous but accessible treatment of a range of topics, including Euclidean geometry with all diagrams constructed using dynamic geometry software.

Using the Internet and websites

The use of the Internet and websites can variously be *incidental*, *tactical* or *strategic*. The term "browsing" applies well to the Internet and websites; its use can often be *incidental*, just having a look and seeing what is there (many a good site, forum and/or resource is discovered this way). Sometimes its use will be *tactical* — targeted for a particular purpose, such as finding a range of activities related to a particular topic in

the curriculum to complement other materials. On other occasions its use will be *strategic*, for example as part of a faculty or department project to gather web-based information and resources related to a particular area of the curriculum, such as the development of rational numbers, within the broader context of Number, or the notion of "proof" in geometry/space. In such a context a team of teachers might systematically coordinate their efforts to gather:

- relevant curriculum documentation, materials and advice from a system (authority, board, council, department or region);
- mathematical background material;
- mathematics education research material on approaches to teaching and learning;
- resource materials including activities, presentations, use of technology;
- anecdotal material from teachers' experience;
- information on opportunities for related professional learning;
- suitable assessment tasks; and
- other relevant information and materials.

Such an approach, supported by active use of a school intranet, would enable teachers to develop a core set of key ideas, approaches and materials, which could be progressively enhanced, refined and periodically reviewed. In addition to this, individual teachers could collect their own repertoire of web-based materials that they find particularly interesting or useful.

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Mathematics in the primary years: Mechanisms of change

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Improving mathematics education in the primary years offers some particular challenges. Primary teachers can have a view of mathematics, and themselves as mathematicians, that restricts the range of practices and the level of innovation in practice in the classroom.

Maths for Learning Inclusion is operating in 8 clusters of schools where locally based specialist teachers support their colleagues to become more confident, engaging, and inclusive, teachers of mathematics. The program used the Most Significant Change tool to collect qualitative data as part of its evaluation strategy. From this process came teacher narrative writing identifying the key factors that prompted teachers to try new practices, develop mathematical expertise, rethink relationships with students not performing well in mathematics, and begin to see themselves as confident mathematicians. This workshop will work directly with the writing of the teachers to uncover what practices changed, how they changed, and what were the critical program componants that supported the development of engaging pedagogies and improved student outcomes.

Introduction

This paper describes some of the mechanisms that have operated in changing both teacher mathematics practice and students experience of mathematics learning within Maths for Learning Inclusion (MLI), a South Australian Department of Education and Childrens Services initiative.

The data has been collected using a qualitative methodology called Most Significant Change. This is one data set within an extensive programe evaluation plan developed within a realist evaluation framework. Realist Evaluation Theory recognises that programs are theories for action and change, and each requires the articulation of its theory. Realist Evaluation also recognises that programs exist in contexts and that these contexts will interact with program inputs, in a variety of ways.

Maths for Learning Inclusion: What are its goals and structure?

Improving mathematics education in the primary years offers some particular challenges. Primary teachers can have a view of mathematics, and themselves as mathematicians, that restricts the range of practices and the level of innovation in practice in the classroom. Improving mathematics outcomes for students from low socio economic backgrounds and Aboriginal students continues to present a challenge for education systems.

Improvement in outcomes for these groups of learners was a particular goal of MLI. The program is operating in 44 schools working in 8 clusters where locally based specialist mathematics teachers support their colleagues to become more confident,

engaging, and inclusive, teachers of mathematics. Known as Cluster Co-ordinators, they operate across a cluster of schools. They have no class teaching commitment, and each one has developed a variety of ways of working with teachers, with some drawing on existing mathematics programs, including First Steps and Maths 300.

Most Significant Change methodology was applied to collect samples of teacher narrative writing that helped identify the key factors which prompted teachers to try new practices, develop mathematical expertise, rethink expectations of students not performing well in mathematics, reshape learning relationships and conversations, use data for learning, and begin to see themselves as confident mathematicians.

Maths for Learning Inclusion: What was the program theory?

The overaching program theory for MLI suggests that the path to improved student outcomes in mathematics lies in changed teacher thinking and practice. From the writing of teachers, we hoped to uncover what practices changed, how they changed and what were the critical program components that supported the development of engaging, inclusive pedagogies and improved student outcomes. The mechanisms of change that are captured, through the MSC narratives, give insights into teacher responses to the program inputs and strategies. Teachers varied responses show not just if a program "works", but how and for whom.

Changing mathematics teachers, changing mathematics practice: which comes first?

There is much evidence that programs which attempt to change teacher practice have limited impact on practice, as teachers have a high degree of both privacy and autonomy over what actually happens in a classroom. Equally, many programs find themselves in a position where program inputs are applied across a range of settings and find, at the conclusion that, for some participants, the experience has been significant while other teachers barely show evidence that they have even been part of a program. For any program to "work", those involved must actually "participate".

MLI applied a range of inputs including: practical support from a collegue; both practice and theory based professional learning; opportunities for teacher release; participation in teacher networks; and the construction of clusters to work collaboratively on the program. The evaluation plan of MLI also included the collection and analysis, of multiple sets of student outcomes data that teachers were asked to both interpret and act on. In this they were supported. A key issue for schools and systems attempting to change teacher practice exists around a "chicken and egg" question: which comes first: a change in thinking and understanding or a change in practice? Most Significant Change offered a window into what were the significant activities, events, relationships and learning of the program and when, or how, did these happen.

Identifying changes in thinking and practice: How was this achieved?

Most Significant Change is a participatory evaluation process that has been developed by Jessica Dart and Rick Davies and adapted for use in educational settings by the University of South Australia in collaboration with the South Australian Department of Education and Children's Services.

MSC differs from other forms of evaluation in that it:

- creates space for stakeholders to reflect and to make sense of complex changes that occur as result of the application of a range of program inputs.
- provides opportunities for dialogue, between program participants, to help them make sense of both the changes experienced by others and the value they place on these changes.
- facilitates dynamic dialogue: "What do we really mean by learning inclusion and why do we teach mathematics in the way that we do?"

It is particularly useful for participatory programs, with diverse and complex outcomes. It attempts to deal with the complexity of schools and the practices of teaching and learning through both the writing and processing of teacher narrative. It provides a participant's interpretation of the program they are involved in and, as participants can select any event, new learning or new thinking to write about, it can challenge the view that program designers may have about how a program is experienced by participants.

Most Significant Change: What does the process involve?

In brief the Most Significant Change Process involves a number of steps:
- determine sorts of change to monitor;
- collect narrative writing pieces;
- review and filter writing;
- collate selected narrative pieces for review;
- monitor the process and verify the writing;
- look for patterns, trends and next moves.

In MLI a professional learning session was held for teachers who would be writing narratives, to help them become familiar the process, grapple with the idea of a "significant" change or insight and, also, to highlight the sorts of changes we were trying to monitor. One strategy often used to collect data about topics of concern is to identify domains of change and ask participants to write to these. In this case, the domains were left open.

Within MLI, narratives were used with two particular purposes in mind. The first was for all stakeholders to understand the processes of change for individual teachers, as they themselves understand it. The second was to reinforce a sense of progress and learning for teachers, through reflection and writing.

In analysing the narratives that were collected a number of domains were identified to assist with collation. These reflect the key areas of focus for the program and the areas that teachers had actually chosen to write about. The domains were old teacher beliefs, old teacher skills, teacher mechanisms and new beliefs, changed teaching strategies, issues, learning inclusion, teacher outcomes, student outcomes, and project strategies.

The Most Significant Change narratives: What did they tell us?

Within these domains a number of common mechanisms of change emerged as issues for primary teachers in teaching mathematics, and some insights into the program inputs and the way these inputs were configured, that supported teachers in improving both practice and student outcomes.

For the purpose of MLI the term "program mechanisms" is used to describe the processes of change that have occurred when program strategies or resources are taken by teachers and, in responding to these strategies, changes in "reasoning" occur. Such changes might be in thinking, attitude or motivation that underpin changes in behaviour: "resources + reasoning = mechanism."

A selection of these mechanisms that emerged through the MSC narratives have been identified and named by Gill Westhorpe, a evaluation consultant for MLI, and include what might be described as:

- the "changed purpose" mechanism, in which the teacher's understanding of the goal moves from "all students at same level" — an unachievable goal destined to reinforce a sense of failure — to a goal of "all students make progress." In the quotation that follows, this is supported by an "appreciation" mechanism.

    "I am constantly surprised by the difference in the level of thinking that children exhibit and the different paths they take to arrive at an understanding of the concept… I have greater appreciation of the different levels that children are starting from and a better understanding of the notion that they don't all have to end up at the same point: as long as they have all made some measurable progress from where they were." (MSC)

- the "optimism" mechanism, in which access to ideas and resources triggers a belief that change is possible (here, that positive teaching of mathematics is possible and that all students can access mathematics curriculum and be successful). In the second quotation, this is supported by a "validation" mechanism.

    "From scepticism to optimism — every now and again something comes along to change your way of thinking… The richness of diversity, the wealth of ideas and abundant learning opportunities make this program work for me." (MSC)

    "What I think affects what I do! Attitudinal change. If I think I can do maths with my class and that they will be engaged, enjoy what they are doing and learn, I will do it. My change in headset due to validation (demo lessons, T&D) and quality resources / lesson plans. And I have done it. Maths is integrated more into other curriculum areas, just as Literacy has been in the past." (MSC)

- the "seeing the maths" mechanism, in which recognising mathematics content supports integration across the curriculum.

    "A change? Me realising how much maths was involved in this activity and finding maths ideas for space in the other art ideas I used in this theme." (MSC)

- the "reinforcement" mechanism, in which positive responses to early efforts from students build confidence to progress further.

    "I began small and found some quite immediate, positive changes in students' reaction to maths lessons… I felt more positive, more relaxed about how I was teaching." (MSC)

Mechanisms for change of levels of student achievement and engagement were identified through the MSC teacher narratives. Just as teacher level change incorporates a change in thinking, so does student level change require a change in thinking about mathematics and themselves as mathematicians.

Teachers identified mechanisms for change, for students, that were operating in classrooms. One might be described as "infectious enthusiasm."

"Well, if my teacher's happy about math it can't be all bad". (MSC)

A second might be described as "mathematical self-efficacy" — the confidence to try to work out mathematical problems.

"The attitude of the kids towards the testing and doing the maths. They're not as fearful, not as worried, not as give-uppy, they're quite resilient and… 'I'm going to work this out.'" (MSC)

The third incorporates both self-respect and the desire for the teacher's respect and approval.

"He wants me to care, he wants me to like him and he thinks if he does this then I'll honour him in some way… the difference between day one and now is that he cares enough, he has the respect of himself to want to finish the test… to try."(MSC)

The fourth might be described as the "portable learning" mechanism or the "recognition of mathematics" mechanism.

"It used to be, 'It's maths time, this is when I learn maths.' Like: 'When do you learn about literacy?', 'Oh, all the time, we see literacy everywhere, there's this, that'. 'Where do you do maths?'. 'In maths time'. Whereas now, it's 'Well there's maths here and there's maths there and you use it when you're shopping and you use it when you're getting up in the morning' and all that sort of stuff, so it's just a big change of thinking in kids' minds." (MSC)

The MSC narratives highlight other — or perhaps associated — mechanisms. Two of the most commonly mentioned involve the use of concrete resources and the use of mathematical games. These new resources were associated by teachers with increased engagement in, and confidence in, mathematics:

"Each lesson (as my programming has evolved to) begins with a game. Everyone is always keen to start maths as a result — motivation has increased."

"We prepared coloured pencil cases containing materials including table cards, counters, a range of dice, numbers cards, playing cards and 100 and 200 number charts… This proved to be particularly successful for students requiring concrete materials." (MSC)

A number of teachers particularly associated concrete materials and games with increased engagement in problem-solving which, in turn, was associated with success in mathematics. Two changes in reasoning for students were described here:

- one was a change from seeing the "goal" as "right answers" to the goal being "reasoning through a problem";

- the other, a change from seeing mistakes as "failure" to seeing mistakes as a normal part of the learning process.

"My methodology now encompasses a wider use of concrete materials, more open-ended and fun activities, and more discussion about the outcomes/results… Children who previously probably rarely enjoyed 'success' in maths now know that their answers don't have to be 'right' in the traditional sense; as long as they show evidence of having thought about the problem and represent their understanding. (This has involved a change in mind-set for the teacher too!)" (MSC)

"The Aboriginal students have often been reluctant to try new tasks. They now love the contextualised activities; accept the making of mistakes as a natural learning process instead of as a failing. These students are volunteering information, opinions and answers, whereas before they would have been 'too shamed'." (MSC)

Concrete materials and games based approaches were also commonly associated with increased group-based learning and increased discussion. This, in itself, provides new resources for students — access to a range of different ways of thinking about problem solving, and peer support for learning:

"Problem solving was done as a group effort which led to more open discussion and experimentation. Students began to see more than one way to 'solve' a problem through our sharing." (MSC)

"The talk around the computers and between students who are playing games is now maths talk and you can hear them teaching each other concepts as they come up in a game." (MSC)

As well as highlighting the underlying mechanisms that led to changes for teachers and students, the use of MSC also gave teachers a chance to comment on a wide variety of aspects of both the program design and the program implementation. Some of these included the model itself, the impact of mentoring and modelling, the impact of particular professional learning sessions and the importance of leadership involvement and support. These comments have been used to inform the ongoing planning and implementation of the program, as well as providing useful feedback about how teachers are experiencing their participation in MLI.

It would appear that some similar mechanisms of change operate for teachers and students, as well as some different ones. "More hands on" teaching and learning strategies, in group settings, provide access to new resources (concrete materials, ideas, information, ways of thinking), encourage problem solving and experimentation, and provide peer support for learning. These build motivation and engagement which:

- provide experiences of success;
- reinforce changed behaviours;
- over time, contribute to improved learning outcomes.

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Using Dimensions of Learning
to increase student autonomy[*]

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The Dimensions of Learning *framework has been used to create an environment in which students can develop as autonomous learners, aware and in control of their learning.*

The essential goal of education is to transfer responsibility for learning to the students, so that they can act in an autonomous and self-directed manner. Thus, they need to learn how to develop knowledge and skills to reach deep understanding. Yet, according to the literature, the typical behaviours of high school students indicate a lack of involvement in, and commitment to, learning (Boaler, 1998). Too often, they are unaware of how they learn and their learning is mechanical and instrumental, based on memorising rote procedures, working and reworking de-contextualised examples, and recalling information only for assessment purposes.

To make a positive difference to learning outcomes, teachers should establish an environment that maximises learning opportunities through responding to students' individual needs; empowering students to become independent learners; motivating students to improve their understanding and become enthusiastic about, and interested in, mathematics; valuing active engagement; and encouraging collaborative learning (*Standards for Excellence,* AAMT, 2006). The goal of the Mathematics B syllabus (QBSSSS, 2000, p. 2) is for students to reach the level of competence required for informed citizenship and life-long learning, to have confidence in using mathematics to solve problems, and to develop a basis for a wide range of further studies.

Dimensions of Learning

This paper discusses using the *Dimensions of Learning* framework (Mid-continent Regional Education Laboratory [McREL], 2005) to work towards the goal of creating a learning environment in which the students are engaged in, and in control of, their learning. They should "know why they do what they do and can do it without the teacher", as stated by Jane Doty at a *Dimensions of Learning* training workshop on 9 November 2005.

There are five dimensions (McREL, 2005).

- **Dimension 1: Attitudes and perceptions**

  To learn effectively, students need to feel accepted by teachers and peers and experience a sense of comfort and order. For them to realise that they have the ability and resources to be successful, the learning tasks should be related to their

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[*] Paper accepted by peer review

needs and goals and be of appropriate difficulty. The students should be actively engaged in their learning and have the opportunity to exercise some personal choice about how they learn.

