

*Mathematics
shaping Australia*

**Proceedings of the Eighteenth Biennial Conference of
The Australian Association of Mathematics Teachers Inc.**

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Mathematics: Shaping Australia

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PAPERS

No More (Red-Pen) Marking!

Tony Allan

Can computers be used to test students of mathematics and mark their answers?

Issues include:

- What hardware is available? What software?
- How secure is the software/hardware configuration?
- Which kinds of questions can be tested by a computer? Which cannot?
- What level of difficulty can be achieved in computer testing?
- What about part marks for working out?
- What can computers do that mere mortals cannot? (Like giving each student different numbers in the questions).

This paper describes one product which addresses some of these issues.

No more marking (with a red pen) sounds like one of those over-blown claims for the computer age that pop up periodically in the techno sections of newspapers and from the minds of populist television pundits. The holy grail of teachers! It reminds me of the confident claim made by the IT outfit I worked for in the 1980s in England. We were at the forefront of Image Processing. The software and hardware engineers were developing circuit boards, ROM chips and programs to implement the first tools for manipulating images — on some of the very first IBM-compatible PCs that were then emerging. We demonstrated a computer with a scanner card, an Image Processing card, a high-resolution video card, an optical disk reader and a printer card. A hand-written letter was scanned in; it appeared on the screen; we moved the word 'not' to a different part of the letter and printed the 'fake'. The machine got too warm to demonstrate for more than a few minutes, but on this premise we trumpeted the first steps towards the paperless office. We also dabbled in optical character recognition; I do not think voice-recognition was on the horizon then. Shortly, all forms, correspondence etc. would be largely redundant in the office.

That was nearly twenty years ago. More trees are being turned into paper than ever before. Word processors, e-mail, CDs and so on have not significantly reduced the paper mountain; they have only increased the amount of correspondence — and its triviality.

So it is in this context that I now knowingly trumpet 'No more (red-pen) marking'. I confidently — foolishly? — foresee the day when a class of mathematics students will be assessed on what that have been studying *by sitting at a work-station* to take your test. Furthermore the computer will mark their answers and do everything else with the class marks that you currently do with a calculator and spreadsheet.

This scenario begs a whole list of questions, and the more you think about it, inevitably, the longer the list becomes. This is the same psychological exercise the paper-less office pundits have been experiencing now for decades. Nevertheless I invite you to imagine this dream scenario — there you are, supervising students beavering away at computer terminals, knowing that before the bell goes your students will have in their hands a print-out of their result at each question, their percentage mark and their rank in the class. How is this to be achieved? What are the pre-requisites for this dream to come true? What are the issues? What are the limitations? What would you say if I said the dream is realisable now?

Teachers in most schools will straightaway say they do not have suitable hardware, or they have it but don't have sufficient access to it. The computer labs are fully booked out by classes of kids trawling the Internet for stuff to print out for inclusion in an assignment. Computing facilities are clearly a necessary prerequisite of what is being suggested here. Well, if it is true that the tail can wag the dog, then if and when you can acquire suitable software, it will give impetus to your push for better access to appropriate hardware.

So where is the software? Given the largely mathematical basis of all computer processes it is surprising how precious little software there is that's any use in maths education. We all know how to use *Equation Editor*, of course — and its big sister *MathType* — which is great for mathematical word processing. It has greatly enhanced our productivity and sophistication when writing tests for printing on paper. Then there are a number of tools for doing statistics and other calculations, many of which simply couldn't be done without computers.

A number of packages have been around a while for the home market. You know the kind of thing. A jigsaw sized box containing a CD and a user manual. There are 'hundreds of questions on math for all ages'; not, sadly, suitable for assessing our students' progress.

There have been for some time packages with banks of questions, from which you make a selection in constructing a paper test. These have the answers, which saves a little time. I have used one package that printed multiple choice answer grids.

On the Internet you can find pages which use DHTML to give instant (well, line-speed permitting) checking of your answers to mathematical questions. This idea is great for self-testing, and in general the home market will be what drives most development in this area.

So, what about a package for teachers? What are your functional requirements for a maths-testing product? What do you pride yourself in when constructing a paper-based test? In general terms:

- quality — at least of same standard as current paper based tests
- security — equivalent to 'don't leave your marks book lying around'
- variety — different questions each year?
- rigour — precision
- imagination — your individual touch

- complexity — more than simple arithmetic
- appearance — matching your current products

Arithmetika 1.1

Arithmetika 1.1 is a multi-user program. This means that any registered user on a network can log in to it. Teachers who log in are given the tools to construct tests from a bank of questions, and then either print them out for reproduction (and red-pen marking) in the normal way, or assign them to students. Students who log in are given a list of assignments; they submit their answers and the computer collates the results for the teacher.

A test is in essence just a list of questions, whether it be for paper printing (TMA) or for assigning to users (CMA). There are many issues to decide upon if printing a test, such as one column or two, and there are many issues to decide upon if assigning to users, such as security. Leaving those to later, let us concentrate on question construction.

Arithmetika holds a bank of question templates, such as ‘add a and b’ (do not worry, they get more sophisticated!). If you wish to include a question based on this template, you find it using the flexible file retrieval system and add it to your test. ‘a’ and ‘b’ are placeholders for variable quantities. When you select a question template you are shown an example of what it will look like when the placeholders are replaced by values. You can remove a question later, or change the order of questions. There is no limit to the number of questions. See Figure 1.

The screenshot shows the Arithmetika 1.1 interface. At the top, there are four options for organizing questions: 'Strand, Difficulty, Topic', 'Strand, Topic, Difficulty', 'Difficulty, Strand, Topic', and 'Difficulty, Topic, Strand'. Below this is a tree view titled 'Select one of the following Questions'. The tree is organized as follows:

- Addition
- Angle
- Area
- Capacity
- Circumference
 - Basic
 - Measurement
 - [916] Find the circumference of a circle with radius x (metric)
 - [917] Find the circumference of a circle with diameter x (metric)
 - [923] Find the circumference of a circle with radius x (imperial)
 - Standard
- Compound Interest
- Currency

At the bottom of the interface, a list of five selected questions is shown:

1	[916]	Find the circumference of a circle with radius x (metric)
2	[917]	Find the circumference of a circle with diameter x (metric)
3	[918]	Find the area of a circle given the radius x (metric)
4	[923]	Find the circumference of a circle with radius x (imperial)
5	[925]	Find the area of a circle given the radius x (imperial)

Figure 1

It's not just the numbers going into placeholders that are randomly generated. Take a template such as 'Find the base length of a parallelogram with perp. height x units and area y units _ (metric)'. The units may be any one of mm, cm, m and km.

Some templates have pictures associated with them. There is a template 'Find the perimeter of a sector of a circle with radius r (metric) and angle θ ($90^\circ < \theta < 180^\circ$)'. In addition to variable r and a variable θ , the drawing of a sector comes from a bank of suitable pictures. See Figure 2.

1	[916]	Find the circumference of a circle with radius x (metric)
2	[917]	Find the circumference of a circle with diameter x (metric)
3	[918]	Find the area of a circle given the radius x (metric)
4	[923]	Find the circumference of a circle with radius x (imperial)

Figure 2

You then set a few parameters. For example, what range of numbers do you want? 'Add 4 and 5'? 'Add 17 and 91'? 'Add -4.3 and $+101.67$ '? And so on. How accurate do you want your students' answers to be? Nearest whole number? One decimal place? But note that there are some forced restrictions on variables. For example, the magnitude of θ is restricted to the range given regardless of the parameters you chose above.

What happens next depends on whether you are developing a TMA or CMA. Arithmetika allows considerable customisation of the presentation of paper-based tests, but we concentrate on issues around the concept of computer marking. But remember that a test can be printed OR assigned (OR both — see later, for why!) so you could use Arithmetika now for TMAs and later, when you get the hardware, for CMAs).

You will want to preview the test before inflicting it on your students. A typical page (this one shows the last question) (with its solution) might look like Figure 3:

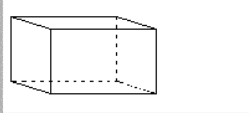
Arithmetika 2000		13/11/00 15:45:43	Test prepared for RedBackSpider - 10 questions.
Error allowed in answers ± 0		Arithmetika test: Everything on Perimeters	
Answers must be rounded to 2 decimal places if applicable			
Fraction answers may be common, mixed			
Q No.	Questions	Graphic	Answers
10	Find the T.S.A. of a rectangular prism given length = 44.54 mm, width = 2.64 mm & height = 74.15 mm		<div style="display: flex; align-items: center;"> <input style="width: 40px; text-align: center;" type="text" value="2"/> <div style="margin-left: 5px;"> ▲ ▼ </div> </div> <div style="border: 1px solid black; padding: 2px; width: 100px; text-align: right;">7231.97 mm²</div>

Figure 3

At this stage you can alter the relative value of this question in the test, by changing the 'Out of' box.

When you are ready you assign the test to a class your screen looks like Figure 4:

Class 4B Ms J
Anitopolis Grade 7 Arithmetic Group
up 10Mat8 Yr 11 Mathematics in Practice Year 9
Level 1 of Whitta

Select one or more of the Classes in the upper list, then de-select from the lower list any Students you do NOT want to be assigned the Test.

- 11MATA
- Staff

- JBrewster
- LLin
- SBottomly
- SAgnew
- CJohnson
- GEct

Figure 4

When a student takes this test the screen is similar to the one you saw when checking. See Figure 5:

Arithmetika 2000 13/11/00 15:57:06 Test prepared for tallan - 10 questions.
 Error allowed in answers ± 0 Arithmetika test: Everything on Perimeters
 Answers must be rounded to 2 decimal places if applicable
 Fraction answers may be common, mixed

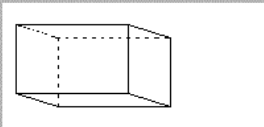
Q No.	Questions	Graphic	Answers	Units	Mark	Out of	Solutions
10	Find the T.S.A. of a rectangular prism given length = 3.21 mm, width = 60.31 mm & height = 32.81 mm		<input type="text"/>	<input type="text" value="mm"/>	<input type="text" value="2"/>		

Figure 5

Note that there are boxes for entering the answer and for selecting units of measurement.

When the student submits the test for marking the screen will look like Figure 6:

Arithmetika 2000 13/11/00 15:57:06 Test prepared for tallan - 10 questions.
 Error allowed in answers ± 0 Arithmetika test: Everything on Perimeters
 Answers must be rounded to 2 decimal places if applicable
 Fraction answers may be common, mixed

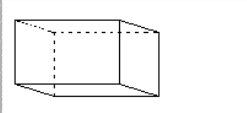
Q No.	Questions	Graphic	Answers	Units	Mark	Out of	Solutions
10	Find the T.S.A. of a rectangular prism given length = 3.21 mm, width = 60.31 mm & height = 32.81 mm		<input type="text" value="4555.37"/>	<input type="text" value="mm<sup>2</sup>"/>	<input type="text" value="2"/>	<input type="text" value="2"/>	<input type="text" value="4555.37 mm<sup>2</sup>"/>

Figure 6

The final illustration (Figure 7) shows the collated results of a class with some results still to come in:

Monday November 13 2000 at 3:30 PM
Addition test

No.	Questions	Date and Time submitted	Result (%)
1	Agnew, Spiro	Not yet submitted	
2	Bottomly, Sally	13/11/00 18:18:41	60
3	Brewster, Jennifer	Not yet submitted	
4	Ectoplasm, Ghosta	Not yet submitted	
5	Johnson, Carol	13/11/00 18:19:25	90
6	Lin, Lynnette	13/11/00 18:17:53	100

Figure 7

Questions

This brief tour of part of Arithmetika prompts many questions, such as:

- How do you control this process?
- What about part marks?
- What about questions that don't have a numerical solution?
- How difficult do the questions get?
- Can students copy from each other?
- Can students look at a test, go away and study, then take the test later?
- Can students write their own tests?
- Can a student log on as another student?
- Can a student give the answers to another student who will take it later?
- What about questions not in the computer?
- Can the students take the test any time?

If you would like to know the answers to these questions, or your own questions, then try out the product. It is on the AAMT 2001 conference CD (under the directory Arithmetika) or can be downloaded from www.redbackspider.com.

About the author

Tony Allan cooks, gardens and plays golf and bridge (anyone for cards this week?). In his free time he teaches mathematics at Daramalan College in Canberra and writes education software at home. He specialises in making maths for the less able more interesting, and making teaching less onerous. He has published two interesting text books, *Flying Start* and *Flying High*. He is the author of *OutcomesMarks*, software that makes marking easier in WA schools. His current software project, *Arithmetika*, is designed to make both writing tests and marking less onerous.

Learning about Learning in Mathematics

Anna Austin

This paper provides an overview of action research undertaken in mathematics at junior secondary level. The research forms part of a wider Wesley College initiative that focuses on learning. Herrmann's whole brain theories of learning are examined and applied with a view to creating a mathematics classroom environment where 'integral learning' (Aitkin, 1998) takes place. Integral learning involves making connections to the work by integrating experience, feelings, imagination, information and action into lessons to help students relate to the work more closely. The study is informed by students' perceptions and feelings about mathematics, obtained through interviews, journal writing and classroom observation.

Introduction

Mathematics teachers are always working to discover ways of developing higher levels of student understanding and enjoyment of the subject. Despite this, students do and will continue to feel differently about learning mathematics. In trying to understand why these feelings would exist it is worthwhile considering some of the ways in which their personal learning styles might differ. Greater awareness of these differences can help us to understand what we see happening in our classrooms. This knowledge can assist us to develop strategies that cater more effectively for a wider range of learners in mathematics.

In 1999, Wesley College embarked on a strategic initiative to assist staff and students to discover more about the way humans learn. Two full time Learning Specialists have since been appointed to the College to work across the three metropolitan campuses. A role of the Learning Specialist is to assist students and teachers to reflect on their current practice in light of the recent research into human learning and the theories on the ways in which the mind works. The Learning Specialists offer support and assistance to staff, observe students, conduct interviews, share insights and facilitate sessions on learning.

Initially one year 7 class from each campus was selected to become involved in the project. The teachers of these classes participated in professional development to find out more about their own personal learning preferences. Each teacher undertook the Herrmann Brain Dominance Instrument and was provided with a profile. The profile provides an overview of the teacher's preferred thinking styles and an opportunity for reflection on the ways in which he or she operates. Teachers were encouraged to consider their ways of thinking, how they approached issues and events, the questions they asked themselves, the types of things they liked and disliked, their personal qualities, teaching styles and the ways in which their thinking preferences might influence what happens in the classroom.

Herrmann's theories on whole brain learning

The Herrmann Brain Dominance Profile is organised into four quadrants shown in Figure 1: A-Upper Left, B-Lower Left, C-Lower Right, D-Upper Right. The upper half of the circle indicates the person's preference for using the cerebral mode and the lower half, the preference for the limbic mode. The left and right modes refer to preferences for using the left and right brains respectively. The polygon imposed on the circle in Figure 1 gives the brain dominance profile. The degree to which the polygon extends into each quadrant provides an indication of the person's preference for using each mode. Some typical characteristics of each quadrant are given in Figure 2. The profile illustrated in Figure 1 shows a preference for cerebral thinking with a slight right brain preference over the left.

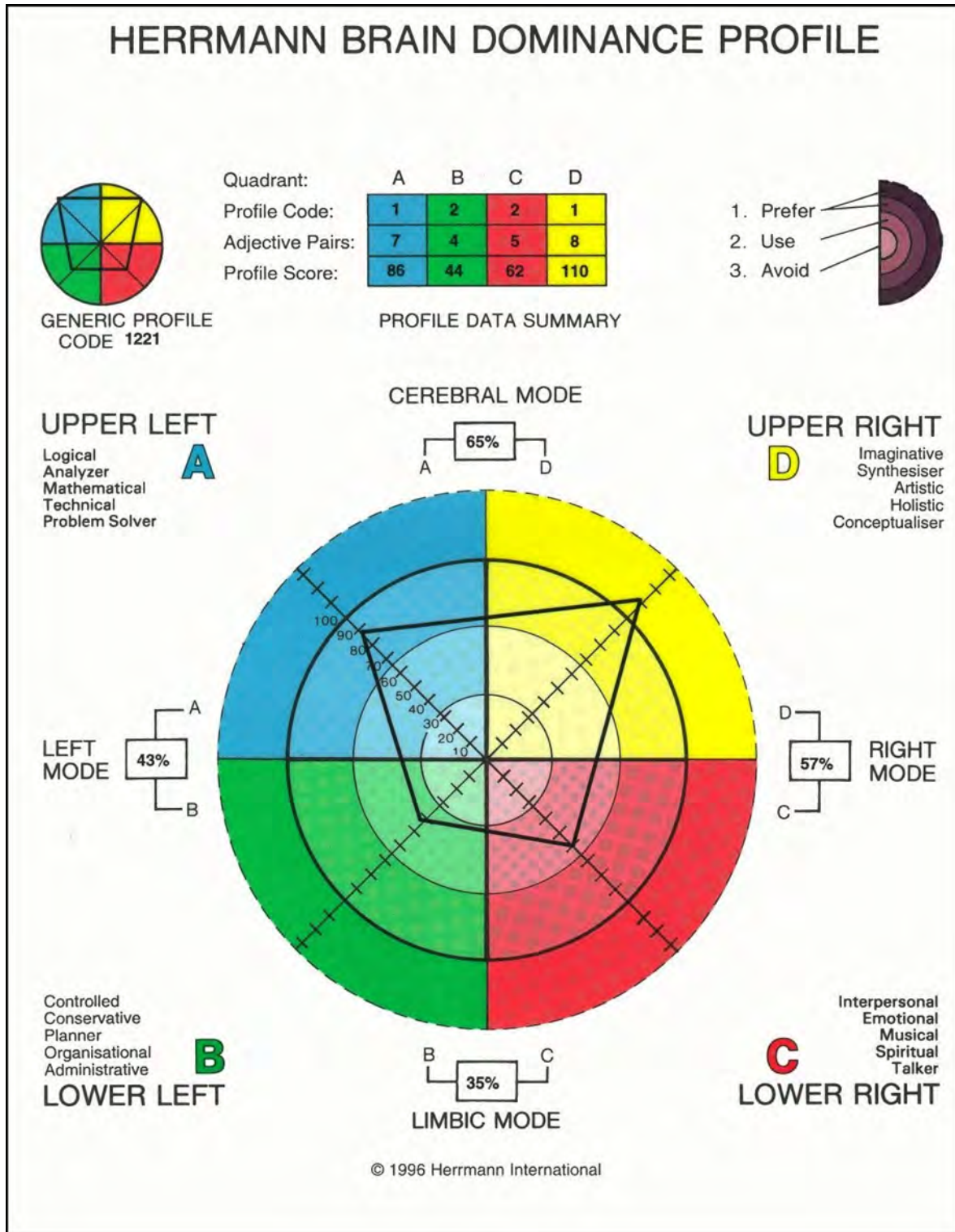


Figure 1: Herrmann Brain Dominance Profile.

What types of things would the person with the profile shown in Figure 1 enjoy? What subjects would he or she prefer at school? Would this person be one of those people who liked or disliked mathematics?

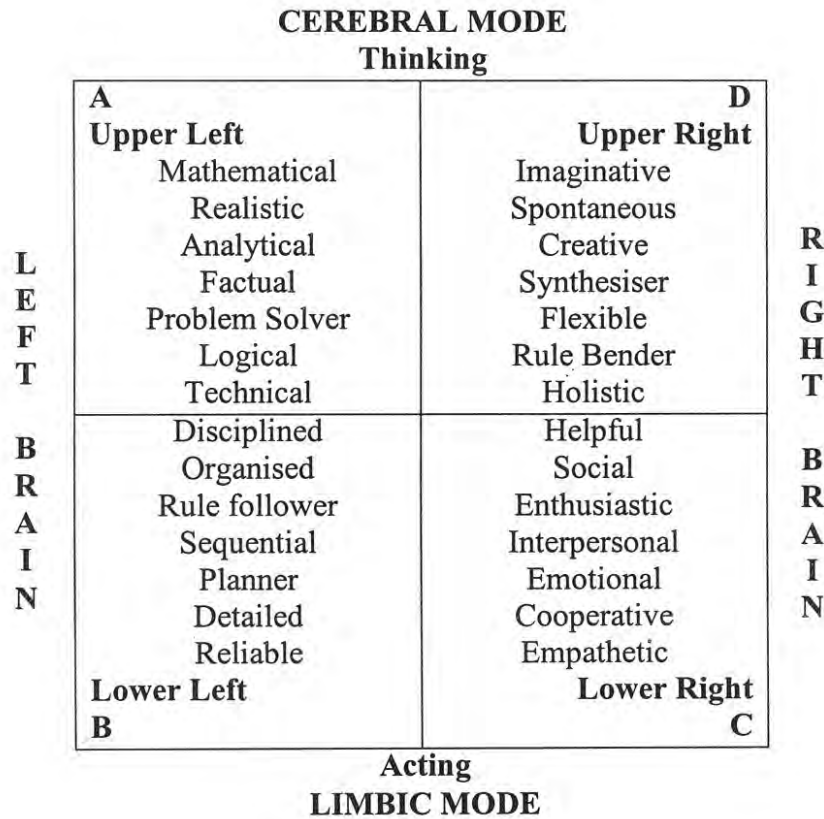


Figure 2: Typical characteristics of the quadrants of Herrmann’s model.

If we consider the characteristics of each quadrant given in Figure 2, we realise it would be likely that the person would enjoy mathematics. The profile shows a fairly strong preference for quadrant A thinking, which means that activities involving analysing, problem-solving, logical thinking and testing ideas would be favoured. It also shows an even stronger preference for quadrant D thinking, which means the person would be inclined to think conceptually, synthesise ideas, take risks, work on open-ended activities, be artistic and bend the rules. It would be likely, then, that the person would be able to see the bigger picture and recognise how mathematical ideas and principles fit together. He or she would probably not see mathematics as a set of disconnected rules and procedures. When solving problems, the person would probably be able to draw together the essential elements and find creative solutions. Because there is a low preference for quadrant B thinking, repetitive activities are unlikely to be favoured, and the person may not be so well organised. This does not mean that he or she is unable to be organised, it means that this is not a preference. A lower preference for quadrant C implies there is less interest in emotional, interpersonal and spiritual involvement.

Some of the functions of the left and right sides of the brain are shown in Figure 3. A student who enjoys mathematics would usually have well developed left brain capacities. This side of the brain controls number, symbols, logic and the ability to analyse. Use of the left-brain may not be the preferred mode of thinking for that person though. He or she may be good at mathematics, but have a stronger

preference for using the right side of the brain as the profile shown in Figure 1 suggests.

Left Brain	Right Brain
<i>Busy</i>	<i>Creative</i>
<i>Logical</i>	<i>Intuitive, Imaginative</i>
<i>Orderly commonsense</i>	<i>Relaxed</i>
<i>Systematic, sequential</i>	<i>Spontaneous inspiration</i>
<i>Slow, detailed processing</i>	<i>Rapid scanning and processing</i>
<i>Deals with one thing at a time</i>	<i>Deals with many things at once</i>
<i>Sees small details</i>	<i>Sees in whole pictures</i>
<i>Controls language, number and symbols</i>	<i>Controls recognition, aesthetics, memory</i>
<i>Expressive</i>	<i>Receptive</i>
<i>Analyses</i>	<i>Synthesises</i>
<i>Mechanical</i>	<i>Artistic</i>
<i>Speaking</i>	<i>Musical</i>
<i>Writing</i>	<i>Spatial</i>

Figure 3: Functions of the left and right sides of the brain.

Sensory preferences for learning: visual, auditory and kinaesthetic modes

The ways in which we use our senses also influences how we learn. Kinaesthetic learners learn best through movement, touch and practical work. They frequently fidget and fiddle in the classroom and can often find difficulty staying in their seat. These students need to move around to learn, and if they do not have this opportunity their capacity for learning reduces. Auditory learners learn by listening to instructions and information. They are very aware of sound and tone and will often speak aloud when thinking. Visual learners learn by seeing, they remember images and pictures. In every classroom there will always be students with different preferences for learning. It is very important for teachers to be aware of these preferences and make provision in their curriculum planning for kinaesthetic learners, as well as those with visual and auditory preferences.

The learning program

To assist students to develop a better understanding of the ways in which they learn, a program on learning was built into the normal classroom schedule, delivered as part of the pastoral care program. The process was developmental and as the year progressed, specific learning and thinking techniques were introduced. Students were given activities to work on and they were asked to explore their feelings and experiences as they worked through these tasks. We found that by having students work in groups discussing their feelings, thoughts and ideas, and by answering

process-related questions, they became more aware of the ways in which they were learning.

In one of the earlier sessions, students were asked to explore their perceptions of learning by drawing a picture and writing about the experience. Since the issues were introduced in a general, non-subject-specific way, it was surprising to find mathematical calculations and formulae featured in many of the drawings. It was clear that many students already had well-established views about learning, and particularly about learning mathematics in a school. A number of drawings showed mathematical symbols and formulae being tipped into or protruding from the head. In other drawings, students were sitting in rows working independently. If there was a blackboard, it frequently showed mathematical calculations. Few students drew or described the holistic type of learning that occurs naturally, and when they did, it was not related to mathematics.

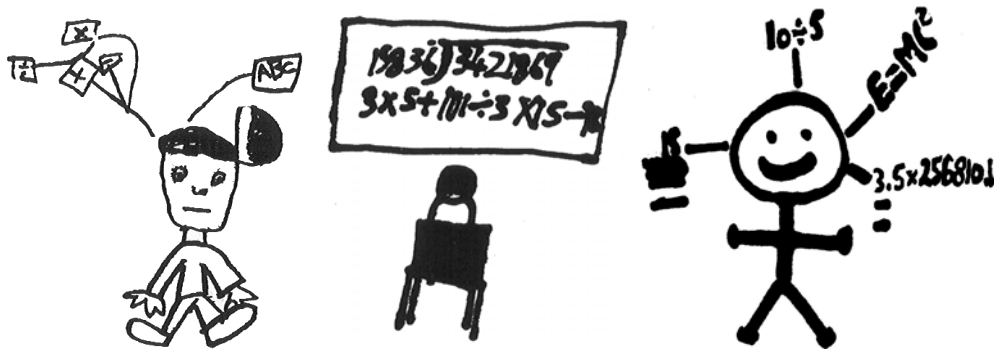


Figure 4: Year 7 students' drawings on their perceptions of learning.

From a mathematics teacher's perspective these drawings are concerning. What are the students saying about mathematics and the way they are being taught? Why are they shown isolated from one another, working on tedious calculations? Does rote learning and repetitive calculation characterise their mathematical experiences?