- **Dimension 2: Acquire and integrate knowledge**

  To recall and use results and procedures, students need both declarative and procedural knowledge, as well as ways of organising and storing this knowledge. Tasks should elicit thinking and meaningful discussion so that students communicate mathematically, construct reasonable solutions to complex problems, and find and describe patterns. Students need to value understanding why a particular approach would be appropriate to solve a problem, not just memorise rules and procedures.

- **Dimension 3: Extend and refine knowledge**

  To see the "big picture" and achieve conceptual understanding, students need to develop complex reasoning processes including comparing, abstracting, inductive and deductive reasoning, constructing support, and analysing errors and perspectives. By focusing on understanding, rather than content and correct answers, students become more confident and more competent, and thus better able to transfer their knowledge to unfamiliar situations.

- **Dimension 4: Using knowledge meaningfully**

  Students should have opportunities to apply their knowledge for a purpose, using the processes of decision making, problem solving, experimental inquiry and investigation. Thus, the teacher needs to provide authentic tasks that emphasise reasoning and higher order thinking skills, allow a range of solution strategies and results, and require the construction and justification of mathematical arguments.

- **Dimension 5: Habits of the Mind**

  The Habits of the Mind can be classified into critical thinking, creative thinking and self-regulated thinking. To work autonomously, students need knowledge about themselves as learners, as well as about effective learning strategies. Otherwise, they remain dependent on the teacher to tell them what to learn and how to do it. Too often, their only learning strategy is memorisation by doing an example over and over again, without understanding why they are using the algorithm and so being unable to apply this knowledge in another context. The requirement is to have students who can reflect on and analyse their thinking, and develop as independent and self-directed learners.

This paper argues that the value of the *Dimensions of Learning* framework is in increasing teachers' awareness of their knowledge so that they incorporate this into their planning.

When one has been teaching the same subject, to the same year level, for a considerable time, much becomes automatic. Work plans, and resources and assessment tasks, have been developed over time. If these seem effective, they are re-used, generally without any explicit consideration as to why the particular strategy or resource was originally used. A focus on each dimension causes the teacher to reflect on what student outcomes are sought through the teaching of the unit, why these are important, and how these might be achieved. Thus, there is increased teacher awareness of the content to be covered, often through referring to the syllabus to see exactly what it is expected that the students experience in the unit, not just recall of what has been habitually covered. Planning includes ensuring what students should master. By listing

what it is assumed that the students already know and have experienced, the teacher ensures discussion takes place with the students to confirm the accuracy of these assumptions. It also allows the students to connect the new knowledge to their previous learning. This planning about content also helps the teacher situate the particular unit in relation to the whole course. Subsequent sharing of this with the students helps them develop a sense of the"big picture" of their mathematics learning, and why the unit is of value.

The second aspect of teacher awareness relates to the teaching strategies to be used. Instead of teaching in an automatic way, the teacher considers the strategies available. The *Dimensions of Learning* manual describes a wide range of strategies that can remind teachers of approaches that they seldom use, or suggest new ideas to try. Approaches include didactic whole class teaching, inquiry-based learning, student contracts in which the students are required to complete a set amount of work, and combinations and modifications of these. Problems might be solved individually, in different sized groups, or as a whole class enterprise. By using the framework to consider how the different options will contribute to student learning in each dimension, the teacher is more likely to implement a greater variety of teaching approaches.

The third aspect of the unit planning involves determining the appropriate cognitive and metacognitive strategies. By reflection on personal strategies, for recalling the content and solving the problems related to the unit, the teacher can develop a list of cognitive strategies that can then be given to the students. Having such a list can assist less confident students start solving a problem, instead of initially assuming they are unable to do it. The explicit focus on the cognitive learning strategies reinforces the students' use of existing strategies and may enable their adoption of new strategies. Students may initially be unfamiliar with metacognitive learning strategies. The teacher needs to plan their development throughout the year. Thus, planning for an early unit may centre on introducing the concept of thinking about one's thinking, and starting to become aware of how one goes about learning.

The use of the framework for teacher planning can also directly affect the students' learning. Sharing the outcomes of the planning with them allows discussion about learning itself and about the Dimensions of Learning, as a framework for considering the different aspects of learning. This impacts on their learning in three ways:

Firstly, learning can be viewed as a joint enterprise involving a *community of learners* (Brown, 1994). The teacher and students are active participants in developing knowledge, rather than the students being passive recipients of whatever it seems the teacher has decided they need to know. They realise that the teacher cares about what, and how, they learn and has put considerable thought into the planning. However, it is then their responsibility to ensure that they achieve the outcomes listed on the plan. They become aware of what the teacher assumes they already know. If there are gaps in this knowledge, it is their responsibility to let the teacher know so that this can be dealt with. As there are explicit statements about what they need to be able to do, they can exercise more control over their learning and continually check whether they are achieving these outcomes.

Secondly, the students are encouraged to become more aware of themselves as learners, not just to learn without considering how this is done. They are told about the learning strategies that they will be encouraged to use. Instead of trying to improve through "working harder", they have a range of strategies that they can implement and, so, can exercise greater control over their own learning.

Thirdly, effective learning and understanding, rather than content knowledge and correct answers, are seen as the main objectives of the course. The framework provides ways of talking about learning.

Thus, sharing the planning with the students contributes to the development of a "community of learning" in which learning, not course coverage, is the focus. The students are able to exercise greater control over their learning as they know specifically what they need to be able to do, have a list of learning strategies to use to achieve this, and know how they will be expected to demonstrate their understanding.

How this approach contributes to the growth of student autonomy

This approach is being used with a Year 12 Mathematics B class in a Catholic coeducational college in a regional Queensland city. Mathematics B is an academically rigorous subject that is a prerequisite for many university courses.

The introductory lessons and unit of work, covered during the first five weeks of the school year, are discussed. This unit involves functions. The focus is the development of an understanding and appreciation of relationships between variables; algebraic, graphical and numerical representation of these relationships; and application of functions to solve problems in life-related situations (QBSSSS, 2000, p.12, p. 17).

The students receive handouts on the content and strategies and the associated Dimensions of Learning (see Appendix).

The contribution to the growth of student autonomy is considered through three areas: the explicit focus on learning, the increased student control over learning, and the development of a community of learning. However, these aspects are connected, rather than discrete.

Explicit focus on learning

The first two lessons of the year focus on awareness of learning. The students reflect on attitudes and strategies that they have found to contribute to effective learning by responding individually to the following stems:

- I learn well when I…
- I learn well when the teacher…
- I learn well when the other students…

The responses are shared in small groups and, then, with the whole class. They repeat the process considering what causes them to have difficulty learning. The students also write about their goals for the year, and how they plan to achieve them.

The discussion in response to the students' reflection on learning makes thinking more visible than it usually is in classrooms, giving the students more to build on and learn from (Perkins, 2003, p. 1).

The students receive the handout that lists the assumed knowledge and skills, the concepts to be developed and assessed, and related cognitive and metacognitive learning strategies. They are also given notebooks to use as journals to encourage reflection on both what they are learning and how they are doing this.

Reflection on understanding occurs as the students write, in their own words, about what they are doing and why, and why they have chosen particular strategies to solve problems. This gives them a valuable resource for later study. At the end of some lessons, they are asked to list what they have learnt in the lesson, what they do not yet

fully understand, and what they plan to do about this. Some students use the journals as a medium to ask the teacher about a particular type of problem with which they are having difficulty.

Reflection on how they are learning occurs through both class discussions and individual journal writing. Students are asked to consider how effectively they are using both class and home time, whether they learn better working with others or individually, and what they are doing to get the best value from the time they spend doing mathematics. In their journals, they can admit their difficulties and concerns, set goals and later determine the extent to which these have been achieved, acknowledge their successes or failures, and consider why these have occurred.

The goal is to make the students aware of their learning, realise their strengths and weaknesses, and have some control over their learning environment through their input into how the class is run. As students realise that they are the ones who achieve the results, they can "own" their behaviours and the resulting outcomes.

Increase of control over one's own learning

The importance of exercising control over learning is discussed in the initial lessons. The students are told that the beginning of a new year provides an opportunity for a new start. Keeping up with the work from the very beginning, ensuring that they are using time effectively, asking questions when they did not understand, and contributing understandings and strategies to the class, are all stressed.

There is also a discussion on what it means to "know" and to "be able to do" mathematics. Generally, the class consensus is that it means knowing what to do and why it might be done that way, and being able to explain this to another person.

As the plan lists previous knowledge required for the unit, students can prepare by reviewing that work and so connect the new and old knowledge. By referring to the concepts to be covered in the unit, they know what they need to be able to master.

Students are also told, at the beginning of each lesson, what will be covered during the lesson. This is then summarised at the end of the lesson and, generally, reviewed at the beginning of the next lesson. Providing questions, cues and advanced organisers so that students realise the objectives or purpose of new concepts, and what knowledge they will need to acquire, has been found to enhance achievement (Marzano, Pickering & Pollock, 2001).

It is emphasised that it is not sufficient just to find a correct answer: they should understand why they are using a particular procedure of approach. Students are encouraged to write, in their own words, the steps that they are taking when working through new types of questions. They can then construct their own algorithm, or procedure, to use in similar questions, rather than learning it as steps of a recipe. They can also develop rules, instead of just memorising a rule given in the textbook. They can then use their own instructions when solving similar questions.

They are thus able to access knowledge from their own sources, without needing to rely on the teacher as much. Similarly, if the teacher's response to students' questions is questions that lead the students to solve the problem themselves, the students become more independent.

Community of learning

The third aspect of developing student autonomy is the move towards being actively engaged with the teacher in "constructing knowledge."

In a *community of learners,* the students engage in thinking, reasoning, analysing and theorising; learn the content and strategies for using that content simultaneously; assume responsibility for their learning; and learn collaboratively in a socially supportive learning environment. The teacher facilitates learning through selecting and posing appropriate work, sharing information, when necessary, and developing a classroom culture in which students work on problems and discuss and reflect on answers and methods (Hiebert et al., 1997). The teacher guides the students towards forms of disciplined inquiry that would not be reached without expert guidance, but does not constantly monitor, direct and supervise students' learning and behaviour. A changed understanding of the roles of teacher and students occurs, as teacher and learners work together.

Sharing the plan with the students, and explaining its different aspects, make this partnership between teacher and students explicit.

Because of the focus on a community learning together, students are encouraged to work together and learn collaboratively. They extend their knowledge as, together, they relate new and old knowledge, analyse problem contexts, acknowledge what is not known and then seek to find this out. Thinking is clarified and understanding consolidated by explaining ideas to others. They are exposed to different approaches and modes of thinking. They mostly enjoy this more social type of learning and are more motivated to learn, but must be willing to verbalise the processes and learning strategies being used. Whole class problem-solving also allows a supportive environment for the students to use problem-solving strategies.

Results

After a term with a focus on a community in which everyone was expected to be actively involved in learning and accept responsibility for their own learning, the students were asked to reflect on the value of the approach. They did so through both individual journal writing and whole class discussion. Aspects that they were asked to consider included strategies used to improve learning, understandings of themselves as learners, and how mathematics had been taught in this period. Their responses have been considered as related to achievement, awareness of learning, and classroom climate.

Although the results of only one term have been considered, 19 of the 25 students in the class improved compared with their results at the end of the previous year. Most of the students believed that they achieved at a higher level. Comments from the students included the following:

"I am actually understanding what we are learning".

"It really satisfies you to know what you can accomplish when you try".

Many students indicated that they have become more aware of how they learn. They found having the plan helped as they know what they would expect during the unit and be required to do for assessment.

"I learn best when I know exactly what I will need to learn".

"This subject needs time and effort. You need to know how to use the work and ask lots of questions. You need to be motivated".

"I go home and practise after learning something new and make sure I understand it before moving on".

The responses also indicated a positive classroom climate in which students thought that they were learning work which was "challenging and interesting" in an "enjoyable environment". Many students commented on the importance of the teacher listening to

them and answering their questions. However, there were no comments stating that the teacher should have been telling them more. Many also found it valuable working with other students, with individual students noting the sense of teamwork and their freedom to choose how they learned.

In conclusion, the *Dimensions of Learning* framework enables the teacher to explicitly plan pedagogy that emphasises making sense of the concepts, using a range of learning strategies, and encouraging the students to accept the responsibility for their own learning. As the resulting plan, and the reasoning behind it, is shared with the students, they have the opportunity to exercise greater control over their learning and develop a sense of working together to achieve to their potential.

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Appendix: Unit on functions

Time: 15 periods
Terminology:
    polynomial, turning point, point of inflection, intercepts, exponential, logarithm
Previous knowledge (to be reviewed):

- curve sketching
    - basic functions x^2, x^3, $1/x$
    - effect of constants on a general curve $y = f(x)$, $af(x)$, $f(bx)$, $f(x + c)$, $f(x) + d$
    - use of graphics calculators — entering function, expected result, suitable windows
- index laws — simplifying exponential expressions and solving equations
- logarithms — change to exponential form and back, log rules, change of base rule
- finding equation of a line given points and/or gradient

New knowledge to be developed (extension of previous knowledge):

- process to sketch $y = af(bx + c) + d$, starting from $y = f(x)$ — emphasis on communicating reasoning
- sketching factorised polynomials $y=a(x+b)(x+c)\ldots$
- attributes to consider when curve sketching: behaviour approaching $\pm\infty$, x and y intercepts, TPs, PIs, discontinuities

- working backwards from curve to equation — emphasis on communication, evaluation of correctness of conclusions
- composite functions — need to understand function notation
- inverse functions — how to check correctness of solution
- regression functions on graphics calculators to fit a curve
- semi-log graphs to fit exponential function given a set of points
- e

What you need to be able to do (will be assessed on):
- sketch $y = af(bx + c) + d$ given $y = f(x)$
- sketch curves of the form $y = a(x + b)(x + c)^2(x + d)^3$
- find a suitable equation for a given curve, explaining your reasoning
- find composite functions
- find inverse functions
- solve word problems involving functions
- simplify logarithmic and exponential expressions
- change from logarithmic to exponential form and vice versa
- solve logarithmic and exponential equations, including word problems
- find an exponential function to fit given points

Assessment (with specific focus on functions):
- Assignment
- Mid-semester test

Cognitive learning strategies:
- draw diagrams
- try small easy values to develop a relationship, then generalise
- use graphics calculator to check solutions
- use unfamiliar (new) questions to practise problem-solving approach
- maintain (and continually review) review sheets
- do some revision every week (as well as homework and working on assignment)
- use the problem solving algorithm:
    - what have I been given?
    - what am I asked to find?
    - what are some possible approaches?
    - is this giving a solution?
    - does the solution make sense?
- use a highlighter to mark homework questions that you want to discuss further

Metacognitive learning strategies:
- Ask yourself WHY you are doing something, do not just do it. Could it be done another way?
- Keep a Maths study/homework timetable. Find out how much time you are putting in each day and week
- Reflect on the efficacy of each learning session — class time and home time

Dimensions of learning associated with unit

Attitudes and perceptions:

- A new year and a new start — keep in control from the beginning
- Control your learning, input into whole class processes/approaches — speak up, ask questions
- Use your problem-solving strategies and algorithms — you should have the knowledge and resources to complete tasks
- Focus on understanding (making sense of what you are doing)
- Collaborative learning ("I know that I understand when I can explain it to someone else")
- Your own learning — be aware of this!
- Tutor sessions

Acquire and integrate knowledge:

- Review sheets — new rules, algorithms, "handy hints"
- Organise maths pads — course outline, page numbers, exercises done, corrections in red ink, instructions to self
- Develop own revision sheets — what should be reviewed again at weekend
- "Think aloud" problem solving — in whole class situation and in pair work

Extend, refine, and use knowledge meaningfully:

- Problem solving approach to learning
- Evaluation and monitoring of solutions (are they meaningful?)
- Develop arguments (communication and justification)
- Authentic assignment task

Habits of the mind:

- Self-regulation — time management, monitoring effectiveness of learning — work smart!
- Goal setting — big goals (hopes for the year) and small goals (plans for individual lesson)

Mathematics curriculum in the age of modern technology: Changes and challenges[*]

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This paper draws upon practical classroom experience as well as recent studies to discuss the changes and challenges associated with the integration of technology into the mathematics classroom.