In another lesson, students were asked to recall something that they had learned recently and the process they had gone through to learn it. This produced a very different response. Most students described something that they had learned outside the school environment. Frequently they described a practical situation where they had made a significant personal achievement. Their personal involvement seemed to add value to the experience. Students were easily able to recall what was learned and the process that they had gone through to learn it. There were very few students, only one or two in each group, who referred to things they had learned at school. (McGraw, 1999)

Student perceptions of learning mathematics

It became apparent that there was a wider need within the College to look at the ways students were learning mathematics. Information gained from interviews with students from various levels revealed that they perceived:

1. few opportunities to express themselves creatively in mathematics;
2. few opportunities to work with others in groups in mathematics;
3. few opportunities to be physically involved in mathematics;
4. difficulty relating mathematics to real life and their personal situation;
5. methods of assessment to be too narrow in mathematics;
6. mathematical language as foreign;
7. difficulty understanding the purpose of mathematics and how it will be useful later in life;
8. gaps between those who are intuitively good at mathematics and those who struggle;
9. limited awareness of strategies to help them break through the period of struggle when learning something new (McGraw, 1999).

The following additional information was gleaned from a series of interviews with students in year 7. Students were asked open-ended questions and their responses have been interpreted in nine areas. They provide some telling insights into students' feelings and perceptions about mathematics.

Student perceptions of mathematics and the influence of community values

- Students have strong ideas about mathematics and its role in life outside school. They believe it is either central to everyday living or else irrelevant.
- Students feel that the community, particularly their parents, highly value mathematics. Some students reiterate clichés about mathematics being important, but they don't appear to really understand, at a personal level, why it is important.

Impact of individual learning preferences

- Some students feel that they are naturally good at mathematics, whilst others feel that they are not. Different learning and thinking styles need to be catered for and there are fewer opportunities for this to occur in mathematics than in other subjects.

Impact of previous experience in mathematics

- Previous learning environments have an impact on students' current confidence and skill level. By the time they enter secondary school students

already have very clear views of what mathematics is about and the way it should be taught. Something quite dramatic is needed to turn an attitude around.

Teacher centred nature of mathematics learning

- There is a strong sense that what happens in mathematics is designed and controlled by the teacher.
- Confident learners of mathematics talk about the subject in a more constructivism way. Other students believe the teacher has a central role, and that it is up to the teacher to explain the different methods and show them how these should be applied. This appears to lead to students blaming teachers if they cannot understand the work.

Effects of self-esteem on depth of learning

- Those students who are confident and capable mathematics learners have more awareness of the strategies they use to solve problems. They can articulate what they do and why more easily, more comprehensively and more fluently.
- The students who believed they were less able in mathematics had longer moments of silence in the interviews and were less detailed in their responses. They hesitated more, were less able to reflect on their experiences or see connections between their experiences in mathematics.

Collaboration

Many students find that collaboration really assists learning in mathematics, particularly when they work in pairs and are on much the same level.

Textbook exercises and problem-solving

Some students prefer to learn via a textbook rather than through problem solving because they don't need to think as much. All they have to do is repeat what the teacher has done on the board.

Variety and choice

Students respond well to a variety of teaching and learning strategies and feel that choice is important in mathematics.

Role of homework

Some students do not have a clear understanding of the role of homework and feel that certain types of homework are more important than others.

(McGraw, 1999)

Clearly there is a need to improve students' self-confidence, their level of engagement and understanding of how mathematics can be applied in real situations. We want to stimulate a sense of natural inquiry and develop deeper mathematical understandings. But how can we do this? How can we facilitate more creative and connected learning experiences? And how can we equip students with strategies to better manage the frustration that often occurs when learning mathematics? Can the learning theory help us to deal with these issues more effectively? Does Herrmann's Whole Brain Model offer mathematics teachers new insights into the complexities of learning?

Applying Herrmann's whole brain theory in mathematics

There will always be a variety of ways that students will begin to understand a mathematical idea. Some will start with the facts and try to make sense of these (Quadrant A thinking), some will try to define or categorise what they are doing and attempt to apply methods or routines that they already know (Quadrant B thinking), others might give the situation personal meaning and build their understanding from this perspective (Quadrant C thinking), yet others may visualise the situation and perhaps draw a picture to start themselves thinking (Quadrant D thinking). Left to themselves, individuals will attempt to use the thinking style that they feel most comfortable with, regardless of whether this is most appropriate for the task. When there are options a preference pattern tends to develop. A low preference in a particular area should not be an excuse for not using this mode. The reverse is true. The most productive form of learning occurs when the whole brain is used and students will benefit by developing their less favoured preferences. They can become more proficient in these ways of thinking by collaborating with others who are strong in these areas. In a whole brain group each thinking style is represented. When students are grouped in this way, they are able to access and develop their less preferred modes of operation, which can create wider opportunities for learning.

In order for the year 7 students to recognise their preferred thinking and sensory styles, they played the Diversity Game (Aitkin, 1998). Whole brain groupings were formed on the basis of the information the students found out about themselves playing this game. Groups comprising three to four students were balanced as far as possible for personality, thinking preferences and gender. Some qualities students recognised were:

Neville	Mathematical, Technical, Knowledgeable — Quadrant A preference
Emily	Punctual, Organised, Planner — Quadrant B preference
John	Social, Organised, Collector — Quadrant C/B preferences
Sarah	Playful, Rule Bender, Helpful — Quadrant D/C preferences

Although the names given here are pseudonyms, these students formed a whole brain group in the class. Through the use of Herrmann's model, it was much easier to relate to the way students were learning. Their actions and behaviour could be more readily understood in light of Herrmann's theories.

Whole brain groupings were found to be most effective in mathematics when used for problem solving, investigations and revision. Grouping students so that their personal qualities complemented each other reduced the pressure on the teacher, as it is far easier to cater for 7 groups of 4 students than 28 individuals. When students were working in groups, the classroom became a dynamic learner-centred environment. Within each group, students were expected to provide ideas, think for themselves, work cooperatively and solve unfamiliar problems. A healthy level of competition can develop between groups of students working on similar tasks. This tends to motivate individuals to reach higher levels of attainment than they would working alone. Members of the group are responsible for its smooth functioning and output. The student with a preference for quadrant B thinking would keep the others focussed. Without this student in the group it was easier for others to lose focus. The student with a preference for quadrant D thinking would generate ideas, but these would not always be relevant to the task as the logical quadrant A student would show. The student with a preference for quadrant C was needed to keep harmony within the group, to ensure everyone participated and had their say.

Strategies for engaging students in mathematics

The use of group work and the shift towards more open-ended, practical tasks provided an opportunity for collaborative exploration. Students indicated that they found the approach more interesting and relevant than when taught under traditional methods. Greater effort was made to involve students physically in group activities and this provided more opportunities to engage the kinaesthetic learners. Some strategies for stimulating student involvement include:

1. Linking student interests to areas of the work. Themes used with year 7 include: politics, cycling, snooker, computers, sport and reptiles.
2. Allowing students to create stories and role-plays to describe mathematical processes and concepts. This creates a lot of enjoyment for students, particularly when they perform in front of the class. Examples include: acting out algebraic problems, writing stories to demonstrate understanding of directed number concepts, making skits to explain graphical interpretations.
3. Using games to teach new concepts. Involving students in the making of games, creating class challenges and quizzes.
4. Encouraging peer teaching and involving older student in the development of coursework for younger students. When students have to write clues or questions it makes them think much more deeply about what they are doing.
5. Making use of the physical environment as much as possible and moving outside the classroom more often. The cartesian plane, directed number, geometry, trigonometry and linear graphs are topics where many physical activities can be undertaken. Have students participate in or develop a Maths Trail within the school grounds or in the local area. Involve students in maths excursions where possible.

6. Teaching students to use concept maps and flowcharts to connect ideas, revise the work, make links between topics and see the bigger picture.
7. Videoing students working in groups and highlighting some of the different strategies that they have used when solving an unfamiliar problem; e.g. drawing diagrams, constructing models, looking for patterns, making tables, working backwards, and trial and error. Students seem to remember the most when they see themselves performing a particular technique or skill.
8. Using questioning to elicit higher order thinking without giving away answers. For example, use questions such as: What are you thinking? How do you know this? What does this show? Can you explain this feature? Where is your proof? When does that occur? Why are you using this technique? Do you have any other evidence? Can anyone add to this idea? Are you ready to formulate a conclusion?
9. Providing some choice by offering a range of tasks, each having different rating levels depending on the degree of difficulty; e.g. standard skills (1 point), problem-solving (3 points), open-ended tasks provided by teacher (6 points), investigations (10 points). Each student should gain at least 20 points to satisfactorily complete a unit.
10. Allowing students more say in the ways they will be assessed. Involving them in the writing and correction of revision sheets and tests. Using self and peer assessment, and providing a range of options including non-traditional mathematics assessment; e.g. class presentations, talks, web page design, multi-media commentaries.
11. Giving students time and space to learn. Reviewing courses to cover fewer topics in one year and allowing them time to dig deeper into each area.
12. Varying the teaching style, tasks and processes regularly.
13. Creating an uncluttered classroom environment and displaying students' work.
14. Encouraging reflective practice by asking students for feedback on how they feel about the work. Acting on the feedback received. Some questions to encourage student reflection include: What have you learned over the past few lessons? How did you get started on the task? What helped you to clarify your understanding? What helped or hindered you in the process? Which parts did you find difficult? Why did you find them difficult? What strategies did you use to break through periods of difficulty or frustration? Can suggest ways to improve your understanding of this topic? How does this work link to things that you already know? How did you contribute to the group discussion? What would you do differently next time?

Unit planning using Herrmann's whole brain model

The different thinking styles of students enrich the group activities, enabling concepts and ideas to be explored from a variety of perspectives. Figure 5 shows some typical actions, questions and suggestions offered by students when working

on a year 7 Algebra unit. The whole brain groupings facilitated more integral learning through the sharing of ideas and skills, and the development of students' less preferred ways of thinking.

At the start of the unit students worked on problem-solving tasks, they acted out the problems, searched for patterns, wrote their own rules to express generality and designed their own problems. Through the process they came to recognise a need for pronumerals. They learned how to substitute values into algebraic expressions, understood the differences between expressions and equations, applied spreadsheet algebra and worked out how to check their answers. The group discussions and collaborative activity tended to facilitate deeper understanding of the concepts. At the end of the unit, students were more readily able to relate to algebraic ideas than if taught more traditionally. They had a better appreciation of why algebra is used and how it may be applied in real situations. The evaluation of this unit and others developed along similar lines revealed that students enjoyed the work and were keenly engaged in the process.

The questionnaire given at the end of the year confirmed that the majority of students had enjoyed collaborative work and that they found the activities more interesting than textbook exercises. By the end of the year, many students in the class believed that mathematics could be creative and fun even though it may not be one of their most favourite subjects at school. There were some students who did not enjoy the group work and would prefer to work alone. These feelings were acknowledged and used to assist them to understand some of the reasons why they might feel this way. It was important to involve these students positively in the collaborative work, and so they needed to be placed with students who had well-developed interpersonal skills. The matching of personalities and learning styles needs to be carefully considered for each class, and a variety of tasks offered in order to cater for the needs of all students.

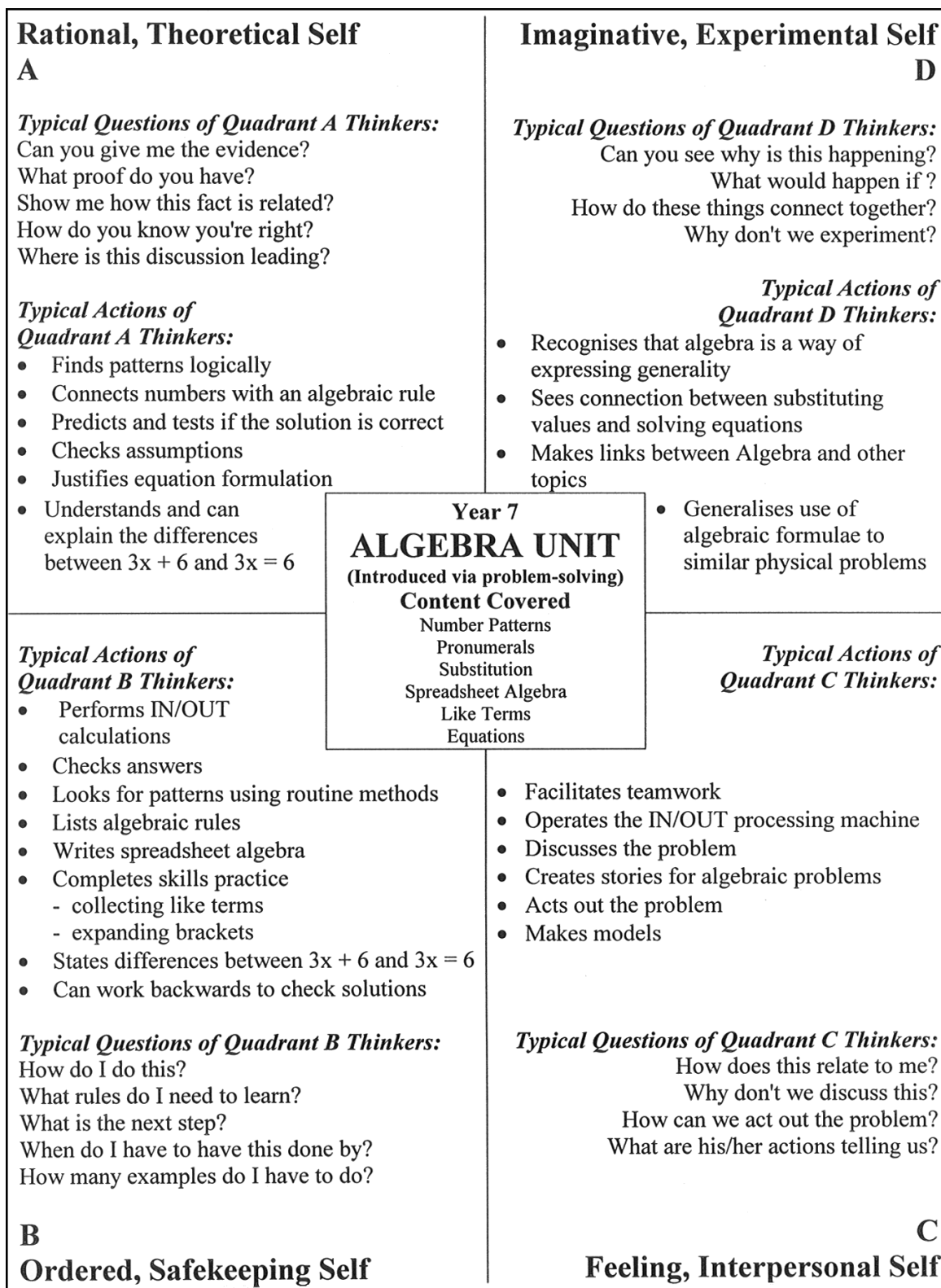


Figure 5: Perspectives on Algebra based on Herrmann's Whole Brain Model.
Adapted from Aitken (1998)

Conclusion

The research described in this paper developed as a result of feedback from students on their experiences in mathematics, an interest in the learning theory and a belief in the benefits of teaching through collaborative inquiry. Through the introduction of practical activities, group work, and awareness of the individual learning preferences of students, we can facilitate more active engagement of students in mathematics. The approach relies on a willingness to experiment with new ideas, to reflect upon the outcomes of lessons, to seek feedback and modify teaching practice accordingly. Through the process of reflection students can come to understand more about the way they like to learn mathematics and this knowledge can assist teachers in their endeavour to provide them with well-balanced and challenging mathematics programs.

Acknowledgement

I would like to thank Wesley College Learning Specialist, Amanda McGraw for her support and assistance in working with me on the research outlined in this paper.

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About the presenter

Anna has been a mathematics teacher at Wesley College in Melbourne since 1997 where she has taken an active role in curriculum development, both as a teacher of mathematics and as the Secondary Curriculum Coordinator of the Prahran Campus. Anna is interested in constructivist approaches to teaching and learning and is a strong supporter of the benefits of collaborative group work in mathematics. She enjoys working with teachers to make mathematics curriculum more relevant and accessible to students. Anna is currently a member of Council of the Mathematical Association of Victoria.

Mathematics Education in Thailand: From Kindergarten to Graphics Calculators

Nittayaporn Bunyasiri and Peter Jones

While most of us as teachers of mathematics have some knowledge of educational practice in countries such as the UK and the US, few of us have any real knowledge of the mathematics curricula of some of our nearest neighbours, the countries of SE Asia. In this session, a senior high school teacher from Thailand will outline the structure of the educational system in Thailand with particular reference to mathematics curricula in the upper secondary school. She will also discuss her recent experiences in introducing the graphics calculator to her classes and the materials she has developed to assist her in this process.

Introduction

Like many countries in South East Asia, Thailand is just beginning to introduce computer based technology into their mathematics curriculum. However, mathematics educators in Thailand are working hard to ensure that, when technology becomes more readily available to students in schools, it is used in a way that improves the quality of their students' education. There are, however, many issues to be resolved.

Senior high school mathematics in Thailand

The Thai Educational System is separated into four divisions: Kindergarten (3–5 year olds), Elementary School (grades 1–6 for children 6–11 years of age), Secondary School (grades 7–9 for children 12–14 years of age) and High School (grades 10–12 for children 15–17 years of age). Previously, the government provided 6 years of free compulsory education at Elementary level. More recently, this has been extended to 9 years (to Secondary level). In accordance with the National Education Law BE. 2542 (AD 1999), compulsory education has now been extended to 12 years (High School level) to take effect in BE 2545 (AD 2002). Further education in university or college is dependent on family finances.

Content

The content of the Thai High school (grades 10–12) mathematics curriculum follows on from the Secondary School curriculum. The curriculum is divided into 6 sections and a textbook has been written to cover each section.

Table 1: Overview of curriculum content years 10–12

Year	Semester I	Semester II
10	Book 1 Sets, subsets and set operations The real number system and basic number theory Solving equations and inequalities Logic and argument Relation; domain and range and inverse The rectangle co-ordinate system; distance between two points, midpoint and slope of a line	Book 2 Conic sections : the circle, ellipse, parabola and hyperbola Functions, composite-functions and inverse functions Trigonometric functions and graphs (radian and degrees), solving trigonometric equations Statistics : collection and presentation of data, mean, median, mode, mid-range, geometric mean, harmonic mean of grouped or ungrouped data
11	Book 3 Exponential and logarithmic functions Applications of trigonometry and trigonometric identities Matrices and determinants Linear programming	Book 4 Vectors and vector operation, applications Complex numbers and complex number operations Statistics : percentiles, quartiles and deciles, distributions, the normal distribution, standard scores.
12	Book 5 Sequence and series Limits of continuous function Derivative of function Definite and indefinite integrals	Book 6 Permutations and combination, the binomial theorem Probability and its applications Share index Regression of two variables

Participation rates

In Thai High Schools, students are separated into 3 streams: the Science/Mathematics stream, the English/Mathematics stream and the General stream. All students study some mathematics. The Science/Mathematics and the English/Mathematics streams have 5 periods of Mathematics per week and study all topics, while the General stream has only 3 periods per week and students tackle a limited number of the topics outlined in Table 1.

Role of technology

Current situation

According to some Thai researchers, most Thai Mathematics teachers still rely entirely on chalk and talk, with no technology (computers and calculators) in the classroom. While there are computers in Thai schools, they are used to access other information and for computer studies, but are not widely used for learning mathematics.

The use of graphic calculators is not widespread, partially due to teachers' lack of familiarity with the technology, their inexperience in adapting graphic calculators into their teaching and the high cost of this technology in Thailand.

The Samsen Wittayalai School initiative: a personal experience

I borrowed a TI-83 from Open Technology Company in Thailand and tried to use it in my mathematics class. I divided the students into groups of 4–5 and gave them some worksheets on different topics, allowing them to discuss and develop the concepts together. This is very difficult in Thailand because most Thai students are used to learning from teachers, rather than learning by themselves. Also, with large student numbers it took a long time to complete an activity in class. Sometimes I used the TI-83 view screen to demonstrate to the class as a way of saving time. This way my students can see graphs of different kinds of functions and relations and the effect of operating on them.

For example, I set worksheets on topics such as:

- graphs of functions
- composite functions
- algebraic functions
- identical functions
- limit functions.

Sample worksheets for three topics are included in Appendices 1–3.

In addition to the above activities, graphics calculators are used to develop programming skills. I believe that if the students are capable of writing a program, they can reason and solve problems step by step.

Although not directly related to mathematics, students can use a graphics calculator to create pictures. In the process, which takes a long time, the students will develop a lot of skill, will learn to be patient and will be very proud of themselves. The types of pictures that can be drawn, for example, the Opera House or the elephant, are shown in Figure 1. Instructions for drawing pictures with the TI-83 are given in Appendix 4.

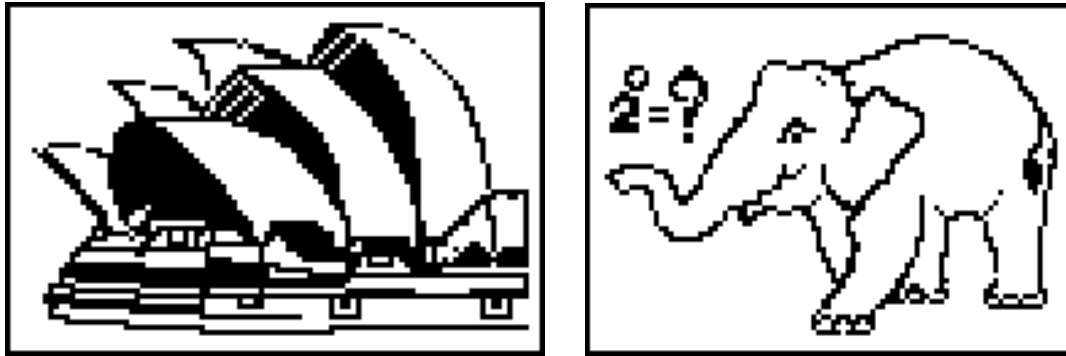


Figure 1. Graphics calculator generated pictures.

Professional development

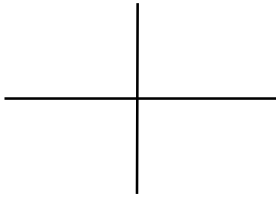
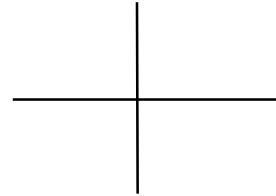
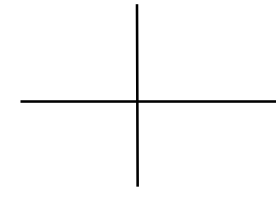
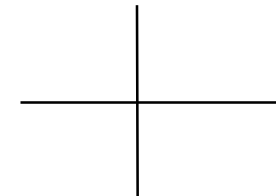
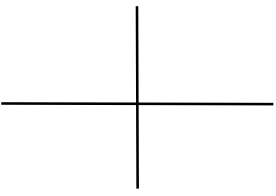
I have been using the graphics calculator for 2 years and I am trying to encourage the spread of this technology by arranging training sessions on the use of graphics calculators for teachers in many Thai schools, the International Schools, Universities and various other Educational Institutes. I have provided hands-on training, mathematics worksheets and support materials for the TI-83 graphics calculator.

I plan to post all of these documents on the Samsen Wittayalai's website in the near future. I firmly believe that if we can convey and exchange innovative ideas with our fellow teachers, we can improve the way mathematics is taught for future generations of mathematics students.

Conclusion

The aim of this paper has been to give Australian readers some insight into the structure of the Thai school system and the mathematics curriculum that is taught to students in years 10–12. It also describes an early initiative involving the introduction of graphics calculators and some of the novel ways in which it is being used. As teachers, we all have much to learn by sharing experiences, particularly with those whose experience is quite different from our own. It is in this spirit that this paper has been written.

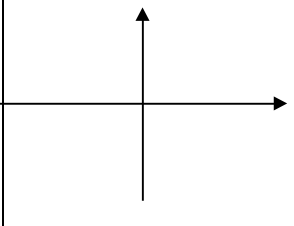
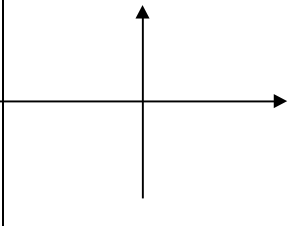
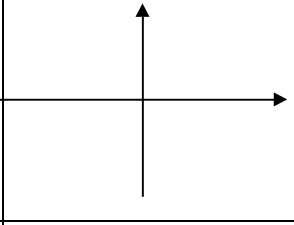
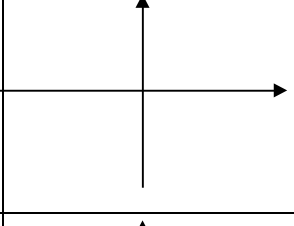
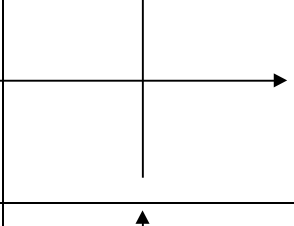
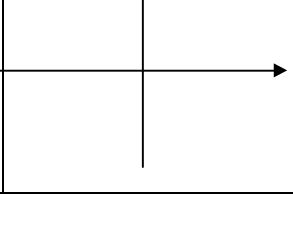
Appendix 1: Graphs of Quadratic Equation (original in Thai)

	Equation $y = a(x - b)^2 + c$	a	b	c	Graph	Vertex
1	$y = x^2$ $y = 2x^2$ $y = 3x^2$					
2	$y = -x^2$ $y = -2x^2$ $y = -3x^2$					
3	$y = x^2$ $y = (x - 2)^2$ $y = (x + 3)^2$					
4	$y = x^2$ $y = x^2 - 2$ $y = x^2 + 3$					
5	$y = -(x + 1)^2 + 2$ $y = 2(x + 2)^2 - 1$ $y = -3(x - 1)^2 - 2$					

Questions

1. What is the shape of graph of each the equations?
2. How does the graph change by the value of a ?
3. How does the graph change by the value of b ?
4. How does the graph change by the value of a ?
5. Conclude that when we have $y = a(x - b)^2 + c$
 the shape of graph is
 the coordinate of the vertex is
 If $a > 0$ the graph is open (above or under)
 If $a < 0$ the graph is open..... (above or under)

Appendix 2: Composite functions (original in Thai)

Function	Graph	1-1	Domain	Range
$y_1 = f(x) = \sqrt{x} + 4$			$D_f =$	$R_f =$
$y_2 = g(x) = -\sqrt{3-x}$			$D_g =$	$R_g =$
$y_3 = y_1(y_2(x))$				
$y_4 = (f \circ g)(x)$				
$y_5 = y_2(y_1(x))$				
$y_6 = (g \circ f)(x)$				

Questions

1. Which Y has no graph?
2. Which Ys have the same graphs?
3. $f \circ g$ and $g \circ f$ have the same graph or not?.....
4. Domain of $f \circ g =$ Domain of g or not?.....
5. Domain of $g \circ f =$ Domain of f or not?
6. If f and g is 1-1 function then $f \circ g$ and $g \circ f$ is 1-1 function or not?.....