Introduction

In the age of modern technology, digital equipment and Information Communication Technology (ICT) have become widely accessible and inexpensive. Many social and educational researchers are, therefore, naturally concerned about the effect of these technologies on school performance and curriculum reform. In response to such concerns, educational authorities, and other stakeholders in the wider community, are attempting to educate and equip students with necessary skills by utilising existing technology and encouraging, and assisting, students and teachers to appropriately use these technologies to enhance the teaching and learning of mathematics, science, and information technology. This paper will use practical classroom activities and review a number of studies in order to highlight the problems and challenges of integrating technology into the mathematics curriculum. In particular, the following questions will addressed:

- How does modern technology affect the teaching and learning of mathematics?
- What kinds of changes are being made to mathematics curriculum?
- What is the role of the teacher in such unavoidable changes?
- How are authorities orchestrating the changes in the mathematics curriculum in the age of modern technology?
- How does the mathematics department-school participate in the ongoing process of preparing students for the next decades in the light of modern technology and, also, in responding to modern cultural, social and economical needs?

How does modern technology affect the teaching and learning of mathematics?

The availability of various kinds of technologies, such as calculators (including graphics and CAS calculators), computer programmes (e.g., Excel and geometric software) and other ICT naturally leads to a discussion of the ways in which parts of mathematics curriculum, classroom practice, and student learning will be affected and may differ from the traditional approach (Arcavi & Hadas, 2000).

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[*] Paper accepted by peer review

Calculator activities

During the last 25 years, handheld calculators became a powerful instrument in mathematics classes, especially because students in most schools had limited access to computers and to computer software (Waits & Demana, 2000). Graphing and CAS calculators became very inexpensive, handheld, and very computer-like. However, the existence of such technology raised the importance of certain ideas, made some problems and topics more accessible, provided new ways to present and handle mathematical information, and provided choices about content and pedagogy that we have never had before (Goldenberg, 2000). However, as mathematics teachers we need to consider the type of technology and the needs of students when preparing mathematics activities, taking into consideration that calculators can cause great changes in the mathematics that we teach. For example, in the presence of calculators, some pen-and-paper applications became obsolete. This can be seen in the following scenarios:

- Calculating a number such as $1250\_(1.04125)^{12}$ or Solving an equation like $3x^3 + 2x^2 - 7x + 9 = 0$ by pen-and-paper is very laborious and is not necessary if the lesson is not about the calculation process but it is about finding and using results.

- Sketching the graph of $f(x) = \dfrac{x^3 - 17x + 7}{x^2 + 1}$ may take 10 minutes of class time, but graphics or CAS calculators can create a graph in just a few seconds. In contrast, the traditional method for graphing included finding derivative $f'(x)$, solving $f'(x) = 0$ and completing a table of values. Ideally, students will use their calculus skills to confirm analytically that the graph they see is accurate.

Even for junior students who work with simple arithmetic calculators, teachers can design activities (such as the "*Broken Calculator*" activities) to help them focus on and analyse the structure and elements of arithmetic and, also, to gain skills along with understanding, rather than letting the calculator replace their thinking. For example, students can be asked to solve the following problems:

- How can you multiply 20 by 50 if the "0" key is broken?
- Calculate 35 _12 by using the "5" and the "7" keys only.
- Calculate 4039 – 49 if the "4" key is broken.

Thus, while calculators are useful when computational labour can get in the way of the purpose of a lesson, calculator use can be a bad idea when the purpose of the lesson is to learn how to perform the computation (Goldenberg, 2000).

Dynamic computerised environments

Dynamic computerised environments enable learners to visualise mathematical situations, see various sides of a problem and make connections between presentations. Working in such environments also allows students to experiment with: measuring, comparing, changing figures, and making supporting figures easily. This can nurture their need to re-inspect their knowledge and assumptions and, thereby, establish opportunities for meaningful learning.

The problem presented in Figure 1 provides a good example. It was presented to Year 9 students as a practical application of Pythagoras' formula (Nelson Maths 9, 2001, p. 163) but can also be presented to more senior classes.

With Year 9 and Year 10 groups, I open the discussion in class by encouraging the students to brainstorm solutions to this problem. Common answers from students, based mainly on guessing, include that: the exit should be in the middle; closer to the right side; and closer to the left. Next, I asked the students to calculate the length of the road "MEC" according to their choice of exit and then to compare their results with those of other students in the class. A few students in the group suggested gathering the collected data into a table of values, so that it would be easier to compare the results and make a decision. Students applied the formula of Pythagoras to calculate the lengths of ME and EC. Normally it is laborious to add any bit of data that comes from a student (to insert new data we need to create a new table or delete all the data we had and start again from the beginning); also, it is a big job which requires many calculations to cover all possible positions of the exit (using values between 0 and 4, as the length of the freeway section is 4 km).

The freeway exit

Building roads can be very expensive so civil engineers try to make them as straight as possible. The line of a new freeway as it passes by Melford and Extown, two small towns, is to be as straight as a ruler. Only one exit is proposed for local residents, and it is to be joined to both towns by straight roads.

Where should the exit be put to minimise the total length of road from Melford, *M*, to the exit, *E*, and back to Extown, *C*?
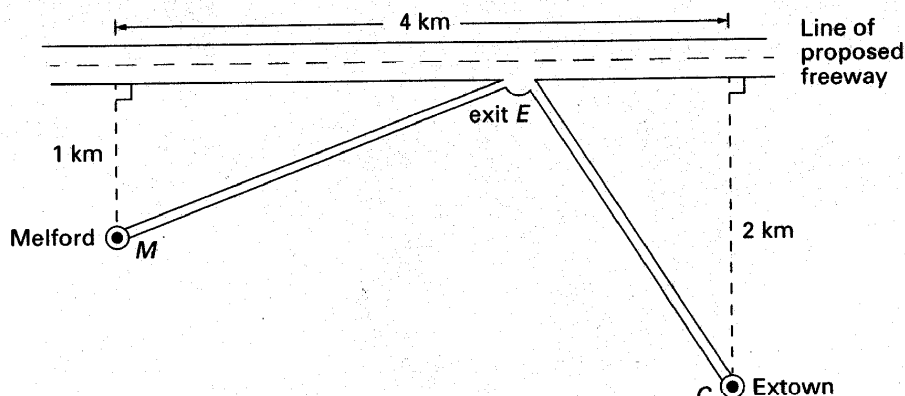

Figure 1

One mathematics teacher in the school suggested giving this problem solving exercise as an assignment for a week, letting students find all possible positions of the exit using 10m elements between 0 and 4 km and presenting data in a table of values to find the best position, or use 1km element to get a rough idea for the best position and then create a new tables to include smaller and smaller values around the suspected position to be more precise. I suggested using an Excel spreadsheet to solve this problem; so that students can save time and pay attention to the problem itself rather than suffer the long and boring calculations. My suggestion did not receive a positive response from other mathematics teachers, possibly because they were not comfortable or familiar with Excel, or possibly because of the challenges associated with having a group of 20–25 students in a computer laboratory or problems with getting access to a computer laboratory in the first place.

However, there was a computer station (two computers) in the classroom, so I put the idea to the class and a few students with prior Excel skills volunteered to attempt the

task, with me assisting them to set up the spreadsheet and write the correct formulas, as in Figure 2.

	A	B	C	
1	x	ME (a)	EC (b)	Y= a + b
2	0	=SQRT((a2)^2+1)	=SQRT((4-a2)^2+4)	=b2 + c2
3				
4				
5				

Figure 2

I later retold this experience to the school principal and the head of the mathematics department who encouraged me to speak with other mathematics teachers about it and to organise a Professional Development session for the department (which I did later). Both Year 9 and Year 10 groups had the opportunity later on to experiment with this activity in a computer lab. They were able to calculate the length of each part of the road ME and EC add them and compare measurements and make a decision.

Where is the best exit on the freeway?

| x	a	b	y=a + b
0.2	1.01980	4.29418	5.31399
0.4	1.07703	4.11825	5.19529
0.6	1.16619	3.94462	5.11081
0.8	1.28062	3.77359	5.05422
1	1.41421	3.60555	5.01976
1.2	1.56205	3.44093	5.00298
1.3	1.64012	3.36006	5.00018
1.31	1.64806	3.35203	5.00009
1.32	1.65602	3.34401	5.00003
1.33	**1.66400**	**3.33600**	**5.00000**
1.34	1.67200	3.32800	5.00001
1.35	1.68003	3.32002	5.00004
1.4	1.72047	3.28024	5.00071
1.6	1.88680	3.12410	5.01090
1.8	2.05913	2.97321	5.03234
2	2.23607	2.82843	5.06450
2.2	2.41661	2.69072	5.10733
2.4	2.60000	2.56125	5.16125
2.6	2.78568	2.44131	5.22699
2.8	2.97321	2.33238	5.30559
3	3.16228	2.23607	5.39835
3.2	3.35261	2.15407	5.50668
3.4	3.54401	2.08806	5.63207
3.6	3.73631	2.03961	5.77592
3.8	3.92938	2.00998	5.93935
4	4.12311	2.00000	6.12311
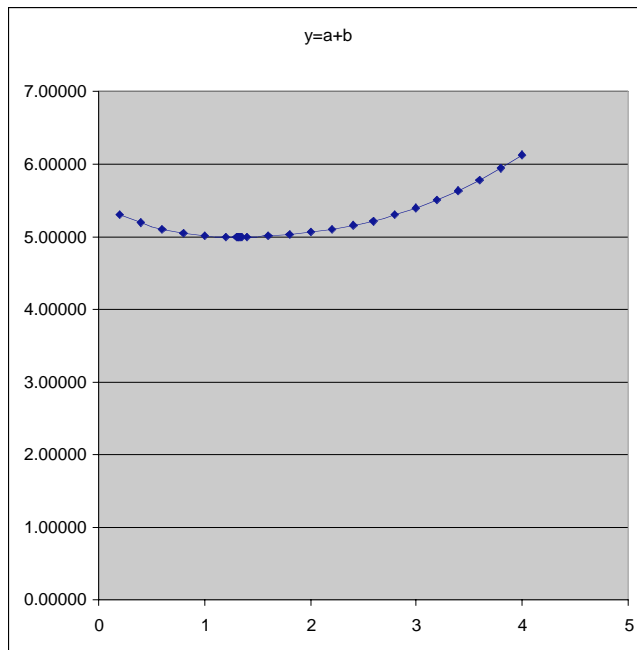
Figure 3

Then by completing this spreadsheet, students were able to graph their data, making connection between presentations, visualise the situation, and make a decision (see Figure 3).

My students also had the opportunity to use one of the dynamic geometry program (Geometer's Sketch Pad or GSP) which enables them to experiment and compare various exit positions. Further, by using another feature of the application called "animation and tracing," they were able to make their decision based on concrete observations and results (see Figure 4).

The availability of different kinds of technology enables teachers to pose advanced questions such as using "calculus" to solve the "Freeway Exit" problem.

Year 12 students in Mathematical Methods, where CAS calculators and GSP are available, were easily able to find the derivative of the function $f(x) = \sqrt{x^2 + 1} + \sqrt{(4-x)^2 + 4}$, that represented the length of the road and to solve the equation $f'(x) = 0$ without wasting their time doing such long and hard calculations. By saving this time, they had more time to analyse and understand the mathematics of the problem, and were also able to discuss the coordination of different procedures that were available to them.

Reflecting upon the above problem, it was clear that middle school students enjoyed using technology in mathematics. It was a different and interesting tool which allowed them to solve difficult questions or questions that required long and laborious processes and complex calculations. In particular, they found GSP — with its animations and quick calculations — to be an exciting application.

For senior students, technology saves time and they can attempt more problems and produce more work, with less effort. They can use the spare time to concentrate on, and understand, the mathematical sides of problems. Mathematics teachers in my school, as well as other teachers who attended my workshops on various occasions, found that with modern technology they had an opportunity to ask students to check different aspects of the problem and to design tasks and activities which take advantage of features and potential of the available technology. This enhanced and supported new ways of learning mathematics (Arcavi and Hadas 2000).
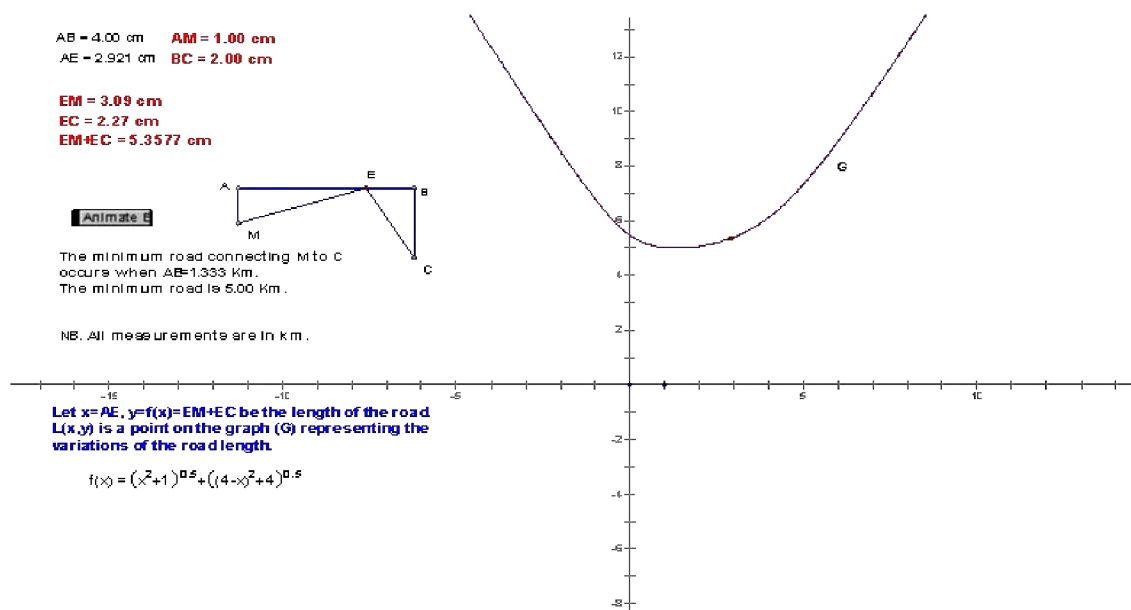

Figure 4

In such a learning environment, the teacher is able to develop different teaching methods that enable students to make good use of modern technology, develop mathematical ideas, produce results and carry out analysis, in a problem solving situation.

What kinds of changes are being made to the mathematics curriculum?

As seen in the above examples, technology can lead to changes in teaching and learning. In the presence of modern technology, some mathematics becomes less important (e.g., some arithmetic and symbolic techniques) while other aspects become more important (e.g., discrete and nonlinear mathematics and data analysis).

"Before calculators we studied calculus (applications of the derivative) to learn how to obtain accurate graphs. Today we use accurate graphs produced by graphing calculator to help us study the concept of calculus" (Waits and Demana, 2000, p57).

Thus, teachers should balance paper-and-pen approaches and technology when preparing their mathematics curriculum. Yet,traditional arithmetic and algebraic skills are still very important as we move to more computer-intensive learning environments (Waits & Demana, 2000). For example, students need to know how to use the main operations, need to know how to simplify and factorise expressions, and should be able to estimate answers before using technology or, even, before doing any computation. A curriculum should provide sufficient time for appropriate practice of these skills. To achieve a good balance, teachers are advised to have students, from time to time, use each of the three strategies:

1. Solve problems using paper and pencil and then support the results using technology
2. Solve problems using technology and then confirm the results using paper-and pen techniques
3. Solve problems for which they choose whether it is most appropriate to use paper-and-pen techniques, calculator techniques, or a combination of both (Waits & Demana, 2000, p. 59).