Appendix 3: Limit of Sequences (original in Thai)

Sequence	Graph	Value a_n as $n \rightarrow \infty$	Limit
1. $a_n = 1 - \frac{1}{n}$			
2. $b_n = \frac{1}{2}(1 - \frac{1}{n})$			
3. $a_n + b_n$			
4. $a_n - b_n$			
5. $a_n \times b_n$			
6. $a_n \div b_n$			

Appendix 4: Drawing a Picture on the TI-83 (original in Thai)

1. Set Home Screen

- Press 2nd / [FORMAT]
- Set CoordOff and AxesOff
- Press 2nd / [QUIT]

You will get the screen dimension is 64×96.

2. Drawing Point

- Press 2nd / [DRAW] / A.Pen
- you get the mark + on the centre of screen
- Press ENTER for drawing each point and press ENTER for eject Drawing and Press ENTER for drawing again
- move + by four ways arrows

3. Delete Point

- Press 2nd / [DRAW] / POINTS / 2:Pt-Off
- move + by four ways arrows to the deleted position
- Press ENTER for deleting each point

4. Store Picture

- Press 2nd / [DRAW] / STO / 1:StorePic
- screen will show word StorePic
- Put the number of picture 0 to 9
- Press ENTER

5) Show Picture

- Press 2nd / [DRAW] / STO / 2:RecallPic
- screen will show word RecallPic
- Put the number of picture 0 to 9 that you want
- Press ENTER

6) Clearing Picture on Screen

- Press 2nd / [DRAW] / 1:ClrDraw
- screen will show word ClrDraw

- Press ENTER
- The picture on screen was cleared but it still has it in memory.

7) Delete Picture in memory

- Press 2nd / [MEM] / 2:Delete...
- screen will show menu , select 8:Pic...
- select Pic # by move arrow key
- Press ENTER and it is immediately deleted (be careful). If you change your mind, press 2nd [QUIT]

8) Printing Picture

Used TI Graphlink.

About the authors

Nittayaporn Bunyasiri has been a Mathematics Teacher in Government High Schools in Thailand for more than twenty years. In 1969 she was awarded the Teacher of the Year Award from Ministry of Education and, in 1994, the most Exemplary Mathematics Teacher Award from the Teacher Committee of Bangkok. She has a B.Sc. (Mathematics) and a M.Sc. (Teaching Mathematics).

Professor Peter Jones is Head of the School of Mathematical Sciences at Swinburne University of Technology, Melbourne, Australia. He is one of the pioneers in investigating the use of graphics calculators in the teaching and learning of mathematics and is a regular speaker on the subject at conferences around the world. In addition to his university work, Professor Jones is a Chief Examiner of Year 12 Mathematics in the state of Victoria, where students are now assumed to have access to a graphics calculator at all times.

Unsolved Problems and the Mathematics Challenge for Young Australians

John Dowsey and Mike Newman

In this session some problems from the Mathematics Challenge for Young Australians will be presented for discussion. The Challenge seeks to foster mathematically talented youngsters in years 5 to 10 and encourage their continuing involvement with mathematics. One of the aims is to develop their mathematical skills and knowledge, in particular via a focus on problem solving and critical thinking. Challenge problems are generally designed to attract students to interesting mathematics. This anticipated attraction is supported by extension material which sometimes leads to deeper and occasionally unsolved problems. Some of this extension material for the Challenge problems discussed will also be presented.

Introduction

The Mathematics Challenge for Young Australians seeks to foster mathematically talented youngsters in years 5 to 10 and encourage their continuing involvement with mathematics. It aims to develop students' mathematical skills and knowledge, in particular via a focus on problem solving and critical thinking.

The Mathematics Challenge is rather different from many other mathematics competitions. The problems require sustained effort — students have three weeks to work on them and submit their answers. For some problems, students can work in small groups though their final submissions must be individual. Student work is teacher marked from marking schemes supplied and certificates are awarded based on marks submitted to the Australian Mathematics Trust in Canberra. Papers are set at three levels: Primary (years 5 & 6, four problems), Junior (years 7 & 8, six problems) and Intermediate (years 9 & 10, six problems) with some problems common to two or more papers.

Challenge problems are generally designed to attract students to interesting mathematics. In addition, teachers are provided with extension questions and notes which they can use with their students after the Challenge. Some Challenge questions have been inspired by more advanced mathematics and known research problems; occasionally some of the extensions have led to some rather deep and even unsolved problems.

Three challenge questions

Three examples of unsolved problems and the mathematics involved are addressed in the following section. In this section, the original Challenge question and its mathematical intent are stated.

Challenge question: Pirates (1992)

When pirates go ashore to dig up buried treasure, each pirate in the digging party receives a different-sized share of the plunder. The treasure is always completely shared out with the captain getting the largest share, the first mate the next largest share and so on down to the cabin boy, if he is with the digging party.

As pirates are not smart, the fractions they use for sharing the treasure always have 1 as the numerator. [Pirates using fractions such as $\frac{2}{3}$ or $\frac{4}{7}$ are punished by being made to walk the plank!]

- (a) Show that there is only one way to share out the treasure amongst a digging party of 3.
- (b) Find the six different ways which may be used to share out the treasure amongst a digging party of 4.
- (c) Show that there is at least one way to share out the treasure amongst a digging party of any size greater than 2.

Note that a fraction with numerator 1 is often called a unit, or Egyptian, fraction.

The question is essentially: Show that it is possible to write 1 as the sum of n unit fractions for $n > 2$.

There were four (related) extension questions given, including the following:

Can every fraction of the form $\frac{4}{n}$ with $n > 3$ be expressed as a sum of three different unit fractions?

Challenge question: Boxes (1996)

A rectangular prism (box) has dimensions x cm, y cm and z cm, where x , y and z are positive integers. The surface area of the prism is A cm².

- (a) Show that A is an even positive integer.
- (b) Find the dimensions of all boxes for which $A = 22$.
- (c) (i) Show that A cannot be 8.
(ii) What are the next three even integers which A cannot be?

The question is essentially: Given a positive integer A find the lengths of the sides of a rectangular prism with integer sides whose surface area is A .

Clearly there can be no such prism unless A is even and greater than 4. Numbers of this form for which no such prism can occur were subsequently named *O'Halloran Numbers*, in memory of Peter O'Halloran (1931–1994), co-founder with Bruce Henry (the current Director) of the Mathematics Challenge for Young Australians. Note that for all numbers of the form $4k + 2$ rectangular prisms of the required kind do occur (consider a 1 by 1 by k prism). So the question revolves around numbers of the form $4k$. In fact, Andy Edwards, the creator of the original problem, found 16 O'Halloran

numbers: 8, 12, 20, 36, 44, 60, 84, 116, 140, 156, 204, 260, 380, 420, 660, 924. This led him to ask whether there are any more.

Challenge question: Pharaoh's Will (2000)

As he lay dying, the first Pharaoh of Ufractia proclaimed:

'I bequeath to my oldest child, one third of my estate; to the next oldest child one quarter of my estate; to the next oldest child, one fifth of my estate; to each succeeding child, except the youngest, the next unit fraction of my estate; and to the youngest the remainder.'

When the Pharaoh died and his estate was divided, the youngest child received the smallest share which was worth 27 000 gold grikkles.

(a) What was the value of the oldest child's share?

The Pharaoh's successor was so impressed with this method that he proclaimed: 'In future, a Pharaoh's estate will be divided according to these rules:

each surviving child, except the youngest, will be bequeathed a unit fraction of the estate;

the oldest child will be bequeathed the biggest fraction;

if a child is bequeathed $\frac{1}{n}$ of the estate, the next oldest child, except the youngest,

will be bequeathed $\frac{1}{n+1}$ of the estate;

the remainder is to be at most the next unit fraction and is to be bequeathed to the youngest child.'

(b) The second Pharaoh had eight children living when he died. What fraction of his estate was bequeathed to the oldest child?

(c) When the third Pharaoh died, he had more surviving children than the first Pharaoh but less than eleven surviving children. It was found that the rules could not be followed precisely.

How many children might have survived the third Pharaoh?

The question is essentially: Find the sum of successive unit fractions (starting with the fraction $\frac{1}{n}$) under the condition that the sum should be close to but less than 1.

Some related extension questions led to an interesting conjecture, of which more in the next section.

All the Challenge problems for the years 1991–5 together with much of the extension material have been collected in Henry et al. (1997). A subsequent collection for the years 1996–2000 is in preparation.

Unsolved problems

As outlined earlier, the major purpose of the Mathematics Challenge for Young Australians is to introduce students to problems which require sustained effort. The

Problems Committee also wants, via teacher discussion and reference notes, to create an awareness that there are related problems to which it does not know an answer and even ones to which no-one knows an answer.

The mathematics involved in the three questions described above comes under the general heading of *Number Theory* or, more precisely, *Diophantine Equations*, that is, equations where one is looking for integer solutions. Such equations are named after Diophantus (c. 200–284). A substantial account can be found in a monograph on Diophantine equations by Mordell (1969).

The last chapter of Mordell's book deals with miscellaneous results. In the first section, the following conjecture of Erdős and Straus is found.

The equation $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ where n is an integer greater than 3 is solvable in positive integers.

The second section deals with the equation $yz + zx + xy = d$. Mordell points out that this second equation is connected with the class number for binary quadratic forms. That is a mouthful — let us not go into the details. We will just describe briefly what happened in relation to the Boxes question.

The mathematics content boils down to finding the positive integers d (greater than 3) for which this equation has positive integer solutions. The only 16 values of d for which it is known that there is no solution correspond to the 16 O'Halloran numbers listed earlier. Are these the only ones?

A few enquiries led to the Mordell reference and more. With a vacation student, Chris Tuffley, these leads resulted in the second author making contact, by email, with Stéphane Louboutin in Caen, France. The exchange of emails resulted in some joint notes in which it is shown that apart from the known values there is at most one more — and no more if one assumes the generalised Riemann hypothesis. [The Riemann hypothesis is a major unsolved problem which is one of seven challenges for each of which the Clay Institute has recently offered a million dollar prize.]

Over the last few years, some papers have been published on this second equation. The most recent appeared in a Chinese journal this year and gets the same result as above except with 'one' replaced by 'two'.

Let us now return to the Erdős-Straus conjecture. It is closely related to the mathematics involved in the Pirates question. Both ask about decomposing fractions, or (positive) rational numbers, into sums of unit fractions. Erdős and Straus, in part inspired by Neugebauer's history of pre-Greek mathematics, made this conjecture around 1948. It was known that every fraction $\frac{m}{n}$ can be written as the sum of at most m distinct unit fractions. It was also known that for both $m = 2$ and $m = 3$ this result is the best possible. They tried the case $m = 4$ and found it hard. Straus showed that for all n less than 5000, three unit fractions suffice.

In the teacher discussion and reference notes for the 1992 Challenge, it was shown how to solve the equation for all n except those which leave a remainder of 1 when divided by 24.

Mordell refers to a number of papers which appeared between 1950 and 1965 which culminated in the result that the equation can also be solved for all n less than 10^7 . Mordell also gives a proof that the equation has a solution unless the remainder on dividing n by 840 is one of 1, 121, 169, 289, 361 or 529.

There has been more activity recently. There is a statement, on the world wide web, that Swett (in 1999) has increased the general bound to 10^{14} by doing some extensive computations. [We give no specific web address; using a good search engine such as www.google.com will enable the reader to find many articles relating to unit fractions. The ones we found vary considerably in quality.]

The conjecture in full generality remains unsettled. Of course it cannot be settled just by doing more computations. One can only hope that computations will shed enough light to lead to a general solution (if there is one).

There are by now quite a lot of results and questions about writing fractions as sums of unit fractions. Here are some in the form of statements that do not, as a challenge to the reader, indicate which are results and which are questions!

- Every fraction can be written as a sum of distinct unit fractions.
- Every fraction can be written as a sum of distinct unit fractions with even denominator.
- Every fraction with odd denominator can be written as a sum of distinct unit fractions with odd denominator.

Can the truth of these statements be found using in the *greedy algorithm*? For example, with respect to the first statement, the greedy algorithm works like this: start with the given fraction, subtract the largest unit fraction less than it and repeat the process on each remainder until a remainder is a unit fraction. The sum of the unit fractions that result is the answer.

For example: $\frac{2}{9} = \frac{1}{5} + \frac{1}{15}$; $\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}$; $\frac{17}{29} = \frac{1}{2} + \frac{1}{12} + \frac{1}{348}$.

Why is there this interest in decompositions into sums of unit fractions? The short answer is that the questions seem intriguing and challenging. It is a long story. It goes back at least to 1650 BC. Two other critical dates are 1202 and 1880.

Leonardo of Pisa (Fibonacci) describes the greedy algorithm for the first statement above in his *Liber Abaci* of 1202. He implicitly assumes that it always gives the desired result. The 1880 date is when Sylvester proved one of the statements above. He was inspired by Moritz Cantor's account of some of the mathematics in the Rhind Mathematical papyrus, which had been rediscovered in 1858 and was translated into German in 1877. This papyrus is dated to about 1650 BC. In it, decompositions into sums of distinct unit fractions are shown and used.

The conjecture of Erdős and Straus inspired interest in the mathematics associated with unit fractions. There has also been interest in the history of their use which poses equally intriguing and challenging questions.

For example, we find that the Rhind Mathematical papyrus correctly gives unit fraction decompositions for all $\frac{2}{n}$ with n odd and $3 \leq n \leq 101$. There has been quite a lot of debate on how these decompositions were found. It is certain that it was not simply by use of the greedy algorithm, because the papyrus gives the decomposition $\frac{2}{9} = \frac{1}{6} + \frac{1}{18}$.