Nevertheless, the mathematics curriculum, in both written documents and practice, needs to stress that students, in the technology environment, should develop a set of skills, strategies, rules or guidelines for actions, routines and ways of doing things, together with tactics, methods and algorithms that can be used in various situations (Bosco, 2004). In the presence of technology, the mathematics class becomes different: students and teachers will have more time for discussions; students can work in groups, can do much more work than before, and work much faster than they could imagine (Connors and Snook, 2001).

Great changes have occurred in the mathematics curriculum through taking advantage of technology such as CAS calculators. Here some topics have become more accessible, and new courses have been introduced, such as Mathematical Methods with CAS in Victoria, Australia (Stacey, 2003). Such curriculum change is influenced by the use of education to improve the economy, the efforts of government initiatives, and the influence of commercial curriculum resource providers (Textbook and software publishers, and calculators and computer manufacturers), and international organisations (McCormick, 1999).

What is the role of the teacher in such unavoidable changes?

In the environment of modern technology, authorities expect teachers to integrate the appropriate technology into their teaching. However, research show that new teachers, with shorter length of service and better qualifications, are more competent in using technology then more experienced teachers (Andrews, 1997).

As schools plan their use of technology, not all teachers are both willing and able to align their teaching with the new teaching strands, nor with the integration of technology. Teachers should be provided with sufficient time and opportunity to become fluent with the tools, so that they can be flexible and more confident in their ability to use them (Goldenberg, 2000). Teachers' fears about technology need to be understood and addressed. Professional development in technology, especially around tools, can play a great role in the process of change. Teachers need to be updated as well as be made keenly aware of the limitations of the technology, so that they are able to keep the balance while they develop, in their students, the ability to appropriately decide when to pick up a pen, when to pick up a calculator, or when to go to a computer workstation (Connors & Snook, 2001).

Educators need to avoid letting a powerful technology tool become a crutch to poor understanding. They need to develop teaching strategies which enable students to make sensible choices about when to use, and when not to use, technology and become judicious users of technology (Ball & Stacey, 2005). Ball and Stacey described these strategies in their Classroom Vignette:

1. Promote careful decision making about technology use, where teacher and class discuss the nature of human and technology contributions.
2. Integrate technology into the curriculum, focusing on mathematics rather than on the calculator's syntax and include examples that really require technology use, and examples where technology use would be inefficient.
3. Tactically restrict the use of technology for a limited time.
4. Promote habits of using algebraic insight for overview and monitoring (model this routinely at the start of each problem and encourage the continual monitoring of technology outputs against expectations). However, when preparing a mathematics activity using technology (calculators or computer); teachers need to make sure that they have access to the right equipment at the right time.

Nevertheless, more work is still needed to involve all teachers in the process of change, especially those who need to be convinced of the need to use technology in their teaching (Andrews, 1997). To initiate change and achieve the full potential provided by technology, we need to put the potential for change in the hands of everyone, and provide additional teacher in-service training on the use of technology, particularly addressing conceptual and pedagogical issues. Having said that, change has to come from within the teaching profession and be supported both from within and from without, as it is very hard to change the teaching practice from the outside alone.

How are authorities orchestrating the changes in the mathematics curriculum in the age of modern technology?

On the international scene, educational authorities hold the responsibility of monitoring curriculum developments and implementations. They work closely with other organisations to develop new teaching strategies and resources to cater for the needs of

the wide range of student abilities. They orchestrate changes in the curriculum and organise the entire assessment processes in each country. Authorities take the initiative of setting the curriculum's goals and the standards framework, provide schools with the necessary resources and assist with the necessary support, monitor and assess teachers' performance, and assess the curriculum implementation (Prawat 1996).

In recent times, mathematics education has been directly affected by the wide-ranging availability of, and easy access to, new technology. As a result, educational authorities have needed to update school curriculum, structure and performance in order to enhance the teaching and learning of mathematics and ensure that students are well prepared and equipped for their future endeavours (Kilpatrick and Silver, 2000). In making changes, authorities are faced with challenging decisions. Integrating technology into the mathematics curriculum requires particularly complex decisions (Connors & Snook, 2001).

In Victoria, the Victorian Curriculum and Assessment Authority (VCAA) holds responsibility for school curriculum and like other similar bodies around the world, VCAA has expended a great deal of energy in recent years to support schools in integrating new technologies into their curriculum.

The mathematics CSF II (2000) and the VELS (2005) in Victoria, support these developments and encourage teachers to integrate technology into the mathematics classroom. The objective is to enable students, using ICT, to have access to resources and information to ensure that they are making sensible use of calculators and computers. Teachers are advised to use technology effectively and appropriately to support the student's learning i.e. to help students think and work mathematically and carry out mathematical activities and solve problems. The choice of technology, when it should be used and to what extent, should be guided by the degree to which technology can efficiently and effectively assist students in their learning. However, the ultimate objective is to equip students with the mathematical and numerical skills for successful employment and prepare them for further studies, should they desire it.

The Victorian Certificate of Education (VCE) Study Design (2000; 2005) stipulates that: "The appropriate use of technology to support and develop the teaching and learning of mathematics is to be incorporated throughout each VCE mathematics unit and course, and in related assessment. This will include the use of some of the following technologies for various areas of study or topics: graphing and CAS calculators, spreadsheets, graphing and numerical analysis packages, dynamic geometry systems, and computer algebra systems" (VCAA, Mathematics VCE Study Design 2005, p. 14). Thus, teachers are required to develop courses that encourage students to appropriately select and use technology, where they are required to respond to outcome 3 which is based on the appropriate use of technology: to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modelling or investigative techniques or approaches in the area of study of each course and unit.

School participation in the ongoing change

When schools plan to integrate technology into the curriculum, issues arise such as the interaction of technology with different elements in the school (Bosco, 2004).

The school holds the main responsibility for creating and maintaining a healthy and effective technological learning and teaching environment. The school is responsible for providing the necessary resources to achieve curriculum goals and meet learning needs of students, and the teaching needs of teachers. Resources include: class sets of

calculators; computer labs; computer pods; printers; data projectors; computer software; school networking and Internet connection; as well as necessary maintenance and upgrading of software or hardware. Also, to ensure that the technology is smoothly and effectively integrated into the curriculum, the school administration needs to:

- allocate a sufficient budget to run the curriculum;
- provide teachers with necessary information on software and hardware;
- organise relevant and necessary workshops and encourage teachers to attend in-services courses;
- make official information available to students and teachers, especially information that relates to course design and exam regulations (e.g., VCE Maths exams technology active or technology free); and
- provide flexible timetables that enable mathematics teachers to have access to technology on a regular or temporary basis.

Decision makers and curriculum reformers need to be aware of the different roles for technology, to think clearly about classroom goals, about the particular needs of students and to choose technologies expressly to further these goals, rather than merely adding technology to the classroom in ways that may be attractive, but tangential or even detrimental to set goals (Goldenberg, 2000). The responsibility on schools is a great one given their key role in enabling students to respond to ongoing cultural, social and economic changes.

Conclusion

Technology enhances the teaching and learning of mathematics. To be able to achieve this goal, however, teachers need sufficient training in using the available technology, and must possess sufficient knowledge of computer facilities that are provided by their department or by the school. Teachers should be able to prepare and deliver teaching materials and learning activities that are interesting and relevant to the topic students are learning in ways that open new windows of opportunities for students to improve their learning and to build up new and additional skills and knowledge which are necessary for future work or further studies. Finally, the integration of technology into the mathematics class should be part of the global curriculum which is fully supported by the school community and also by the wider community.

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Mathematics through learning inclusion: Learning inclusion through mathematics

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If primary teachers see mathematics as content-driven and challenging to teach, they may take refuge in teacher-centred pedagogies that do not engage students or make mathematics meaningful for them. The Maths for Learning Inclusion Program in South Australia seeks to address the achievement gap in schools by developing teachers' capacity to teach mathematics in ways that are engaging, inclusive and make high intellectual demands on students. In doing this, the program draws on strategies that have proved successful in improving disadvantaged students' literacy achievement, and applying them to mathematics. This workshop will examine how the program integrates mathematics content with a focus on learners and inclusive learning and discusses the tensions and issues involved in this approach.

Introduction

The Maths for Learning Inclusion Program (MLI) was developed in South Australia by the Learning Inclusion Team within Curriculum Services, in the Department of Education and Children's Services (DECS). It seeks to address the achievement gap in schools by developing teachers' capacity to teach mathematics in ways that are engaging, inclusive and make high intellectual demands on all students, particularly students from low socio economic status (low SES) and Aboriginal (ATSI) students.

There are eight clusters with a total of 44 primary schools in the program, each one with a full time cluster coordinator (CC).

In designing the program, the Learning Inclusion Team chose a three part approach which included mathematics curriculum, learning inclusion and students in the primary years.

Underpinning the design of the project was an assumption that teachers want all their students to be successful in mathematics. However, data suggests that for many low SES and Aboriginal students, success in mathematics seems a long way off. The logic for the program was that by providing teachers with opportunities to develop their expertise in a high stakes subject like mathematics, teachers would not only increase their repetoire of successful practice, improve their confidence and understandings, skills and resources, but have the opportunity to focus on those learners who were currently not being successful in mathematics and ask themselves; Why? What was it about the teacher's own beliefs, assumptions and practices in mathematics teaching that resulted in the largely predictable learning outcomes?

The plan was to support teachers,through programming and planning, with the state curriculum framework, the critical analysis of data, teacher reflection and questioning as strategies, to begin to address the issues of equity and inclusion and imbed these

changed practices and beliefs into their pedagogy and, ultimately, into their identity as teachers of mathematics.

The focus on mathematics in 2006

Professional learning is a vital component of the program and has included conferences, workshops, network meetings, school team meetings, and one to one sessions with teachers and CCs. The professional learning program offered to address each of the three focii, i.e., mathematics, primary years learners and learning inclusion, and has included the following topics in one or more sessions:

- *A focus on the students in the primary years*, their changed/changing nature especially in the area of digital technology (i.e., students as digital natives); the importance of knowing your learners and engaging with their realities and world views as the starting point for engagement; questions raised about what a relevant mathematics curriculum for this group of learners might look like in the 21st century.

- *Data collection and analysis* — has included topics which addressed issues about the importance of using multilple data sets to get a "truer" picture of what is working for whom; questions to facilitate critical analysis; whole school approaches; analysis of data to support teachers' planning and programming, including for individual special focus students, e.g., the Low SES and Aboriginal student(s).

- *Developing professional learning communities* — has included input from Etienne Wenger introducing the concept of Communities of Practice; promoting structures to support learning communities and professional dialogue, e.g., structured teachers dialogue, moderation of student evidence in schools and clusters, etc.

- *Challenging and engaging mathematics curriculum* has been adressed by a wide variety of key speakers and practisioners including Robyn Zevenbergen, Thelma Perso, Doug Williams, Sandra Knox, George Booker, The First Steps program, etc.

What did we find?

While this broad range of professional learning was on offer throughout the program, the focus for the day to day work in the clusters was predominantly focussed on mathematics content, pedagogy, resources, etc. Teachers were eager to develop new understandings and skills and saw immediate rewards back in their classrooms.

Every data set including (the PAT Maths tests, LaN tests, SACSA achievement data) suggested that the urgency for explicit intervention for Low SES and Aboriginal learners had to become a priority for ACTION.

Educational rhetoric ("We need to close the achievemnt gap," etc.), certainly supports the need for action. Given that the vast majority of schools in the program had high numbers of both ATSI and low SES students, it was surprising to us that the learning inclusion focus had not become more central to the work of teachers and leaders. Even though there were high levels of goodwill and commitment to equity, there seemed to be general confusion and uncertainty about what *learing inclusion* was, and how you might do *it in mathematics?* CCs reported a general anxiety about stopping or interrupting the mathematics focus to "do" learning inclusion. This

highlighted that, despite the professional learning program and 12 months in the program, for many, learnng inclusion was still seen as some *optional extra* and not integral to the pedagogy of mathematics. The learning inclusion agenda was clearly more problematic than the mathematics agenda for many teachers, CCs and some school leaders, who struggled with the concepts and ways in which to integrate learning inclusion into mathematics teaching and learning.

How did we respond?

Clearly, we needed some mechanism to make explicit what we meant by learning inclusion. In professional learning sessions, we had previously worked with teachers to analyse the work of Martin Haberman and his "Pedagogy of Poverty" article in which he describes what you see when you observe "good teaching." The work of Phillip C. Schleckty, however, provided a framework for our thinking. The Learning Inclusion Indicators were adapted from Schlechty's work, *Shaking up the Schoolhouse* and *Working on the Work*. Schlechty helped provide us with the concept of a framework that would describe the various aspects of learning and teaching that contributed to more equitable outcomes, particularly for Low SES and Aboriginal learners. It is an attempt to provide a framework to describe the inclusion work teachers and schools already do, and also provide some direction and priority for future planning and actions needed to make outcomes less predictable and more evenly spread across race, gender and class.

The topics covered include, for example:

- student engagement
- student achievement
- curriculum content and substance.

There are 12 topics in total, and each has an introductory statement and a number of indicators which describe in some detail, for example, *student engagement*.

> Statement 1: Classes are generally engaged, and when they are not, teachers make every possible effort to redesign the pattern of activity in the classroom so that all students are authentically engaged
> 1. Most students, most of the time are engaged in learning tasks.
> 2. Teachers intentionally plan the work they provide to students in ways that builds on those qualities that show the most promise of increasing authentic engagement.
> 3. When patterns of engagement differ from what teachers want or expect, teachers talk to and question students and analyse their work to discover what might account for the difficulty.
> 4. Teachers commonly work together to analyse the work they are providing students and give each other help and advice to make the work more engaging to students.
> 5. There is evidence that over time the level of engagement has increased and the amount of rebellion, retreatism, and passive compliance have decreased.

Working with the indicators in 2007

While still in draft, the Learning Inclusion Indicators are being used in clusters in various ways including:

Teachers in the Beach Road Cluster have used the indicators to design inquiry questions for classroom research. The central question of teacher inquiry is aligned to one or other aspects of the indicators; e.g., How can I improve my questioning techniques/skills so that I am better able to find out what student x knows about measurement?

In the North Yorke Cluster, leaders worked with the Learning Inclusion statements to brainstorm how they might support the development of the indicators in their schools. The results of the brainstorm helped leaders determine the actions necessary in order for them to support teachers and help them to address inclusion issues in their classrooms and the school, as a whole.

The Torrens Cluster used a number of approaches including:

- As part of the induction for new staff;
- In network meetings teachers focused on *Improved Student Engagement*. Teachers shared how they have changed what they do to increase engagement of low SES and Aboriginal students. Posters and bookmarks were produced in these sessions, with key statements, and were distributed to all teachers to display in the rooms to keep the focus on inclusion;
- One teacher used the "engagement "indicators to help her identify what to look for in observing a particular student. She developed a rubric to monitor a student's engagement and asked the CC to observe, and help her observe, class interactions.

Beyond 2007

The Learning Inclusion Team is currently developing a number of ways in which to work with teachers, school leaders and CCs in implementing the indicators.

We have been encouraged by the level of enthusiasm with which the indicators are being used across the program. It is reflective of the level of commitment to taking action, when clear direction and opportunities for input are provided. This has encouraged us to continue and broaden our consultation and share the collective learning with colleagues across DECS and beyond.

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Mathematics to motivate

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This paper presents examples of mathematical activities as trialled and proven successful in an Australian Government Quality Teacher Programme project aimed to motivate students and teachers in their interest and learning of mathematics. The success of these activities in the development of student motivation was examined in various contexts and is presented for information in the areas of Working Mathematically, Fractals and 3D Animation. An activity which has students writing their own lessons instead of a teacher "driving" the class is also illustrated. The activities are not difficult for the teacher to conduct nor are they equipment intensive. This sample of activities will be expanded on during the presentation.

The activities presented in the session are ideas and lessons taught, shared or examined during an Australian Government Quality Teacher Programme (AGQTP) titled "Teachers as Learners." This teacher professional learning project centred on in-situ learning using a "mathematics educator in residence" who conducted teacher professional learning, visited classes, observed specific lessons, taught model lessons and was involved in team teaching with staff. The project involved six week-long visits to a school in rural Victoria spaced across two years. A structured learning program was developed and teachers were paired with professional learning partners. The basis of the professional learning model being conducted was "learn – practice – review." For each teacher, half an hour was spent with the mathematics educator in residence prior to each lesson to be observed. The lesson was discussed and any other concerns or interests regarding the teaching of mathematics were addressed. Then a lesson of approximately one hour was taught, with the mathematics educator in residence and the teacher's professional partner viewing and participating as appropriate. After the lesson, half an hour (or more) was spent reflecting on the lesson with the mathematics educator-in-residence and the professional partner. The aim of the model was to promote teacher learning, with an individual focus on goal setting.

Student outcomes and student motivation were a major initiative of the project. It was felt that through examination and reflection, teachers were able to trial new ideas to improve student outcomes. This outcome based reinforcement was expected to lead to a change in teachers' practice (Rogers, 2007). Having students genuinely interested in mathematics became one of the key performance indicators across the school (P – Year 6) and throughout the duration of the project. To measure this; problem solving, open-ended tasks and links to students' worlds were examined. These links became important as they illustrated to students the need for mathematical learning and its relevance as well as importance in their lives.

Presented in this paper is a sample of the activities taught at the school. The focus is predominately Geometry, although other areas will be explored during the presentation. While these activities were conducted within the P – Year 6 setting, many could easily

be altered for use with secondary students, and suggestions have been included as appropriate. The presentation will include an expanded discussion focussed on the comments resulting from the "review" of the lessons with the teacher, professional partner and the mathematics educator in residence.

Cut ups

Equipment:
- one large piece of paper
- three small pieces of paper
- tape
- scissors
- a ruler
- a piece of string
- a pencil

The aim of the activity was to make all of the objects below:
- an object that holds water
- an object that measures the height of a member of the group
- a waterproof name card for the group
- a party decoration
- a Christmas decoration.

Students worked in small groups of three or four and the activity was conducted as a whole lesson with a Year 3/4 class.

This activity resulted from the exploration of the area of problem solving at the school which was the focus of one of the mathematics educator-in-residence week-long visits. The lesson was taught by the classroom teacher. After the teaching of the lesson, both the teacher and the professional partner commented that it was an interesting activity as all student groups produced quite different results. For example: with the waterproof container a variety of shapes were produced such as an open box, a cone and a cup shape, and all of the groups tested the container with varying degrees of success. Teachers noted that the object for measuring the height of a member of the group easily reflected students' understanding of the concepts of measurement and equal divisions; which was a topic that had been explored recently in the teaching of formal measurement.

The discussion in the review after the lesson focussed on how the activity itself required students to plan mathematically, work in groups and use existing knowledge. The professional partner commented that students were also encouraged to think more broadly and challenge the limitations set by activity.

Fractals

Adapted from the Math Rice website http://math.rice.edu/~lanius/frac/, the following two activities were conducted as model lessons by the mathematics educator-in-residence with Year 5/6 students producing and exploring fractals. The ideas were then used by one of the teachers viewing the model lesson with their own Year 5/6 students.

Drawing Sierpinski's triangle

1. Draw an equilateral triangle of side length 20 cm. Connect the midpoints of each side to create a triangle in the centre.

2. Shade the triangle in the centre.

3. Now connect the midpoints of the sides of the unshaded triangles to create smaller triangles and shade the triangle in the center as before. Continue the process.

Both the set of observing teachers and the teacher who later taught similar lessons found the activity valuable as it provided students with the opportunity to use geometric construction as well develop their use of mathematical terms such as equilateral, midpoint, etc. Students recorded in tables, information about the side lengths and the areas of the constructed triangles and looked for patterns. For example: for each iteration students found the area *not* shaded as a fraction. This produced the pattern of 3/4, 9/16, 27/64…which students attempted to generalise by identifying the relationship 3/4, $3^2/4^2$, $3^3/4^3$… for each iteration.

Teachers found the activity also allowed students to be creative and students could be further challenged. As an extension, a comparison between Sierpinski's Triangle and Pascal's Triangle was also made using a template on triangular paper from the website. Students completed Pascal's Triangle, after examining and learning about the patterns, and then they shaded all of the triangles except for the odd numbered ones and an image was found that was the same as Sierpinski's Triangle. Students appeared to like the examination of the patterns of the fractals and participated enthusiastically in the discussion, which required them to express their thinking and understanding in a logical manner.

In the review after the lessons — modeled and taught — teachers commented that they found students were inspired and motivated by the development of the fractals. The students were actively engaged in the activity and the discussion throughout the session. In subsequent classes teachers later shared, students completed fractals in similar colours and these were combined into one design for display purposes. Students also looked at different sized equilateral triangles for the starting point, creating both large and small versions of Sierpinski's Triangle.

A similar activity was also completed with students drawing the Koch Snowflake. In this case the drawing is completed on the outside of the equilateral triangle instead of the inside.

Drawing the Koch Snowflake

1.   Start with a large equilateral triangle.
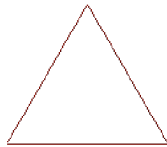
2.   Make a star.
       Divide one side of the triangle into three parts and remove the middle section.
       Replace it with two lines the same length as the section you removed.
       Do this to all three sides of the triangle.

3.   Continue this process.

       During this activity students investigated the area of the different iterations as the Koch Snowflake was developed. Again patterns were established, shared and examined.
       Teachers reflected that students responded well to these activities and they liked the visual aspect of the developed images. It was an opportunity to bring a variety of mathematical skills together into one activity including geometry, fractions, patterning, working mathematically and some of the students began looking at generalising the patterns in algebraic terms. One of the classes became so interested, that the activities were taken from working on paper to students completing the drawing on the computer. (Note this could be completed using programs such as Geometer's Sketchpad for older students.) Initially, to gain students' interest, examples of fractal images were found on the Internet on such sites as Wikipedia: http://en.wikipedia.org/wiki/Fractal. Students then went on to find many relationships of fractal images on the Internet such as natural fractal patterns evident in Romanesco broccoli.

3D animation

In one of the classes students were completing a unit on drawing 2D shapes and 3D objects within the area of Space. Through the practice of team teaching of the teacher and the mathematics educator-in-residence, ideas of how to draw 3D objects were shared. One of the examples was with the "overlap" method which is where two

congruent shapes are drawn slightly overlapping each other and then each of the corresponding pairs of points are joined with straight lines to complete the prism as seen in Figure 1.

Figure 1. Congruent squares are overlapped and corresponding points are joined.

This activity led to students asking, "When are we ever going to use this?" The application of 3D animation was explored. This idea immediately "grabbed" the students' attention. Different web sites such as http://www.garyharbo.com/activity.html and http://www.anim8or.com/main were presented which illustrated images for students to look at. There were also programs that students downloaded allowing them to complete the animation process.

Teachers involved reflected that it was terrific how students devised their own images on screen (computer) and animated them, so they could view the images from any perspective. They found that this development from paper to computer allowed students to use their geometrical knowledge to apply it to a real situation which was of interest to them. Many elements of mathematics were brought together including working mathematically, two and three dimensional shapes, perspective, problem solving and programming.

Writing your own activities

In many classes teachers set the questions and the activities. In this case students came up with their own ideas. Students were randomly provided with elements of a topic on separate pieces of card. For example: for a topic on measurement, the elements were:

- numbers, e.g.. 4, 0, 8
- equipment, e.g., Unifix
- context, e.g., area
- symbol, e.g., ×

Students worked in small groups and used a large piece of paper to:

- discuss what each of the elements were
- link each of the elements (e.g., some groups used a concept map)
- develop their own activity / lesson that used all of their provided elements.

This activity proved exciting and interesting. The teachers involved felt the activity provided students with ownership of their own learning. It allowed the teacher to see any student misconceptions and it provided an insight into what students may like to learn about. The completed ideas were collected together into a class book, but they could also be run in the classroom. In the review time teachers felt that this activity could also be modified for younger students by providing them with some numbers and having them write everything they know about the numbers on their piece of paper.

Alternatively the activity could be extended for older students by including equipment such as calculators, protractors and compasses.

Discussion

The "review" component of the "learn – practice – review" model allowed the teacher, the professional partner and the mathematics educator-in-residence to reflect on the taught lesson, the impact of the learning and the students' responses to the lessons. As the lessons were conducted during the week-long visits, between visits teachers could try further ideas resulting from the discussions, which were again shared and discussed during the following visit. The model lessons, in this case the fractals activities, could also be tried by teachers viewing the lesson, and ideas could be developed and explored if students displayed interest.

The use of the mathematics educator-in-residence provided teachers with support in their use of lessons aiming to motivate students. This support was in the form of feedback, ideas, conducting model lessons and team teaching. The use of the professional partner again provided support for the teacher to attempt innovative lessons. It also aided in the rich discussion occurring after the lesson (review), as well as creating the beginning of the on-going dialogue regarding mathematics teaching to continue between the visit weeks of the mathematics educator-in-residence.

The activities presented in this paper and during the presentation form part of a larger study of the effectiveness of the teacher professional development model "learn – practice – review" briefly explained in the paper. It appeared that the activities provided the teachers with the opportunity to learn about a new idea, concept or take a different perspective when looking at a topic. This motivated the teachers to try different approaches and activities in the classrooms with their students with the support of their professional partner and the mathematics educator in residence. Finally it was through the review process that teachers could examine if the students were engaged, interested and motivated. The concept of motivating students remained a focus during the two years of the project.

Conclusion

All of the activities presented in the paper used minimal and low cost equipment making it easy to keep these ideas ready for an appropriate moment. All of the activities have been used in classrooms, with positive results. It appeared that both the students and the teachers became motivated in the learning and teaching of mathematics through trying new ideas that engaged the students. The "learn – practice – review" model was utilised throughout the project with a range of teaching models including model lessons and team teaching. The use of the model allowed teachers to explore new and/or different ideas, to try them and then to reflect on the lessons and their own teaching. The use of the mathematics educator in residence and professional partners provided teachers with the support to try the new ideas with their own classes. It then allowed professional conversation to develop between teachers about mathematics and teaching in general.

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Incorporating dynamic geometry software within a teaching framework[*]

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There have been numerous studies that have characterised the nature of levels of understandings in geometry. Some studies have extended this work and focused upon developmental pathways and identified hurdles leading to higher order thinking. There is an urgent need, however, to explore teaching/learning practices that facilitate student developmental progression. This paper considers previous work by presenting dynamic geometry software as a teaching tool linked to the van Hiele teaching phases framework. Hence, the teaching sequence presented links theory and practice within a technological environment in the mathematics classroom.

Introduction

The philosophical stance taken by van Hiele in regards to teaching geometry is grounded within the notion of insight. The opportunity to exhibit and develop insight is described by van Hiele (1986) as the aim of teaching mathematics. Thus, for the promotion of growth in understanding, learners require geometrical tasks that allow them to control their individual problem-solving environment (Hoffer, 1983, p. 205). Essentially, dynamic geometry software (DGS) provides the potential for student-centred problem-solving tasks that remain in the control of the individual student. DGS allows the "continuous real-time transformation often called "dragging." This feature allows users, after a construction is made, to move certain elements of a drawing freely and to observe other elements respond dynamically to other altered conditions" (Goldenberg & Cuoco, 1998, p. 351).

Numerous studies have highlighted the difficulties associated with students' attempts to understand and utilise networks of relationships in geometry (Burger & Shaughnessy, 1986; Fuys, Geddes & Tischler, 1985; Mayberry, 1981; Usiskin, 1982). The van Hiele Theory (van Hiele, 1986) grew from a desire to address this need through the provision of a five level framework of development in Geometry from which to view students' understandings. The level pertinent to this paper, is described as Level 3. This includes:

- the formation, and awareness, of a network of relations among properties and figures;
- the ability to provide descriptions of figures and properties based upon known relationships; and
- the recognition of class inclusion concepts and their implications.

A typical example of a focus on the network of relationships among quadrilateral properties is the relationship between parallelism of sides and equality of angles and

sides. An example of class inclusion within the context of quadrilaterals is the ability to explain and justify the square as a member of the class of rectangles in terms of property relationships.

Background

In response to the recognised difficulties identified above, Serow (2002) highlighted the reasons students find class inclusion concepts in geometry difficult to grasp. Serow (2002) detailed the hurdles encountered by many students through the characterisation of the development of relationships among figures and relationships among properties. A natural progression would involve the presentation of suitable teaching strategies to assist students in meeting and rising above the identified cognitive hurdles.

The tools within The Geometer's Sketchpad, Version 4.0 (an example of DGS), designed by Jackiw (2001), provide teachers with the opportunity to explore the relationships of quadrilateral figures and properties both intuitively and inductively. Goldenberg and Cuoco (1998, p. 396) found the "dynamic nature of these tools makes them both exciting and accessible, even to elementary students" (p. 396). Dynamic investigations are possible when students have time to consider their mathematical ideas as opposed to "spending that time on the tedious effort required to perform numerous constructions" (Tikoo, 1998, in Coffland & Strickland, 2004, p. 347).

It is evident that "few studies have been published on the effectiveness of technology within geometry classrooms" (Coffland & Strickland, 2004, p. 347). In light of this, the following teaching sequence utilises technology as a facilitator of students' growth in understanding geometrical concepts. This need in the mathematics classroom is emphasised by Fitzallen (2005, p. 253) through the recognition of "a need for teachers to gain an understanding of how Information and Communication Technology (ICT) can be used to extend students' thinking and problem-solving skills, rather than just a publication and research tool". Many classroom teachers confidently use technology as a presentation or display tool, but remain unaware of the potential for ICT to promote concept development in the mathematics classroom.

McGehen and Griffith (2004) in Fitzallen (2004, p. 360) "contend that is it important that teachers develop an understanding of how the technology promotes mathematical thinking to ensure teaching and learning strategies promote a better understanding about mathematical concepts". A teaching framework that suitably addressed this need is the basis of the work of Dina van Hiele-Geldof. The five teaching phases represent a framework to facilitate the cognitive development of a student through the transition between one level of understanding and the next. The phases originate from the idea that "help from other people is necessary for so many learning processes" (van Hiele, 1986, p. 181). This idea stems from the notion that students find it very difficult to move unassisted from one thought level to the next.

The van Hiele model acknowledges that progress is easier for students with careful teacher guidance, the opportunity to discuss relevant issues, and the gradual development of more technical language. The phases are organised in such a way that they acknowledge the assumptions underpinning the van Hiele levels, while providing students with the opportunity to exhibit insight. The van Hiele teaching phases address the concern that "teachers often feel reluctant or uncomfortable because their pedagogical knowledge perhaps does not include a framework for conducting technology-based activities in their lessons" (Chua & Wu, 2005, p. 387).

The five phases of teaching assist in maintaining student ownership of ideas throughout the learning process. During this process, students can seek clarification

from each other and from the teacher concerning the language used. In particular, language plays a central role. It is only after students have identified and described concepts using their own language that the more technical language is introduced. A description of the phases is provided in Table 1 with an emphasis on the changing role of language as the student progresses through the phases.