Pharaoh's Will also involves unit fractions. It leads naturally to the following question:

Given a positive integer n , for what m is it true that:

$$\frac{1}{n} + \dots + \frac{1}{m} \leq 1 < \frac{1}{n} + \dots + \frac{1}{m} + \frac{1}{m+1}$$

where ... indicates all unit fractions between $\frac{1}{n}$ and $\frac{1}{m}$.

For small values of n this can be done easily just using the definition, but even with a computer one runs out of steam quite soon — try doing $n = 10^6$ that way.

One of the Challenge moderators, Barry Harridge, asked for a formula for m in terms of n ; or in other words, better ways of calculating m from n . He proposed a formula equivalent to

$$m = \text{IntegerPart}(1+(e-1)(2n-1)/2) + n - 2$$

and noted that this formula holds for n up to 5000 except for $n = 36$.

It is not too difficult to see, using upper and lower bounds for the appropriate integral, that the exact m for the starting fraction $\frac{1}{n}$ is either the number given by the Harridge formula or 1 more than that. The point of such a formula is that it can give a very good approximation for larger n with much less effort than summing fractions.

There remain some questions:

- When is the formula exact?
- Is there another formula which gives the exact answer with less calculation than summing, though not necessarily less than that needed for the Harridge formula?

Using just summation one can find that the formula is not exact also for $n = 9045$. Other places where the formula is not exact have been determined by Ralph Buchholz and Michael Smith. They have found 48 values of n for which m is 1 more than given by the Harridge formula — the largest of these is about 10^{201} . Their work involves rather more sophisticated mathematics and use of refined software though only moderate computing resources. They use convergents to the continued fraction for $(e-1)/2$ and approximations to sums of reciprocals of integers involving Bernoulli numbers. They compute with 500 decimal digits.

It would be interesting to know whether all the convergents give values of n for which the formula is not exact; also whether every n for which the formula is not exact is such a convergent.

The third Buchholz-Smith number is 5195512. The second author has checked that it is the third exception by summation, taking less than 70 minutes. The summation itself takes about 52 minutes of this time. Each instance of the Harridge formula takes less than a millisecond to evaluate.

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About the presenters

John Dowsey has been associated with the Australian Mathematics Trust since 1995 writing questions for Mathematics Competitions run by the Trust; he is currently Deputy Director of the Mathematics Challenge for Young Australians. He has been involved in mathematics education for many years.

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I Can Do Maths Too — Count Me In!

Rhonda Faragher

For sound educational and social reasons, more and more students with special needs are being included in regular classrooms. However, many teachers undertook pre-service training before inclusion was common practice and many feel they lack the skills and strategies to effectively teach students with special needs. This paper will present a rationale for inclusion, discuss potential sources of difficulty with mathematics and suggest strategies to help teachers effectively manage the learning of students in their classes.

Some years ago, a friend of mine was telling me about her daughter who has cerebral palsy. When she was in year 7, the class teacher commented they would need to teach her to type because she would not be able to handwrite fast enough at university. This was a significant moment for my friend as at the time, university entry by people with disabilities was not common.

Since the early 1980s, it has become increasingly common for children with special needs to be included in regular classrooms. This paper will consider a rationale for inclusion, discuss potential sources of difficulty with mathematics and suggest strategies to help teachers effectively manage the learning of students in their mathematics classes.

Terminology

Before we consider how to help children with special needs, a comment on correct use of terminology is important — not just for political correctness, but for the potential role for teachers.

The World Health Organisation has defined the terms ‘impairment’, ‘disability’ and ‘handicap’ to have specific and separate meanings. (These are currently under review, perhaps moving from the current definitional emphasis on a medical model).

Impairment — an abnormality in the way organs or systems function, usually of medical origin, for example, short-sightedness, heart problems, cerebral palsy, Down syndrome, spina bifida or deafness.

Disability — the functional consequence of an impairment. For example, because of the impairment of short-sightedness, the disability may be that a person is unable to see clearly without glasses.

Handicap — The social or environmental consequences of a disability, for example, inability to follow television news because of deafness. (Foreman, 1996, p. 404f)

Impairment is at the level of diagnosis and is of little assistance to teachers. The functional consequences of the impairment are much more important for us. It is at

the level of handicap, though, that we can make a big difference. For example, if the person with a hearing impairment is watching a captioned news service, he or she may have no handicap in the situation. This is an important distinction for teachers. By modifying the learning environment, it may be possible to completely or partially eliminate the social or environmental consequences of the disability.

The teacher who was remembered many years later by my friend saw a disability — slow handwriting, the functional consequence of the impairment of cerebral palsy — and set about reducing the resulting handicap by arranging for her student to learn to type. Situations such as these afford teachers an exciting opportunity. It is an example of an ordinary teacher, seeing an educational need, setting out to meet the need and in the process becoming an extraordinary teacher in the life of a child.

Towards a rationale for inclusion

It can be said that special education has undergone a paradigm shift in the last decade. The WHO definitions provide a hint of the reasons behind this change. Previously, the focus of assisting children with special needs was on identifying the source of the problem and attempting to correct it. The shift has come with changing the focus from the individual to the curriculum (Ainscow, 1994). Whereas once the focus was on the child's deficits, the emphasis of the curriculum view is on changing the educational environment. (Sykes, 1989)

Part of the motivation for the change came from research which suggested that schools which were successful with including students with special needs were successful in meeting the needs of all their community. The same was noted with teachers: 'The evidence seems to support the view that teachers said to be successful in meeting special needs are to a large extent using strategies that help all pupils to experience success. Indeed we are probably referring to the very same teachers.' (Ainscow, 1994, p.24).

As a result of these findings, the UNESCO project 'Special Needs in the Classroom' views the special needs task reconstructed as school improvement. (UNESCO, 1994 and discussed in Ainscow, 1994). In the process of achieving the best possible learning outcomes for the students with special needs, the entire school — students and staff — will benefit.

Certainly entire school improvement is a fairly significant outcome from including students with special needs! However, there are other benefits mentioned in the literature. Some of these are listed below.

- Students with special needs educated in regular classes do better academically and socially than comparable students in noninclusive settings. (Baker, Wang and Walberg, 1994, p. 34)
- When special needs are being met, the learning of all students in the class is enhanced. (Ainscow, 1994, p.24; Goodlad and Hirst, 1990, p.230)
- Schools should mirror society, allowing children the opportunity to 'learn and grow within communities that represent the kind of world they'll live in when they finish school.' (Sapon-Shevin, quoted in O'Neil, 1994, p.7)

To achieve these benefits, the process of inclusion must be well done. Ainscow and Muncey have identified the following features common in schools experiencing success with including students with special needs:

- effective leadership from a headteacher who is committed to meeting the needs of all pupils,
- confidence amongst staff that they can deal with children's individual needs,
- a sense of optimism that all pupils can succeed,
- arrangements for supporting individual members of staff,
- a commitment to provide a broad and balanced range of curriculum experiences for all children, and
- systematic procedures for monitoring and reviewing progress. (Ainscow and Muncey, 1989, cited in Ainscow, 1994)

It has also been shown that preparing the school community for the inclusion of a child with special needs improves the likelihood of a successful outcome. This may involve preparing a written school policy on inclusion, explaining the benefits to other parents (including benefits to the learning of all students and extra resources which may come with the child) and preparing the students (if necessary, explaining unusual behaviour, communication methods and how other students can help).

Resource provision has not been shown to be a key factor in the success of inclusion. Surprisingly, the resourcing issue can be a source of difficulty instead. (Ainscow, (1994, p.20). Indeed, Sykes (1989, p102) notes, '...the belief that integration necessarily requires specialized and expensive physical and educational resources is erroneous.'

A related issue concerns the provision of teacher aide assistance for students with special needs. When teacher aides are used effectively, they can assist teachers to support the learning of all children in the class. However, effective practice is not always achieved. Giangreco, Edelman, Luiselli and MacFarland (1997) point out the difficulties which may arise when the teacher aide is seen to be responsible for the instruction and management of the child with special needs. Problems include:

- The child being effectively excluded by working on a separate program, isolated from other students.
- The teacher not being responsible for the teaching program — the learning of the student is not being managed by the professionally trained educator.
- Limits on the receipt of competent instruction.
- Teacher aides have been observed to over-assist students leading to the development of learned helplessness. The child can also become dependent on the adult.
- Interactions with peers in the absence of adults may be prevented. (Everyone needs time to talk to friends without being overheard!)
- Interference with the teaching of other students.

Effective use of teacher aide time can occur when the teacher aide is not viewed as attached to the child with special needs but instead as a resource to assist the teacher to meet the needs of all in the class. Successful strategies include:

- the teacher aide works with a group of students, including the student with special needs;
- the teacher aide supervises the other students working on previously set material while the class teacher works with the student with special needs, or a small group having difficulties;
- the teacher aide prepares materials for use by the whole class;
- the teacher aide works with the high achieving students in the class, using material planned by the class teacher.

What might be the difficulties with mathematics?

In this section, I will move to consider students with intellectual disabilities — in particular, Down syndrome. Down syndrome is the most common congenital abnormality resulting in intellectual disability, occurring in approximately one in 600 births in Australia. Although a great deal is known about how children with Down syndrome learn, at this stage little is known about the source or extent of the difficulties with mathematics. It can be said with little argument that difficulties may be universal in the population and, for some children, may be profound. (Bird and Buckley, 1994)

However, current research is shedding light on this problem (Faragher, 2000b). It has been known for some time that people with Down syndrome can do mathematics (Cruikshank, 1948). While they have difficulty developing their own strategies, when carefully taught, they are able to use strategies such as counting on for addition (Irwin, 1991). What is more, some students in Irwin's study were shown to still be using the strategy in the following year, without reinforcement from their new teacher.

The source of the difficulties people with Down syndrome face with mathematics is at the heart of what it means to learn, know and do mathematics itself. Mathematicians have often described their field as the study of pattern and doing mathematics to be making conjectures, finding connections and spotting and explaining patterns.

From the literature, it is known that people with Down syndrome experience great difficulty in developing strategies. As a result, much of their learning of mathematics is restricted to learning what appear as unconnected concepts. Compounding their difficulties, it is also known that many people with Down syndrome have restricted working and short-term memory capacity. These two factors combine to make the learning of mathematics difficult for students with Down syndrome. (Faragher, 2000b)

Another related problem arises from the difficulty found with developing strategies. It is known from research (and the experiences of many parents) that ordinary

children develop a deep sense of number in the preschool years. The new mathematics syllabus being developed in Queensland acknowledges that young children have their own strategies and that these should be encouraged. (Ilsley, 2000). This is an important and laudable development but it does present a significant problem for children with Down syndrome. It is unlikely they will develop strategies on their own and will need to be directly taught effective strategies for performing the mathematics required. This is an important teaching issue and will be considered in the next section.

Some strategies to help.

When planning to teach students with Down syndrome, it is important to remember that they can (and do!) learn mathematics. It is also important to realise that there is no 'miracle method'. Good special education is good teaching. As Ainscow (1994, p.19) notes:

My conclusion now is that no such specialised approaches [special ways of teaching children with special needs to learn successfully] are worthy of consideration. Whilst certain techniques can help particular children gain access to the process of schooling, these are not in themselves the means by which they will experience success.

Directly teach strategies.

Mathematics teachers are good at directly teaching strategies! The difficulty lies with deciding which strategies are worth teaching. Careful thought about the purpose of the mathematics, where the topic leads and future needs of the student should inform the decision.

Many students with learning difficulties have an individual education plan (IEP). Strategies to be taught should be noted on the IEP and when performance has been demonstrated, the accomplishment should be recorded. This will allow teachers in the following year to reinforce the strategy rather than interfere with previous learning — a problem noted in Irwin's 1991 study.

1. Aim for over-learning

An unfortunate characteristic of many children with Down syndrome is the failure to consolidate newly acquired cognitive skills into the repertoire (Wishart, 2000). The technique of over-learning can help students overcome this deficiency. After a student has demonstrated mastery of a particular strategy, further opportunities to practise, reinforce and learn are offered. Frequent revision will also be beneficial.

2. Use error free learning

Many children with Down syndrome take longer to learn than ordinary children and take longer still to overcome misconceptions they may have developed. The aim of error free (or errorless) learning is to avoid the misconceptions developing in the first place.

3. Ensure adequate time for learning

More time for mathematics will be needed. Unfortunately, for many children with learning difficulties; less is often the result. Sometimes program planners feel they are being kind by reducing time allocated to a potentially frustrating subject in the mistaken belief that progress will be limited.

Making time within the mathematics program can be achieved by filtering out the unnecessary. Careful and informed decisions will have to be made about the areas which can be omitted without disadvantaging the future progress of the student. (Faragher, 2000a).

4. Use a calculator

Over twenty years ago, Koller and Mulhern (1977) demonstrated that students with Down syndrome could be taught to use a calculator effectively. There is no excuse for not allowing them to do so.

Help!

So you have found out you are going to be teaching a child with Down syndrome (or another special need which will make learning mathematics difficult) — where to from here? The following suggestions might be helpful.

- Ask the student. Depending on their age, they will be able to tell you how they like to learn, where they have trouble and what they want to learn.
- Ask the child's parents. Parents will not expect you to know a great deal about their child's difficulties but they will expect you to want to know. In recent years, early intervention has been readily available in most places in Australia. Parents have been expected to play a large role in the therapy and learning development of their children. In the process, most learn a great deal about what motivates their child, how they learn best, what will not work and what has been accomplished in previous years. Parents should be seen as an invaluable resource.
- Ask the previous teacher and read the IEP so you know what has been accomplished and what the current aims for learning are.
- Contact your local special school or unit, (in Queensland, the Low Incidence Support Unit of the Department of Education Queensland can assist) or support associations such as the Down Syndrome Association of Queensland.