Table 1.2 Descriptions of the van Hiele teaching phases

Phase	Description of phase focus
1. Information	For students to become familiar with the working domain through discussion and exploration. Discussions take place between teacher and students that stresses the content to be used.
2. Directed orientation	For students to identify the focus of the topic through a series of teacher-guided tasks. At this stage, students are given the opportunity to exchange views. Through this discussion there is a gradual implicit introduction of more formal language.
3. Explicitation	For students to become conscious of the new ideas and express these in accepted mathematical language. The concepts now need to be made explicit using accepted language. Care is taken to develop the technical language with understanding through the exchange of ideas.
4. Free orientation	For students to complete activities in which they are required to find their own way in the network of relations. The students are now familiar with the domain and are ready to explore it. Through their problem solving, the students' language develops further as they begin to identify cues to assist them.
5. Integration	For the students to build an overview of the material investigated. Summaries concern the new understandings of the concepts involved and incorporate language of the new level. While the purpose of the instruction is now clear to the students, it is still necessary for the teacher to assist during this phase.

The phases allow a means of defining and aiding progression from one level of understanding to the next. This does not mean that each time the student passes through the five-phase process within concept development that he/she has reached the next level. The phases do, however, provide students with the opportunity to come closer to meeting the need to move to the next level. It is interesting to note that this teaching process is not centred upon one specific form of instruction. The five-phase process lends itself to many teaching styles, and each phase provides a specific and important purpose.

The five-phase teaching approach provides a structure on which to base a program of instruction. As can be seen, the phase approach begins with clear teacher direction

involving exploration through simple tasks, and moves to activities that require student initiative in the form of problem solving. The phases are organised in such a way that they acknowledge the assumptions underpinning the van Hiele levels, while providing students with the opportunity to exhibit insight.

Teaching sequence

The teaching sequence was designed with two main elements; the developmental framework, and the embedding of dynamic geometry software and other suitable technology. The two-week teaching sequence (eight sessions of 40 minutes duration) was delivered to a Year 9 (ages 14–15) secondary mathematics class. The focus content strand of the teaching sequence from the K–10 Mathematics Syllabus (Board of Studies, NSW, 2002) is Space and Geometry and the target outcomes to be addressed by the teaching sequence are "classify, construct, and determine properties of triangles and quadrilaterals" and "verify the properties of special quadrilaterals" (Board of Studies, 2002, p.39). An outline of the teaching sequence and sample student responses to tasks are outlined below:

Activity 1: Information phase
— Mechanics and recall

Students work through simple constructions in Sketchpad and brainstorm known quadrilaterals. Instructions for individual students, with respect to the constructions involve:
a) Write your name using sketchpad.
b) Create a person and reflect the person. Measure a selection of corresponding sides and angles. What do you notice when you drag one of your people?
c) Create a house design using the six quadrilaterals, namely, kite, trapezium, square, rectangle, rhombus, and parallelogram.
At this stage, the students, in most cases, will construct their figures using the line tool. This will be extended in later phases. When the students are asked to drag (drag test) the quadrilaterals they have formed this way, they will notice that the constructions are not robust (do not remain the intended figures). This becomes the motivation for the following activity.

Activity 2: Explicatory phase
— Robust templates and recording

Students create robust templates for each of the six quadrilaterals on separate Sketchpad pages. If the drag test allows the figure to remain as intended, the construction will involve known properties of each figure. Discussions will begin to occur concerning relationships among figures. For example, comments such as " this is really strange, when I drag the parallelogram it is sometimes a rectangle, square or rhombus". This activity will involve constructions such parallel lines, perpendicular lines, and transformations. It is essential for the students at this phase to describe their construction within a text box on Sketchpad and to record the properties for each quadrilateral on a teacher-designed table.

Activity 3: Directed orientation phase
— Irregular quadrilateral and midpoint construction

Students are instructed to:

1. create any irregular quadrilateral using the line tool;
2. construct the midpoints;
3. join the midpoints to construct another quadrilateral;
4. answer the question, What do you notice?; and,
5. investigate the properties of this shape to justify what they have found, and record their justification in a textbox.

Activity 4: Free orientation
— Further exploration of properties and figures

Students design a spreadsheet where the six quadrilaterals are contained in the first column, and the first row contains all possible properties of quadrilaterals. Particular care needs to be taken to include diagonal properties such as "diagonals meet at right angles." The students record the properties of each figure by ticking the appropriate cell. There is an element of surprise in the classroom when the students notice that the square has the maximum number of ticks.

Activity 5: Free orientation
— Diagonal Starters Game Design

This activity is designed to reinforce diagonals as a property and not merely a feature of the quadrilaterals. Students are given the challenge to create the diagonal formation needed for each of the quadrilaterals. The aim is for the students to construct templates for younger students to complete the figure and explore the properties. Figure 1 contains a student sample for the diagonal starters game.
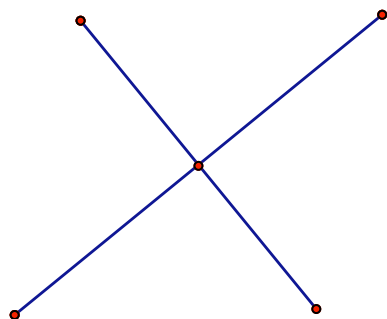
All sides are equal, though diagonals are not! Angles between diagonals are right angles. The shape has two axis of symmetry at least. Join the dots to reveal a - you guessed it, a ......

Figure 1. Sample of a student's diagonal starters diagram.

Activity 6: Free orientation
— Concept maps and flow charts

Students create concept maps and flow charts using suitable software. The chosen software will vary according to school accessibility. Figures 2 and 3 include student samples of a concept map and flow chart.
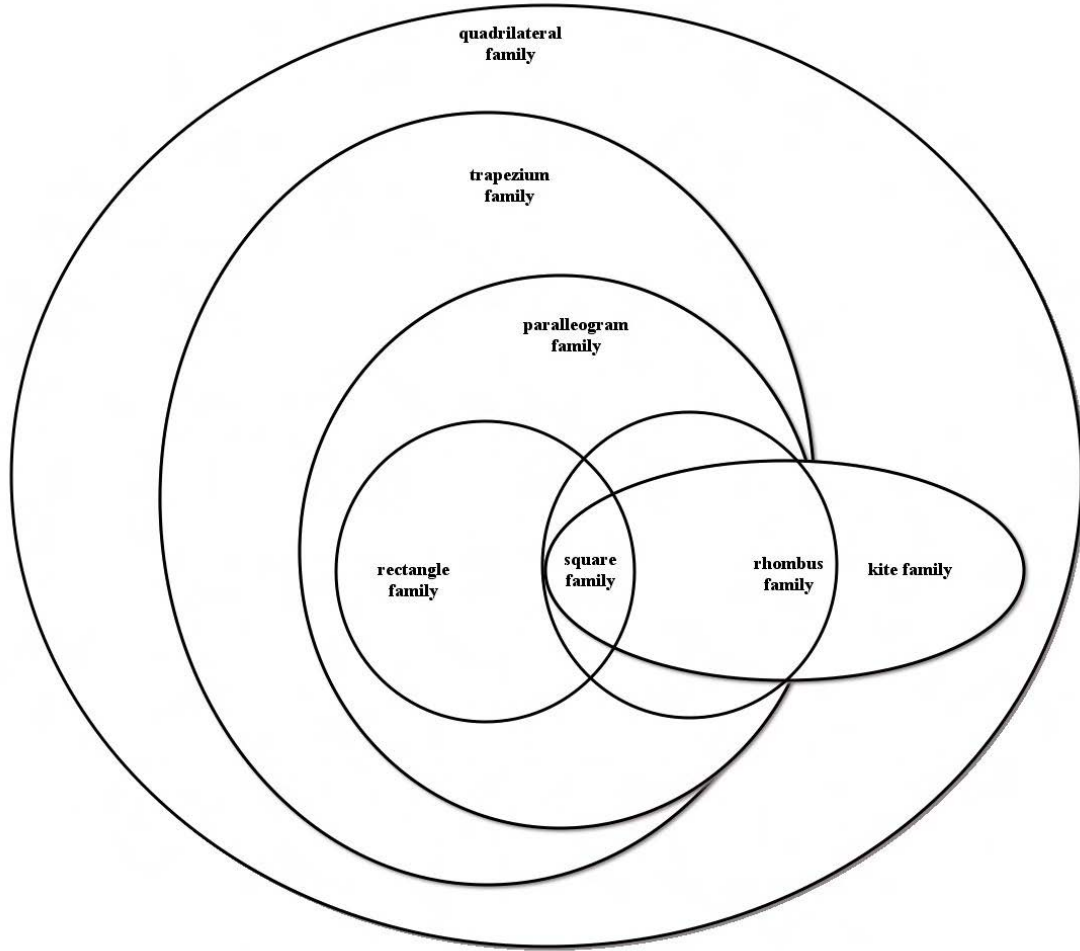
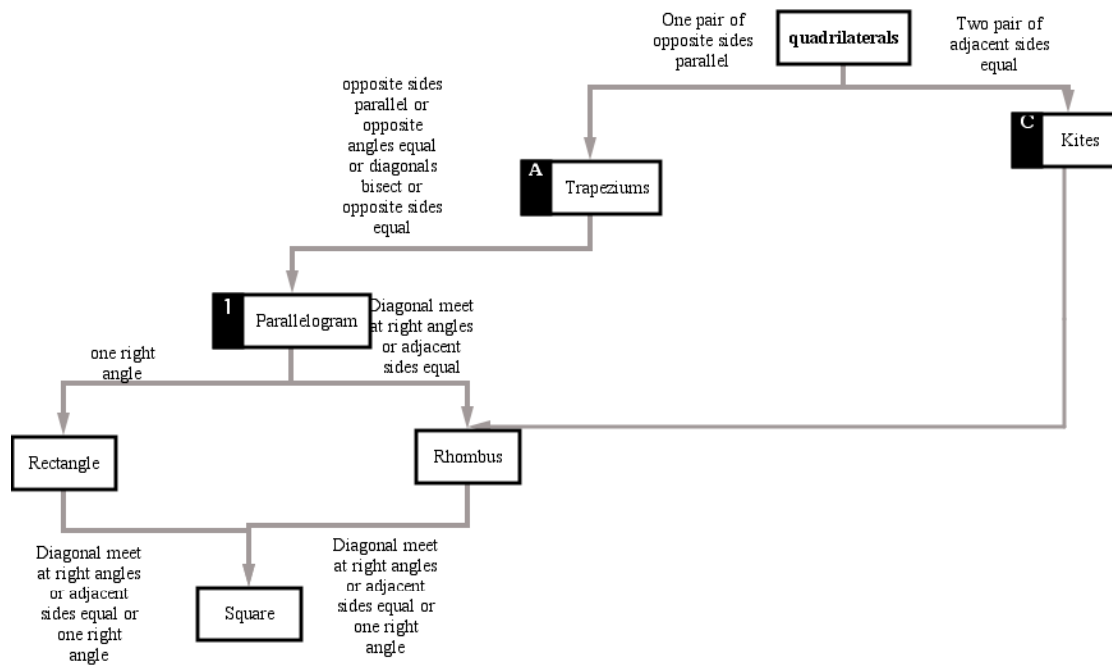*Figure 2. Sample of a student's concept map.*

Figure 3. Sample of a student's diagonal starters diagram.

Activity 7: Free orientation
— Property relationships consolidation

Using Sketchpad, students explore and record the relationships among properties. For example, students may record the relationship among opposite sides parallel, opposite sides equal, and opposite angles equal. Figure 4 contains an example of a student's record of property relationships.
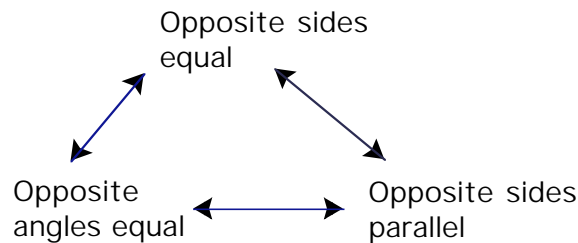

Figure 4. Sample of a student's property relationship summary.

Activity 8: Integration
— Information booklet design

Students organise the constructions that they have made, justifications, tables, spreadsheets, concept maps, and flowcharts to produce an information booklet to explain what they know about the relationships among quadrilaterals and relationships among quadrilaterals figures. Students are instructed to include an overall summary of their findings.

Activity 9: Integration
— Sharing and Routine Questions

Class sharing of booklet designs. Routine questions involving known properties and relationships.

Conclusion

A main feature of the teaching sequence presented is the integration of dynamic geometry software using the van Hiele teaching phases as a framework (van Hiele, 1986) for maintaining student ownership of their mathematical ideas. This is facilitated via student-centred tasks that acknowledge students' individual experiences and the progression from informal to formal language use. The teaching sequence combines a range of effective teaching practises that make use of the potential of technological tools currently available to secondary students.

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Magic maths moments: The power of dynamic and effective numeracy classrooms[*]

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Encouraging students to think mathematically using open-ended questioning, investigation, reflection, and feedback is a key feature of effective numeracy classrooms. This paper describes how open-ended tasks are being used to build students' mathematical knowledge and skills alongside their abilities to inquire and think reflectively. In Tasmania, curriculum reform that values the development of higher-order capacities and the connection of knowledge and skills, across the curriculum, has enabled two teachers to work in a way that matches their own pedagogical beliefs and practices with respect to planning, teaching, and assessment for student learning.

Introduction

Ron Ritchhart (2002) in his book, *Intellectual Character: What it is, Why it Matters, and How to Get it* challenges all teachers to consider their purpose for teaching. Is the role of teaching to fill students with knowledge, to help them succeed on national assessments, to prepare them for high school, or as Ritchhart would propose, are teachers "working to change who students are as thinkers and learners?" (p. 7). Developing classrooms as cultures of reasoning is critical in enabling students to build the intellectual capacities required of them in the twenty-first century.

Numeracy and the role of the teacher

Numeracy is an important aspect of the intellectual character of both students and adults. It has become an essential capability for any individual who wishes to participate fully in a democratic society, to apply not only knowledge and skills, but also critical reasoning capabilities, to learning and to everyday life. In Australia, a view of numeracy, beyond the ability to handle number concepts in context, is reflected by the Australian Association of Mathematics Teachers (AAMT) definition:

> To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life.
> In school education, numeracy is a fundamental component of learning, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:
> • underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic);
> • mathematical thinking and strategies;
> • general thinking skills; and
> • grounded appreciation of context. (AAMT, 1998)

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[*] Paper accepted by peer review

Numeracy is about making meaning of mathematics, at whatever level of mathematical skill. It is about understanding and using mathematics, in all of its representations; for making sense of the world, for considering critically information presented, and for making informed decisions.

Researchers in both Australia and the United Kingdom have studied primary school mathematics with a view to identifying key practices that are indicative of highly effective teachers of numeracy. Askew, Rhodes, Johnson, and William (1997) described "connectionist" teachers as those who emphasised the connections between ideas and concepts in mathematics and whose own beliefs and practices were based upon valuing students' own methods. They encouraged students to describe their approaches and their thinking in solving problems. Clarke and Clarke (2002) similarly found that effective teachers of early years' numeracy exhibited specific characteristics. These included the structuring of purposeful and open-ended tasks: tasks that focused on children's mathematical thinking. In addition, these teachers developed a culture of community in the classroom where students were encouraged to question, explain and share their thinking and evaluate others' mathematical ideas. The teachers themselves reflected upon students' responses, recorded observations, and used a variety of assessment methods.

Striving for excellence in the teaching of mathematics is multifaceted and involves ongoing learning for teachers with respect to their own knowledge, beliefs, and practices. In 2002, AAMT adopted a statement describing the knowledge, skills, and attributes considered to be important for excellence in the teaching of mathematics in Australian schools. Excellent teachers of mathematics do not only have a thorough and relevant knowledge of mathematics and of their students, and a commitment and enthusiasm for the profession, but they also make purposeful decisions that create for students a learning environment that maximises learning opportunities for each student. One aspect of the statement describes teaching, in action, as the capacity to "arouse curiosity, challenge students' thinking, and engage them actively in learning" (AAMT, 2002).

Learning context

The use of open-ended tasks is one way to successfully create a learning environment that supports students in their learning of mathematics. Sullivan and Lilburn (2004) describe three main features of "good" questions in the design of open-ended tasks:

1. They require more than remembering a fact or reproducing a skill.
2. Students can learn by answering the questions, and the teacher learns about each student from the attempt.
3. There may be several acceptable answers. (p. 2)

These types of questions encourage high levels of thinking and emphasise the value of students developing understanding of mathematical concepts, so they can apply their knowledge and skills to new situations. Open-ended tasks also support learning that assists students to follow arguments, evaluating them, and apply reasoned logic to any critique.

Two Tasmanian Grade 6 teachers are using open-ended tasks to support their students in the learning of mathematical concepts and skills. These teachers participated in a research project investigating the positioning of numeracy by teachers, in reform-based learning environments, and the numeracy experiences of their students. Their

teaching has been guided by reform curriculum in Tasmania that places thinking skills and strategies at the core of the curriculum, and encourages the connection of knowledge and concepts across the curriculum. Tasmania's Essential Learnings Framework is centred around five Essential Learnings: Thinking, Communicating, Personal Futures, Social Responsibility, and World Futures, and encourages transdisciplinary activities (Department of Education, Tasmania [DoET], 2002). Other Australian states and territories are also reconceptualising curricula in terms of similar over-riding big ideas. Whilst these reforms are systemic and encompass whole curriculum change, similar reforms are occurring within the mathematics classroom. These reforms focus on the importance of conceptual understanding for students over procedural understanding and encourage models of teaching and learning that improve students' capacities to reason, communicate, and reflect on their own learning experiences.

Three key elements of Tasmania's curriculum are driving the teaching of mathematics in these two Grade 6 classrooms: Being numerate, Inquiry, and Reflective Thinking.

Being numerate. The DoET embraces a comprehensive view of numeracy:

> Being numerate involves having those concepts and skills of mathematics that are required to meet the demands of everyday life. It includes having the capacity to select and use them appropriately in real settings. Being truly numerate requires the knowledge and disposition to think and act mathematically and the confidence and intuition to apply particular mathematical principles to everyday problems… (DoET, 2002, p. 21)

Inquiry and Reflective Thinking. The central Thinking Essential Learning brings the processes of inquiry and reflection to all questions and investigations. In particular, the aim of the "Inquiry" key element is to develop the capacity of learners to pose problems, gather relevant information, consider possibilities, make decisions, and justify conclusions.

> Effective learners need the capacity to ask good questions, persevere in a line of inquiry, be systematic, set goals, and plan and follow a course of action… The ability to communicate what has been learnt and thought about, and to do so in a consistent, coherent, relevant and persuasive way, is essential in enabling learners to participate fully in schools, communities and workplaces. (DoET, 2002, p. 14)

Magic maths moments

Three stories, as told by the teachers, are included. They provide examples of how open-ended tasks have been used by two teachers to support student learning in mathematics, and how such tasks have resulted in many moments where student understanding, of important mathematical concepts, is evidenced: moments that these teachers refer to as "magic maths moments."

Square numbers as a bridge to understanding area and perimeter

Every year I spend a lot of time working with students on square and rectangular numbers and developing understanding of concepts such as composites, primes, factors, and multiples. I have never been convinced that students really understand these

concepts as they seem to be forgotten quite quickly. I thought about a different way to approach it and tried rewording it as an inquiry task (see Figure 1):

Do Numbers Have Shapes?
Make a solid square (not just an outline) using counters.
How many counters did you use?
Can you make different sized squares?
What can you find out about the number of counters used for each square?
Can you make any other shapes?
Explore, record, and be prepared to share your findings.

Figure 1. Do numbers have shapes?

Within a very short time I could see "light bulbs" switching on above children's heads! Students quickly saw the connection between their squares of counters and certain number facts that they already knew. I was hearing comments such as:

"So that's why they are called square numbers!"
"I get it — I don't even have to make the squares because I've found the pattern."
"I don't have to count all the counters — it's 6 rows of 6 counters and I know that 6 × 6 is 36. That means 36 is a square number!"

Students then went on to explore other shapes, not only rectangles and triangles, but also circles and hexagons. Already they were "stepping outside the square." Many connections were made with previous knowledge, things they remembered but did not really understand. The "rectangular" group arrived at prime numbers by themselves when they were trying to find as many different rectangles as possible for each number between 2 and 20. At last the term "prime number" was given a context and had some meaning: "We can only make one rectangle using 2, 5, 7, 11, 13, 17 and 19 counters. I think these numbers have a special name."

The "triangular" group were really excited to discover a pattern, and could see why that pattern worked. They were able to accurately predict triangular numbers and spent a lot of time trying to find the biggest triangular number possible! (An emergency run to the maths resource room was needed when they wanted enough counters to check their prediction!). The "circular group" made some amazing discoveries. It was a real learning experience for me as I had never considered circular numbers before and I am not going to divulge their findings — you will have to explore it yourself! The "hexagonal group" were unable to construct regular hexagons with circular counters and after a while switched to one of the other shapes. This in itself led to some interesting discussion on the properties of shapes.

During the sharing back session some students were able to make the connection between what they had just done and the concepts of area and perimeter. A fairly lively discussion ensued with a number of students trying to transfer the length × width formula, used for finding the area of quadrilaterals, to find the area of circles! The different mathematical concepts explored during this task are too numerous to mention, but from then on whenever I mentioned square numbers I could almost see students visualising rows of coloured counters!

Using a knowledge and understanding of factors to see the relationship between perimeter and area

I have experienced many "light bulb" moments with students during our work in Numeracy. For me, measurement has always been a particularly enjoyable unit to teach, filled with those "light bulb" moments, as it allows students to experiment with a range of concrete materials and to discover the relevance of mathematics concepts in real world situations beyond school. As part of a unit of work focusing on measurement, students were exploring perimeter and area through a range of open ended problems. Students were given the problem in Figure 2 to investigate.

I want to make a vegetable garden in the shape of a rectangle.
I want it have an area of 12 square metres.
How many different shaped vegetable gardens can I make?
What would be the perimeter of each of these vegetable gardens?

Figure 2. Designing rectangular vegetable gardens.

Many students started to draw out various vegetable gardens of different dimensions. All the students had done a lot of number work during the year, including number facts and the importance of factors. One student looked at the problem and very quickly used her understanding of factors to work out how many different shaped vegetable patches she could make with an area of 12 square metres, without having to draw them out.

She wrote down:

$6 \times 2 \quad 3 \times 4 \quad 12 \times 1$

The "light bulb" moment for her was great to watch as she then understood that she could use her knowledge to quickly, and easily, work on a range of perimeter and area problems. She then began to share and explain her thinking to other students in the class who, in turn, understood the efficiency of the process that she had used and were then able to transfer this knowledge to a range of problem solving tasks. Several students rapidly made the connection between perimeter and area and used this knowledge to quickly, and efficiently, calculate the area of a range of shapes.

There are many magical maths moments that I observe on an almost daily basis, far too many to mention. This was just one simple example of the understanding of a concept, the ability to use that knowledge in other situations, and the wonderful chance to share that knowledge with others!

Designing vegetable gardens and discovering pi

As part of a unit investigating measurement the students were given the following problem (see Figure 3):

> **The Vegetable Patch**
> Tim wants to plant a vegetable garden but we have a lot of trouble with wallabies eating everything we plant.
> Tim bought a 60 m length of very expensive wallaby-proof fencing wire to fence in an area of land where he could grow vegetables. He wanted to make his vegetable patch as big as possible.
> What is the largest possible area he can fence in with his 60 m of wire?

Figure 3. The vegie patch.

Students tackled this problem in an amazing variety of ways. A large number of students drew diagrams showing the possible dimensions of the vegetable patch. A lot of them started by setting out blocks in different arrays: 20 × 10, 15 × 15, etc. Some students were able to quickly identify a pattern and completed the calculations without the use of blocks or diagrams: 1 × 29, 2 × 28, 3 × 27, etc. One group took a 60 m length of string out to the school oval and measured out vegetable patches of different dimensions in order to see the relative sizes! Then there was Tom.

Tom immediately saw the pattern and completed the calculations accurately. However, he was not satisfied that a 15 × 15 square, with an area of 225 square metres, was going to give Tim, the vegeatbale garden owner, the greatest area because "We haven't considered any different shapes: what about triangles and circles?" He decided to see if he could work out the area of a 60 m circle. Tom began his investigation by measuring the circumference of a circular table we had in the classroom. The circumference of the table was 4 m. He covered the table in large sheets of paper and cut them to fit exactly. Figure 4 shows Tom and his friend ruling a 5 cm square grid on the paper. Tom counted all the squares and then he estimated how many whole 5cm squares could be made from the bits around the edge. He multiplied the total number of squares by 5 to get the area of the table.

Tom then considered the relationship between the table, with a 4 m circumference, and the original task where Tim had 60 m of fencing wire to build his vegetable garden. So Tom multiplied his result by 15 to get the area of a 60 m circle. This was all without any guidance from me and he was able to explain exactly what he was doing and why every step of the way!

Figure 4. Considering the area of a circle.

However, Tom was not happy. He knew he did not have an accurate answer because of the bits of squares around the edge of the circle that he had had to estimate: "Ms M., I've got as close as I can, but I suspect that you know an easier way to find the area of a circle!" I wrote the term pi on a piece of paper and suggested Tom might like to do some finding out about it. He "googled" it and did a lot of reading. He discovered how to find the area of a circle given the diameter and/or radius and/or circumference. He then used compasses to construct circles of different sizes and test out his new found formulae! Once he was convinced that all was legitimate, he applied his newly acquired knowledge to his 60 m circle and concluded that a circle was going to give Tim, the owner of the vegetable garden, the biggest area for his vegetable patch: 286.62419 square metres! Tom's drawing and calculations for the vegetable garden are presented in Figures 5 and 6.
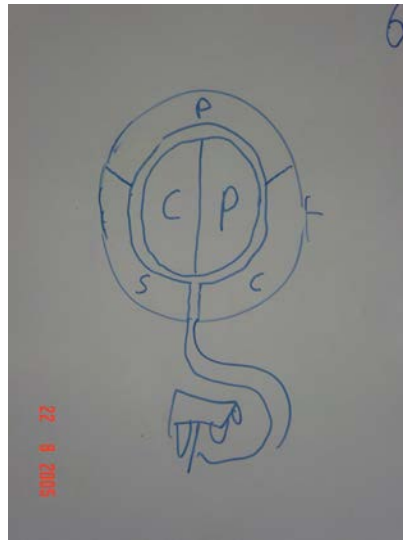

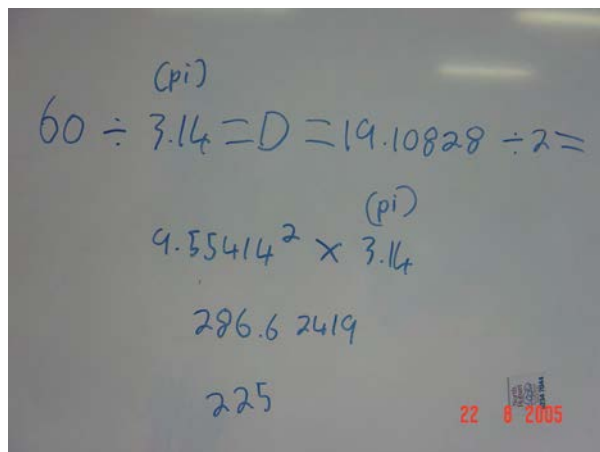*Figure 5. A circular vegie garden.*

Figure 6. Calculating the area of a circular vegie garden.

Conclusion

The stories presented in this paper provide examples of the possibilities that open-ended tasks present, in relation to the learning outcomes of individual students. It has also demonstrated the power of such tasks to unlock student thinking. When students are encouraged to engage in a task, and to share their problem solving strategies and solutions, it changes the nature of the classroom and student learning. Students become problem solvers and strategists who are able to undertake meaningful mathematics activities, to suit their own level of understanding. They are also able to be stretched by the discussions that occur, between teacher and student, and between students themselves.

The use of open-ended tasks also has significant implications for assessment. The use of student work samples, and observational notes, to evidence student understanding are important assessment tools. When students are provided with the opportunity to use their own methods and explain their own thinking, teachers are provided with a depth of assessment opportunities. Students' understanding of important mathematical concepts is uncovered, misconceptions may be revealed, and opportunities for further teaching arise.

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The Context Rich Integrated Mathematics and Science (CRIMS) projects[*]

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What activities can combine the thinking, reasoning and communicating aspects of science and mathematics with the development of essential skills? How can we adapt our existing lessons to generate richer approaches? How can we support teacher professional learning integrating mathematics and science and addressing the AAMT and ASTA professional standards? The CRIMS projects focuses on teacher professional learning, working with middle years teachers from metropolitan and rural schools. The workshop will present the CRIMS model and CRIMS principles, which involve creative activities incorporating science as a way to know and working mathematically.

The Context Rich Integrated Mathematics and Science (CRIMS) projects

The CRIMS projects began as an Australian School Innovation in Science and Technology (ASISTM) Round 1 project. "The ASISTM Project aims to bring about real and permanent improvements to the ways in which science, technology and mathematics are taught in our schools… Project activities will enable schools to come together with industry, science organisations, universities and others to explore ways to encourage a culture of innovation, attract greater numbers of quality students into teaching, improve coordination between primary and high school curricula and provide positive role models for students." (Australian School Innovation in Science, Technology and Mathematics (ASISTM) Project, 2004). The ASISTM project has two main components:

- funding of school cluster projects;
- teacher associates working in schools.

The CRIMS project addressed the ASISTM aims of:

- encouraging innovation in Australian schools;
- promoting world class teaching and learning, improvements in teachers' approaches and techniques and improved student learning outcomes;
- teacher attraction and retention.

The original CRIMS project was an ASISTM Round 1 project, in association with Merici College, St Clares College, St Francis Xavier College and Daramalan College in the ACT. It aimed to produce well-developed and tested context-based teaching resources to promote innovation and student interest. CRIMS activities were developed, trialled and refined collaboratively by partner schools with assistance from non-school partners, enhanced by the engagement of teacher associates who brought enthusiasm

and specialised knowledge into the classroom. Designed for Years 7 to 10, the resources were trialled at multiple partner schools and are available to other schools nationally through the CRIMS website.

The activities from the first CRIMS project are classroom resources for secondary teachers. CRIMS tasks integrate mathematics and science, and are inquiry based and student-centred. CRIMS tasks aim to:

- cross mathematics, science and ICT curricula in Years 7 to 10;
- be practical resources for teachers which include feedback from other teachers;
- vary in length, from one to multiple lessons;
- be in context of the world outside the classroom;
- include elements of constructivism and meta-cognition; and
- promote higher order thinking and help develop problem solving skills.

CRIMS are inquiry based and focus on student-centred learning and problem solving. They emphasise skill development, and thinking and reasoning. The central focus of the CRIMS project is open-ended investigation, scaffolded by the teacher. The interrelated disciplines of science and mathematics come together in the investigation process. CRIMS develop students' understanding of how science and mathematics are used to describe, represent and explain their world. CRIMS assist students to formulate their ideas about the nature of science and mathematics and what it means to be scientifically and mathematically literate by examining the thinking within science and mathematics. Each investigation is accessible at a range of levels so each student can achieve some level of growth, hence there are different entry points and exit points. CRIMS use pedagogical tools which cater for differentiated instruction and contribute to a flexible and supportive classroom environment. Learning is social, and students may work in collaborative groups carrying out investigations.