In conclusion

Working with a student with special needs and learning disabilities in particular presents one of the most rewarding opportunities a teacher can encounter. By attempting to overcome the social and environmental consequences of a disability, the handicap a student experiences in a classroom may be minimised.

This paper has presented a view of education for special needs as one seeking to enhance the learning of all in the school. In mathematics classrooms, some environmental consequences of an intellectual disability result from the nature of the subject itself. Strategies have been suggested to assist teachers to overcome these consequences. Sources of further assistance have also been suggested as no teacher should feel they have to manage on their own.

And finally, a happy ending. My friend's daughter did go to university and has recently completed her degree. May all teachers have the opportunity to make a difference in the life of a child by meeting a special need.

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About the presenter

Rhonda Faragher is a lecturer in mathematics education at James Cook University in Townsville. With a background in secondary teaching, she has taught in Queensland and South Australian schools. She has also been a staff member of The University of Queensland, The Flinders University of South Australia and Central Queensland University. She is a former President of QAMT and a former Councillor of AAMT.

Having an abiding passion for mathematics, Rhonda's research interests have centred around helping others to enjoy the subject. The particular focus on the attainment of numeracy by people with Down syndrome arose following the birth of her daughter with Down syndrome in 1996.

Reading the World with Math: Goals for a Criticalmathematical Literacy Curriculum

Marilyn Frankenstein

[originally published in *BEYOND HEROES AND HOLIDAYS: A Practical Guide to K–12 Anti-Racist, Multicultural Education and Staff Development*, edited by E. Lee, D. Menkart, M. Okazawa-Rey, Washington, DC: Network of Educators on the Americas, 1998]

Marilyn Frankenstein suggests ways that teachers can introduce math as a tool to interpret and challenge inequities in our society. Her teaching methods also make math more accessible and applicable because the math is learned in the context of real-life, meaningful experiences. This article is particularly useful for teachers who are creating an interdisciplinary math and social studies curriculum.

Professor Frankenstein's examples are based on her work teaching at the College of Public and Community Service, UMass/Boston. Her students are primarily working-class adults who did not receive adequate mathematics instruction when they were in high school. Many of them were tracked out of college preparation. Therefore, the ideas presented in this article can be applied to the secondary classroom.

For a more detailed description as well as a more theoretical discussion of the concepts presented in this paper, please refer to the publications listed in the reference section by Frankenstein and those co-authored by Frankenstein and Arthur Powell.

When my students examine data and questions such as the ones shown in the example below, they are introduced to the four goals of the criticalmathematical literacy curriculum.

1. Understanding the mathematics.
2. Understanding the mathematics of political knowledge.
3. Understanding the politics of mathematical knowledge.
4. Understanding the politics of knowledge.

Clearly, calculating the various percentages for the unemployment rate requires goal number one, an understanding of mathematics. Criticalmathematical literacy goes beyond this to include the other three goals mentioned above. The mathematics of political knowledge is illustrated here by reflecting on how the unemployment data deepens our understanding of the situation of working people in the United States. The politics of mathematical knowledge involves the choice of who counts as unemployed. In class, I emphasise that once we decide which categories make up the

numerator (number of unemployed) and the denominator (total labor force), changing that fraction to a decimal fraction and then to a percent does not involve political struggle — that involves understanding the mathematics. But, the decision of who counts where does involve political struggle — so the unemployment rate is not a neutral description of the situation of working people in the United States. And, this discussion generalises to a consideration of the politics of all knowledge.

In this article, I will develop the meaning of each of these goals, focusing on illustrations of how to realise them in their interconnected complexity. Underlying all these ideas is my belief that the development of self-confidence is a prerequisite for all learning, and that self-confidence develops from grappling with complex material and from understanding the politics of knowledge.

Example

Unemployment Rate

In the United States, the unemployment rate is defined as the number of people unemployed, divided by the number of people in the labour force. Here are some figures from December 1994. (All numbers in thousands, rounded off to the nearest hundred thousand.)

In your opinion, which of these groups should be considered unemployed? Why?

Which should be considered part of the labour force? Why?

Given your selections, calculate the unemployment rate in 1994.

1. 101 400: Employed full-time
2. 19 000: Employed part-time, want part-time work
3. 4000: Employed part-time, want full-time work
4. 5600: Not employed, looked for work in last month, not on temporary layoff
5. 1100: Not employed, on temporary layoff
6. 400: Not employed, want a job now, looked for work in last year, stopped looking because discouraged about prospects of finding work
7. 1400: Not employed, want a job now, looked for work in last year, stopped looking for other reasons
8. 3800: Not employed, want a job now, have not looked for work in the last year
9. 60 700: Not employed, do not want a job now (adults)

For discussion

The US official definition counts 4 and 5 as unemployed and 1 through 5 as part of the labour force, giving an unemployment rate of 5.1%. If we count 4 through 8 plus half of 3 as unemployed, the rate would be 9.3%. Further, in 1994 the Bureau of Labor

Statistics stopped issuing its U-7 rate, a measure which included categories 2 and 3 and 6 through 8, so now researchers will not be able to determine 'alternative' unemployment rates (Saunders, 1994)¹.

Goal #1: Understanding the mathematics

Almost all my students know how to do basic addition, subtraction, multiplication and division, although many would have trouble multiplying decimal fractions, adding fractions or doing long division. All can pronounce the words, but many have trouble succinctly expressing the main idea of a reading. Almost all have trouble with basic math word problems. Most have internalised negative self-images about their knowledge and ability in mathematics. In my beginning lessons I have students read excerpts where the main idea is supported by numerical details and where the politics of mathematical knowledge is brought to the fore. Then the curriculum moves on to the development of the Hindu-Arabic place-value numeral system, the meaning of numbers, and the meaning of the operations.

I start lessons with a graph, chart, or short reading which requires knowledge of the math skill scheduled for that day. When the discussion runs into a question about a math skill, I stop and teach that skill. This is a non-linear way of learning basic numeracy because questions often arise that involve future math topics. I handle this by previewing. The scheduled topic is formally taught. Other topics are also discussed so that students' immediate questions are answered and so that when the formal time comes for them in the syllabus, students will already have some familiarity with them. For example, if we are studying the meaning of fractions and find that in 1985, $2/100$ of the Senate were women, we usually preview how to change this fraction to a percent. We also discuss how no learning is linear and how all of us are continually reviewing, recreating, as well as previewing in the ongoing process of making meaning. Further, there are other aspects about learning which greatly strengthen students' understandings of mathematics:

- (a) breaking down the dichotomy between learning and teaching mathematics;
- (b) considering the interactions of culture and the development of mathematical knowledge; and
- (c) studying even the simplest of mathematical topics through deep and complicated questions.

These are explained in more detail below.

(a) Breaking down the dichotomy between learning and teaching mathematics

When students teach, rather than explain, they learn more mathematics, and they also learn about teaching. They are then empowered to proceed to learn more mathematics. As humanistic, politically concerned educators, we often talk about

¹ Thanks to my friend, UMass/Lowell economist Chris Tilly, for this problem.

what we learn from our students when we teach. Peggy McIntosh (1990) goes so far as to define teaching as ‘the development of self through the development of others’. Certainly when we teach we learn about learning. I also introduce research on math education so that students can analyse for themselves why they did not previously learn mathematics. I argue that learning develops through teaching and through reflecting on teaching and learning. So, students’ mathematical understandings are deepened when they learn about mathematics teaching as they learn mathematics. Underlying this argument is Paulo Freire’s concept that learning and teaching are part of the same process, and are different moments in the cycle of gaining existing knowledge, re-creating that knowledge and producing new knowledge (Freire, 1982).

Students gain greater control over mathematics problem-solving when, in addition to evaluating their own work, they can create their own problems. When students can understand what questions it makes sense to ask from given numerical information, and can identify decisions that are involved in creating different kinds of problems, they can more easily solve problems others create. Further, critical mathematical literacy involves both interpreting and critically analysing other people’s use of numbers in arguments. To do the latter you need practice in determining what kinds of questions can be asked and answered from the available numerical data, and what kinds of situations can be clarified through numerical data. Freire’s concept of problem-posing education emphasises that problems with neat, pared down data and clear-cut solutions give a false picture of how mathematics can help us ‘read the world’. Real life is messy, with many problems intersecting and interacting. Real life poses problems whose solutions require dialogue and collective action. Traditional problem-solving curricula isolate and simplify particular aspects of reality in order to give students practice in techniques. Freirian problem-posing is intended to reveal the inter-connections and complexities of real-life situations where ‘often, problems are not solved, only a better understanding of their nature may be possible’ (Connolly, 1981). A classroom application of this idea is to have students create their own reviews and tests. In this way they learn to grapple with mathematics pedagogy issues such as: what are the key concepts and topics to include on a review of a particular curriculum unit? What are clear, fair and challenging questions to ask in order to evaluate understanding of those concepts and topics?

(b) Considering the interactions of culture and the development of mathematical knowledge

This aspect is best described with the following example.

Example

When we are learning the algorithm for comparing the size of numbers, I ask students to think about how culture interacts with mathematical knowledge in the following situation:

Steve Lerman (1993) was working with two 5 year-olds in a London classroom. He recounts how they

were happy to compare two objects put in front of them and tell me why they had chosen the one they had [as bigger]. However, when I allocated the multilinks to them (the girl had 8 and the boy had 5) to make a tower... and I asked them who had the taller one, the girl answered correctly but the boy insisted that he did. Up to this point the boy had been putting the objects together and comparing them. He would not do so on this occasion and when I asked him how we could find out whose tower was the taller he became very angry. I asked him why he thought that his tower was taller and he just replied 'Because IT IS!' He would go no further than this and seemed to be almost on the verge of tears.

At first students try to explain the boy's answer by hypothesising that each of the girl's links was smaller than each of the boy's or that she built a wider, shorter tower. But after reading the information, they see that this could not be the case, since the girl's answer was correct. We speculate about how the culture of sexism — that boys always do better or have more than girls — blocked the knowledge of comparing sizes that the boy clearly understood in a different situation.

(c) Studying mathematical topics through deep and complicated questions

Most educational materials and learning environments in the United States, especially those labeled as 'developmental' or 'remedial', consist of very superficial, easy work. They involve rote or formulaic problem-solving experiences. Students get trained to think about successful learning as getting high marks on school or standardised tests. I argue that this is a major reason that what is learned is not retained and not used. Further, making the curriculum more complicated, where each problem contains a variety of learning experiences, teaches in the non-linear, holistic way in which knowledge is developed in context. This way of teaching leads to a more clear understanding of the subject matter.

Example

In the text below, Sklar and Sleicher demonstrate how numbers presented out of context can be very misleading. I ask students to read the text and discuss the calculations Sklar and Sleicher performed to get their calculation of the U.S. expenditure on the 1990 Nicaraguan election. (\$17.5 million \sim population of Nicaragua = \$5 per person). This reviews their understanding of the meaning of the operations. Then I ask the students to consider the complexities of understanding the \$17.5 million expenditure. This deepens their understanding of how different numerical descriptions illuminate or obscure the context of U.S. policy in Nicaragua, and how in real-life just comparing the size of the numbers, out of context, obscures understanding.

On the basis of relative population, Holly Sklar has calculated that the \$17.5 million U.S. expenditure on the Nicaraguan election is \$5 per person and is equivalent to an expenditure of \$1.2 billion in the United States. That's one comparison all right, but it may be more relevant to base the comparison on the effect of the expenditure on the economy or on the election, i.e. to account for the difference in per capita income,

which is at least 30/1 or an equivalent election expenditure in the United States of a staggering \$30 billion! Is there any doubt that such an expenditure would decisively affect a U.S. election? (Sleicher, 1990)

Goal #2: Understanding the mathematics of political knowledge

I argue, along with Freire (1970) and Freire and Macedo (1987), that the underlying context for critical adult education, and critical mathematical literacy, is ‘to read the world’. To accomplish this goal, students learn how mathematics skills and concepts can be used to understand the institutional structures of our society. This happens through:

- a. understanding the different kinds of numerical descriptions of the world (such as fractions, percents, graphs) and the meaning of the sizes of numbers, and
- b. using calculations to follow and verify the logic of someone’s argument, to restate information, and to understand how raw data are collected and transformed into numerical descriptions of the world. The purpose underlying all the calculations is to understand better the information and the arguments and to be able to question the decisions that were involved in choosing the numbers and the operations.

Example

I ask students to create and solve some mathematics problems using the information in the following article (*In These Times*, April 29–May 5, 1992). Doing the division problems implicit in this article deepens understanding of the economic data, and shows how powerfully numerical data reveal the structure of our institutions.

Drowning by numbers

It may be lonely at the top, but it can’t be boring — at least not with all that money. Last week the federal government released figures showing that the richest 1 percent of American households was worth more than the bottom 90 percent combined. And while these numbers were widely reported, we found them so shocking that we thought they were worth repeating. So here goes: In 1989 the top 1 percent of Americans (about 934,000 households) combined for a net worth of \$5.7 trillion; the bottom 90 percent (about 84 million households) could only scrape together \$4.8 trillion in net worth.

Example

Students practice reading a complicated graph and solving multiplication and division problems in order to understand how particular payment structures transfer money from the poor to the rich².

2 This situation has changed in Massachusetts, which now has a flat rate structure, and my reference did not contain real data for Michigan. So although the context-setting data are real, the numbers used to understand the concept of declining block rates are realistic, but not real.