CRIMS principles

CRIMS aim to use rich contexts to create an environment whereby students can construct their own scientific and mathematical knowledge and develop understanding of scientific and mathematical processes. CRIMS are characterised by:

- interdisciplinary connections / linkages between mathematics and science;
- meaningful worthwhile contexts, e.g., a current event, a social issue relevant to students;
- an open-ended investigative approach, generating first hand data in cooperative groups;
- kinesthetic and visual aspects;
- estimation , thinking, reasoning and communication;
- use of technology;
- multiple entry and exit points.

The main principles which underpin the CRIMS project are now identified and considered:

Linking science and mathematics

A major goal of science is to discover and use relationships between variables (usually expressed mathematically) to understand and make predictions about the world.

Problem solving connects mathematics and science, inquiry is a central activity of both, and they share common tools; for example, collection of data, spreadsheets, scales, instruments, graphs, data loggers, computer-based laboratories (CBLs). Students learn that mathematical thinking, such as the identification of patterns and concepts of probability, underlies scientific investigations.

Constructivist learning theories

CRIMS are consistent with constructivist learning theories. They emphasise understanding, with students actively building new knowledge.

Backward design

The three stages of the backward design planning process are:
- identify desired result
- determine acceptable evidence
- plan learning experiences and instruction (Wiggins & McTighe, 2005, p. 18).

Desired result: Scientific literacy and mathematical literacy

The CRIMS project aims to contribute to the lifelong learning of students, in that they may learn:

> to be mathematically and scientifically literate, to help empower [them] as citizens and skilled participants in a world increasingly dominated by technological innovation. To achieve these outcomes, mathematics and science education must be engaging and contemporary. Mathematics and science education must also be relevant to student's lives, promote deep conceptual understanding and embrace investigative inquiry and problem solving. (Herbert, 2006, p. v)

Acceptable evidence

CRIMS are consistent with constructivist learning theories. The stages of the investigation process can act as guidelines for assessment. Principles of sustainable assessment (Boud, 2000) are consistent with the emphasis on lifelong learning inherent in scientific and mathematical literacies.

Learning experiences

The 2006 Statements of Learning identify essential opportunities for Australian students to learn. The CRIMS project aims to produce tasks which make connections between Science as a Way to Know and Working Mathematically, from these Statements. Science as a way to know is:

> about scientific investigation and the way in which scientific explanations are established. It includes posing questions, planning and conducting investigations, collecting and analysing evidence and communicating findings. It is concerned with evaluating investigations and claims and making valid conclusions. It also recognises that scientific explanations change as new or different evidence becomes available from investigations (MCEETYA, 2006a, p. 5).

Working mathematically involves:

mathematical inquiry and its practical and theoretical application. This includes problem posing and solving, representation and modelling, investigating, conjecturing, reasoning and proof and estimating and checking the reasonableness of results or outcomes. Key aspects of working mathematically, individually and with others, are formulation, solution, interpretation and communication (MCEETYA, 2006b, p. 3).

Inquiry and open ended investigations

CRIMS investigations are a vehicle for students to construct their own scientific and mathematical understandings, within the structure of a relevant and meaningful context. Investigations combine science and mathematics processes. Students may work at any of the three levels of inquiry: structured, guided, or open while learning inquiry skills. As learning becomes more relevant for the student, they are encouraged to take responsibility for their own learning. They choose and use concepts and skills.

Planning and conducting investigations

The contexts chosen have genuine application and relevance to students. Students start with a challenge, a dynamic opportunity to explore. They formulate questions and plan investigations. They clarify what they already know or believe and conduct an investigation which will support or challenge these beliefs. They pose scientific and mathematical questions to guide the investigation and collect qualitative and quantitative information. Students are physically involved in the process which involves visual and kinaesthetic learning. Students understand that the information they obtain can be used as a basis for extended investigations or analysis.

Collecting and processing data

Students make and test predictions, hypotheses and conjectures by designing experiments or using problem solving strategies which model the situation — incorporating creative and imaginative aspects. Observations are made. They collect data and identify patterns and groupings in the information. Science and mathematics both involve searching for patterns, and testing predictions and generalisations: these are inherent parts of scientific and mathematical processes. Investigations and testing attributes lead to the development of generalisations. Abstraction may lead to the development of a formula or a model.

Representing and recording

The concrete materials are manipulated, students represent their observations as diagrams, charts, tables and graphs. These representations are used to clarify thinking and identify patterns. Students translate and move between representations. They use them to model and interpret natural phenomena. They present information in ways that clarify patterns and assist in making generalisations and drawing conclusions. They use mathematical terms to describe relationships.

Communicating

Students use talking and writing to clarify thinking and identify what they do, and do not, understand. They progress from everyday language to scientific and mathematical language and symbols. The teacher models the use of scientific and mathematical language. Students use scientific and mathematical language to give logical accounts of their work and their findings. They report the conclusions of the investigation clearly and identify and communicate the implications of their findings.

Making connections

The context provides a background and framework for exploring science concepts. Students work to develop rich conceptual lattices of understanding. They make connections with their own lives, with the result that concepts do not remain embedded in a particular context. They are able to transfer knowledge and make new connections beyond their current understanding.

Professional standards

The CRIMS projects refer to the ASTA professional standards and AAMT professional standards.

Building on the first CRIMS Project

CRIMS2 PL: An ASISTM Round 3 project

The CRIMS2 PL project focuses on teacher professional learning. Research indicates a relationship between student achievement and the quality of teachers. "This research indicates that the effects of well-prepared teachers on student achievement can be stronger than the influences of student background factors, such as poverty, language background, and minority status" (Darling-Hammond, 2000).

The CRIMS2 PL project involves teachers from 11 primary and secondary schools. Teachers involved in the project meet to reflect on their current practice and engage in professional dialogue. They share resources and information about their teaching and the experiences of other teachers helps to inform their planning. The CRIMS PL project provides opportunities for teachers to share their practice and expertise to "enhance student achievement, teachers need to be supported with sustained and classroom-oriented professional development" (Goodrum, Hackling & Rennie, p. 174).

CRIMSRR: A SiMERR project

The National Centre of Science, Information and Communication Technology, and Mathematics Education for Rural and Regional Australia (SiMERR):

> works with rural and regional communities to achieve improved educational outcomes for all students in the areas of Science, ICT and Mathematics, so that:
> Parents can send their children to rural or regional schools knowing they will experience equal opportunities for a quality education;
> • Students can attend rural or regional schools realising their academic potential in Science, ICT and Mathematics; and
> • Teachers can work in rural or regional schools and be professionally connected and supported. (SiMERR), 2005).

The SiMERR National Survey reported inequities in access to professional development opportunities: "Primary teachers outside Metropolitan Areas indicted a substantially greater unmet need for in-services in science and mathematics than did their metropolitan counterparts. Science Teachers in Provincial and Remote areas indicated a significantly higher unmet need for a broad range of professional development opportunities than did those in Provincial Cities or Metropolitan Areas" (Lyons, Cooksey, Panizzon, Parnell & Pegg, 2006, p. vii).

The CRIMSRR project involves teachers from four rural primary and secondary schools, and focuses on ways of meeting the professional learning needs of teachers in rural schools.

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Mathematics through movement:
An investigation of the links between kinaesthetic and conceptual learning[*]

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Developing strategies for engaging mathematics activities is always a challenge. Teachers seek out new resources and online activities to excite students and support their learning. Mathematics through Movement offers an active learning strategy requiring few resources, and a bit of imagination, to achieve a variety of outcomes across mathematics domains. It is based on sound educational theory and a life time of experience in dance. This paper examines the beginnings of research into this teaching strategy in a remote setting in Western Australia. It shows that this teaching tool can motivate talk, deepen understandings, and engage students in mathematics tasks.

Background

As a dancer, teacher of dance and mother, I had often reflected upon the significant role movement had in my life and that of my daughters. Dancers, dance teachers and parents had often commented that dance, callisthenics or movement of some sort, had supported students in their school work. As a teacher I pondered on Gardner's *kinesthetic learner* and I am drawn to his statement:

> Indeed participation in the arts is so natural and integral a part of human growth that an understanding of this process should provide important clues to many pivotal questions of human development. (Gardner, 1973, p. 23)

Observation of my classroom in northern Western Australia, in a small rural town, showed that students were more engaged with ideas and learning if there was an element of movement involved. Many seemed to be kinaesthetic learners. This prompted the use of interactive technology but also a deeper investigation of how I could use my understanding of movement to enhance some core mathematical concepts. The links seemed obvious in my mind but could I show a direct educational benefit for integration of dance with specific mathematics foci?

Informal action research was undertaken in my classroom and anecdotal notes, video recordings and reflective journal entries gathered. The data were showing that some core mathematical concepts could be clarified and new ideas scaffolded using movement, especially in the Shape domain. This raised questions regarding how effective dance might be in supporting learning in the other mathematics domains, (Number, Measurement, Chance and Data). A program of learning was devised with

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[*] Paper accepted by peer review

dance as the core strategy that addressed all mathematics domains and their links to movement (see Figure 1)

Mathematics through Movement T2-2006				
Outcomes	**Shape**	**Measurement**	**Number**	**Chance and Data**
Level 1 M11.1 **The student:** Makes non-numerical estimates of size involving everyday activities and actions. C&D 12.1 **The student:** Shows some recognition of the element of chance in familiar daily activities and uses and responds to the familiar everyday language of chance. S 15c.1 *Represent transformations* **The student:** Repeats, reorients and turns over things when matching shapes and making pictures and patterns. S16.1 **The student:** Talks about likenesses and differences between things that can be seen or handled and begins to connect shape, movement and function.	Activity – Shape Bags/Body bags Exploring shape and movement.	Activity – Space and Movement Estimating how much space the body or movements take up. Activity – Size when staging a dance.	Activity – Counting the beat. Activity – Rhythm and beat – looking at fractions when dancing.	Activity – Probability. Likelihood of achieving the same things in movement – skills and ability and chance.

Figure 1. Initial planning for Mathematics through Movement

This paper will discuss the findings of that informal research, the testing of the initial outcomes in a new classroom with students with different learning styles and socio-economic backgrounds, and further developments that were derived from the program outlined in Figure 1.

Findings

The following data were derived from observing students in a K–3 cohort of predominately indigenous Australian decent, around 25–27 students at any time, in a small remote township in Western Australia. During 2006, mathematics concepts were integrated into the classroom program using the following outlined in Figure 2.

Theme	Strategy	Purpose	DET Outcomes
Aliens	Elastics Square Dance Stretch Bags	To explore characteristics of 2D shapes. Understanding of a square and fractions of a square Investigate 3D shape Counting Doubling	S 15b.2 *Represent shape* **The student:** Meets simple criteria relating to shape or structure when making and drawing things, creating recognisable copies of arrangements of shapes. N 6b.2 *Understand fractions* **The student:** Understands the meaning of 'half' and 'quarter', splitting quantities into 'equal' shares and partitioning quantities repeatedly into halves. **Number** Students use numbers and operations and the relationships between them efficiently and flexibly.
Motorbikes	March formations Stagecraft	Spatial awareness, conforming to shape Directionality, Language	S 15a.3 …uses order, proximity and directional language associated with quarter and half turns on maps and in descriptions of locations and paths. S 15a.1 *Represent location* **The student:** Uses and interprets familiar, everyday language for the position of things, their movements and paths between them.

Figure 2. Further planning for Mathematics and Movement

The results from using the above strategies will be discussed using the categories, Engagement, Deeper Understandings, Contextual Understandings and Fostering Talk.

Engagement

A primary result noted was the greater engagement of students with a learning task. The movement gave the tasks an element of fun and the students' enjoyment was very obvious in photographs and video tapes of the various activities. The nature of the tasks also made the students see that they could achieve the aims of the activities and therefore they were more easily engaged. For example, the students knew in the Square Dance that they could learn a series of simple steps to music and that the task was complete at the end of the music. They easily understood the directions to move in and the shapes they were creating when moving. This made the mathematics less confronting and more purposeful and the students responded accordingly. Since the students were highly engaged in the activities, the potential to learn was greatly heightened. This was shown in later discussions whereby children responded enthusiastically to questions about the task. The engagement of the students facilitated the next result, fostering talk.

Fostering talk

Talk is a very important aspect of the learning environment (Wood, 2003). Discussion, after movement activities, was enthusiastic and led to in depth talk about the core concepts. This was particularly so with the Stretch Bag strategy. The students were very

interested in the movements and the shapes created and it was a favourite "free time" activity for the students after it was introduced into the program. This activity led to discussions regarding 2D and 3D shapes, including the characteristics of various regular and irregular polygons and how to transform shape. The use of this movement activity allowed for the demonstration all 3 aspects of Bruner's levels of representation, Enactive, Iconic, and Symbolic (Frid, 2001). Bruner contends that by using these modes of knowledge the student is scaffolded to new understandings since the learner is engaged in all aspects of the cognitive system. In this case, the children were actively involved with the concept of shape, making the shapes physically themselves (enactive mode). There were many opportunities to view shape both in a 3D contextual form and in visual representations (iconic) and, finally, due to the engaging discussion there were very powerful language based representations (symbolic).

Contextual understandings

During the Square Dance activity the students were able to use counting and number sense in a specific, purposeful context. The activity was designed to highlight the concept of doubling numbers, in this case doubling 8. Counting was emphasised whilst dancing and later discussion centred on how many counts it took to complete sections of the dance. The correlation and comparison was then drawn to doubles. On this occasion, the understanding of the new concept was not immediate, however there was a very useful reference point created when the concept was discussed in later more explicit teaching regarding the idea of doubling. By placing the concept in a context, I was able to scaffold the students' learning towards new ideas.

A very useful investigation that evolved out of the square dance was the directionality of the dance. The students travelled around the square but also did so diagonally across the square. By representing this diagrammatically the students were led, very easily, into the concept of fractions. A natural progression to cutting half sandwiches and quarters in fruit followed and the students were engaged with, and easily stepped towards understanding, a new concept.

Deeper understandings

By using the above strategies, I was able to give purpose to the students' learning. They did not learn a square had 4 sides by saying it over and over a gain. They knew because they moved around a square, they created one using the elastic (not as easy as it seems), and had to really think about a square and its attributes when doing so. I found this led to deeper understandings and longer term retention of information. By learning through doing, the students gained a thorough grasp of the target understandings.

Summary

This research has shown me that there is an important link between constructivist learning theory, the arts and human development, and learning by doing. My investigation is continuing and I find new and exciting evidence for Mathematics through Movement, with every session I conduct. Current work is showing that by allowing students to become the artistic critics (Gardner, 1973, p. 26) — that is, cultivating their ability to discern features and distinctions — they are able to communicate their understandings very effectively, thereby allowing a more accurate assessments of understandings to be made. The data above were a starting point for my

research and, importantly, have led to reviewing literature, a greater attention to theoretical preface for chosen strategies, and more reliable data for future discussions.

There is, however, enough to suggest a few general applications for classroom practitioners. I should add, at this point, that a teacher need not have a huge knowledge of dance and complicated dance routines to implement a program such as that which I have described. I have used very simple and accessible dances, such as the square dance, to good effect. The challenge is to think laterally and find the place where movement fits in. For example, a new project in my classroom is "Happy Wheat" where the students are learning a simple tap dance routine to insert in their story about wheat in their home town of the wheat belt of WA. There are many forms of dance that can be investigated such as Aussie Bush dances, Social dances such as the "Nutbush" disco line dance, and creative movement with such things as the stretch bag. Finding links to your context will make the strategy more effective and engaging.

Mathematics through Movement offers an interesting and engaging strategy to achieve the vast outcomes that make up Being Numerate. By integrating a well thought out movement strategy, a teacher can achieve many outcomes at once. Mathematics through Movement provides rich tasks that, although seemingly closed in format, offer vast flexibility via discussion as to where an investigation may lead. This important reflective aspect of the process attends to the notion that "it is not sufficient simply to have an experience to learn" (Gibbs, 1988, p. 9).

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