The Rate Watcher's Guide (Morgan, 1980) details why under declining block rate structures, low-income citizens who use electricity only for basic necessities pay the highest rates, and large users with luxuries like trash compactors, heated swimming pools or central air-conditioning pay the lowest rates. A 1972 study conducted in Michigan, for example, found that residents of a poor urban area in Detroit paid 66% more per unit of electricity than did wealthy residents of nearby Bloomfield Hills. Researchers concluded that 'approximately \$10,000,000 every year leave the city of Detroit to support the quantity discounts of suburban residents'. To understand why this happens, use the graph above which illustrates a typical 'declining block rate' payment structure to (a) compute the bill of a family which uses 700 kW/h of electricity per month and the bill of a family which uses 1400 kW/h; (b) calculate each family's average cost per kW/h; (c) discuss numerically how the declining block rate structure functions and what other kinds of payment structures could be instituted. Which would you support and why?

Example

Students are asked to discuss how numbers support Helen Keller's main point and to reflect on why she sometimes uses fractions and other times uses whole numbers. Information about the politics of knowledge is included as a context in which to set her views.

Although Helen Keller was blind and deaf, she fought with her spirit and her pen. When she became an active socialist, a newspaper wrote that 'her mistakes spring out of the... limits of her development.' This newspaper had treated her as a hero before she was openly socialist. In 1911, Helen Keller wrote to a suffragist in England: 'You ask for votes for women. What good can votes do when ten-elevenths of the land of Great Britain belongs to 200,000 people and only one-eleventh of the land belongs to the other 40,000,000 people? Have your men with their millions of votes freed themselves from this injustice?' (Zinn, 1980).

Example

Students are asked to discuss what numerical understandings they need in order to decipher the following chart. They see that a recognition of how very small these decimal fractions are, so small that watches cannot even measure the units of time, illuminates the viciousness of time-motion studies in capitalist management strategies.

Samples from time and motion studies, conducted by General Electric. Published in a 1960 handbook to provide office managers with standards by which clerical labor should be organised (Braverman, 1974).

	Minutes
Open and close	
Open side drawer of standard desk	0.014
Open center drawer	0.026
Close side drawer	0.015
Close center drawer	0.027

Chair activity

Get up from chair	0.039
Sit down in chair	0.033
Turn in swivel chair	0.009

Goal #3: Understanding the politics of mathematical knowledge

Perhaps the most dramatic example of the politics involved in seemingly neutral mathematical descriptions of our world is the choice of a map to visualise that world. Any two-dimensional map of our three-dimensional Earth will, of course, contain mathematical distortions. The political struggle/choice centers around which of these distortions are acceptable to us and what other understandings of ours are distorted by these false pictures. For example, the map with which most people are familiar, the Mercator map, greatly enlarges the size of 'Europe'³ and shrinks the size of Africa. Most people do not realise that the area of what is commonly referred to as 'Europe' is smaller than 20% of the area of Africa. Created in 1569, the Mercator map highly distorts land areas, but preserves compass direction, making it very helpful to navigators who sailed from Europe in the sixteenth century.

When used in textbooks and other media, combined with the general (mis)perception that size relates to various measures of so-called 'significance', the Mercator map distorts popular perceptions of the relative importance of various areas of the world. For example, when a U.S. university professor asked his students to rank certain countries by size they 'rated the Soviet Union larger than the continent of Africa, though in fact it is much smaller' (Kaiser, 1991), associating 'power' with size.

Political struggles to change to the Peter's projection, a more accurate map in terms of land area, have been successful with the United Nations Development Program, the World Council of Churches, and some educational institutions (Kaiser, 1991). However, anecdotal evidence from many talks I've given around the world suggest that the Mercator is still widely perceived as the way the world really looks.

As Wood (1992) emphasises:

The map is not an innocent witness...silently recording what would otherwise take place without it, but a committed participant, as often as not driving the very acts of identifying and naming, bounding and inventorying it pretends to no more than observe.

In a variety of situations, statistical descriptions don't simply or neutrally record what's out there. There are political struggles/choices involved in: which data are collected; which numbers represent the most accurate data; which definitions should guide how the data are counted; which methods should guide how the data are

3 Grossman (1994) argues that 'Europe has always been a political and cultural definition. Geographically, Europe does not exist, since it is only a peninsula on the vast Eurasian continent.' He goes on to discuss the history and various contradictions of geographers' attempts to 'draw the eastern limits of "western civilization" and the white race' (p. 39).

collected; which ways the data should be dis-aggregated; and which are the most truthful ways to describe the data to the public.

Example

Political struggle/choice over which numbers represent the most accurate data. To justify the Euro-centric argument that the Native American population could not have been so great, various 'scholars' have concluded that about one million people were living in North America in 1500. Yet, other academics 'argued on the basis of burial mound archeology and other evidence that the population of the Ohio River Valley alone had been [that] great', (Stiffarm & Lane, 1992) and that 'a pre-contact North American Indian population of fifteen million is perhaps the best and most accurate working number available'. Admitting the latter figure would also require admitting extensive agricultural institutions, as opposed to the less reliable hunting and gathering. Cultivators of land are 'primarily sedentary rather than nomadic... and residents of permanent towns rather than wandering occupants of a barren wilderness'.

Example

Political struggle/choice over which definitions should guide how data are counted. In 1988, the U.S. Census Bureau introduced an 'alternative poverty line', changing the figure for a family of three from \$9453 to \$8580, thereby preventing 3.6 million people whose family income fell between those figures from receiving food stamps, free school meals and other welfare benefits. At the same time, the Joint Economic Committee of Congress argued that 'updating the assessments of household consumption needs... would almost double the poverty rate, to 24 percent' (Cockburn, 1989). Note that the U.S. poverty line is startlingly low. Various assessments of the smallest amount needed by a family of four to purchase basic necessities in 1991 was 155% of the official poverty line.

Since the [census] bureau defines the [working poor] out of poverty, the dominant image of the poor that remains is of people who are unemployed or on the welfare rolls. The real poverty line reveals the opposite: a majority of the poor among able-bodied, non-elderly heads of households normally work full-time. The total number of adults who remain poor despite normally working full-time is nearly 10 million more than double the number of adults on welfare. Two-thirds of them are high school or college-educated and half are over 33. Poverty in the U.S. is a problem of low-wage jobs far more than it is of welfare dependency, lack of education or work inexperience. Defining families who earn less than 155% of the official poverty line as poor would result in about one person in every four being considered poor in the United States (Schwartz & Volgy, 1993).

Example

Political struggle/choice over which ways data should be dis-aggregated. The U.S. Government rarely collects health data broken down by social class. In 1986, when it did this for heart and cerebrovascular disease, it found enormous gaps:

The death rate from heart disease, for example, was 2.3 times higher among unskilled blue-collar operators than among managers and professionals. By contrast, the mortality rate from heart disease in 1986 for blacks was 1.3 times higher than for whites...the way in which statistics are kept does not help to make white and black workers aware of the commonality of their predicament (Navarro, 1991).

Goal #4: Understanding the politics of knowledge

There are many aspects of the politics of knowledge that are integrated into this curriculum. Some involve reconsidering what counts as mathematical knowledge and representing an accurate picture of the contributions of all the world's peoples to the development of mathematical knowledge. Others involve how mathematical knowledge is learned in schools. Winter (1991), for example, theorises that the problems so many encounter in understanding mathematics are not due to the discipline's 'difficult abstractions', but due to the cultural form in which mathematics is presented. Sklar (1993), for a different aspect, cites a U.S. study that recorded the differential treatment of black and white students in math classes.

Sixty-six student teachers were told to teach a math concept to four pupils — two white and two black. All the pupils were of equal, average intelligence. The student-teachers were told that in each set of four, one white and one black student was intellectually gifted, the others were labeled as average. The student teachers were monitored through a one-way mirror to see how they reinforced their students' efforts. The 'superior' white pupils received two positive reinforcements for every negative one. The 'average' white students received one positive reinforcement for every negative reinforcement. The 'average' black student received 1.5 negative reinforcements, while the 'superior' black students received one positive response for every 3.5 negative ones.

Discussing the above study in class brings up the math topics of ratios and forming matrix charts to visualise the data more clearly. It also involves students who are themselves learning mathematics in reflecting on topics in mathematics education. This is another example of breaking down the dichotomy between learning and teaching, a category discussed in the above section on Understanding the Mathematics.

And, of course, Freire (1970) theorises about the politics of 'banking education', when teachers deposit knowledge in students' empty minds.

Underlying all these issues are more general concerns I argue should form the foundation of all learning, concerns about what counts as knowledge and why. I think that one of the most significant contributions of Paulo Freire (1982) to the development of a critical literacy is the idea that:

Our task is not to teach students to think — they can already think, but to exchange our ways of thinking with each other and look together for better ways of approaching the decodification of an object.

This idea is critically important because it implies a fundamentally different set of assumptions about people, pedagogy and knowledge-creation. Because some people in the United States, for example, need to learn to write in 'standard' English, it does not follow that they cannot express very complex analyses of social, political, economic, ethical and other issues. And many people with an excellent grasp of reading, writing and mathematics skills need to learn about the world, about philosophy, about psychology, about justice and many other areas in order to deepen their understandings.

In a non-trivial way we can learn a great deal from intellectual diversity. Most of the burning social, political, economic and ethical questions of our time remain unanswered. In the United States we live in a society of enormous wealth and we have significant hunger and homelessness; although we have engaged in medical and scientific research for scores of years, we are not any closer to changing the prognosis for most cancers. Certainly we can learn from the perspectives and philosophies of people whose knowledge has developed in a variety of intellectual and experiential conditions. Currently 'the intellectual activity of those without power is always labeled non-intellectual' (Freire & Macedo, 1987). When we see this as a political situation, as part of our 'regime of truth', we can realise that all people have knowledge, all people are continually creating knowledge, doing intellectual work, and all of us have a lot to learn.

Marilyn Frankenstein is one of a group of scholars and activists in the field of mathematics education from a critical perspective. She is co-founder, along with Arthur B. Powell and John Volmink, of the Criticalmathematics Educators Group (CmEG) and the author of numerous articles and books on criticalmathematics (see References below.) She is a Professor of Applied Language and Mathematics, College of Public and Community Service, University of Massachusetts-Boston.

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Curriculum Integration in the Middle School: Mathematics Meets History

Merrilyn Goos and Martin Mills

Curriculum integration is one of a number of related reforms to middle schooling currently under way in Queensland government schools as part of Education Queensland's New Basics Project. These reforms involving curriculum (New Basics), pedagogy (Productive Pedagogies), and assessment practices (Rich Tasks) are intended to draw together different disciplinary areas to enable students to tackle real world tasks in an intellectually rigorous learning environment. This paper describes how we are preparing pre-service secondary teachers to engage with these school reform initiatives by planning curriculum units and assessment tasks that integrate mathematics and history in the junior secondary school.

There is mounting evidence which suggests that the middle years of schooling are in dire need of reform (see Barratt, 1998; Queensland Board of Teacher Registration, 1996; see also QSRLS Research Team, 1999). It is important not to see this need stemming from the inadequacies of teachers but to treat it as a systemic problem which has its origins in structural conditions that are not moulded to the needs of students. However, current and prospective teachers need to be made aware of this situation and be provided with a context which enables it to be challenged. Moves towards developing integrated curricula have held significant attraction for those concerned about issues around the 'middle school' (see Beane, 1991, 1993; Brennan & Sachs, 1998; Wallace, Rennie & Malone, 2000). There is some evidence to suggest that an integrated curriculum does work to improve students' outcomes from the schooling process (Barratt, 1998, p. 18).

It is perhaps a little tautological to say that schools which most meet the needs, academic and social, of their students are those which can be considered to be genuine learning communities (Cumming 1998; Seashore Louis, Kruse & Marks 1996). Such communities are places where both students and teachers (and often parents) are considered to be learners. This was the situation we sought to model with our respective pre-service students — one group of prospective mathematics teachers and a separate group of prospective history teachers. We asked these two groups to work together to prepare a unit of work for middle school students which would meet learning outcomes consistent with the requirements of both the Mathematics and the Study of Society and Environment (SOSE) syllabuses in Queensland.

There are a variety of ways in which 'integrating the curriculum' can be interpreted. In some schools integrating the curriculum simply means developing the same themes across a variety of subject areas, whereas in others it might mean the

complete removal of subject boundaries (see Wallace et al., 2000, for a discussion of different approaches). The integrated approach we utilised here sought to break down barriers between the subject domains of mathematics and history in order to encourage the development of a teacher professional learning community amongst the pre-service teachers. Such a community within a school is, according to Seashore Louis et al. (1996) characterised by, amongst other things: reflective dialogue, collaboration and a focus on student learning (see also D'Agostino, 2000). The purpose of this paper is to consider how well integration of these two subject domains worked towards these ends.

A focus on curriculum also necessitates discussions around pedagogy. Throughout the course of the pre-service project we drew heavily upon the notion of productive pedagogies developed by the Queensland School Reform Longitudinal Study (see Hayes, Mills & Lingard, 2000; Lingard, Mills & Hayes, 2000) which is currently at the centre of school reforms in Queensland. This study has in turn informed Queensland's New Basics Project, an educational renewal program that aims to prepare students for the challenge of living in rapidly changing times (see <http://www.education.qld.gov.au/corporate/newbasics/> for more details).

Pre-service program background

The Bachelor of Education program is available to undergraduates as a four year dual degree (e.g. BA/BEd, BSc/BEd), and can also be taken as a single degree, over eighteen months of intensive study, by graduates with appropriate qualifications in two teaching areas. The curriculum specialisations are usually taught as discrete subjects — a practice that mirrors the situation in secondary schools, where disciplinary boundaries are carefully preserved. Attempting to breach these boundaries requires changes to school organisational structures that not only address timetabling and staffing arrangements, but also encourage professional dialogue between teachers in different subject areas. We argue that similar priorities apply to pre-service teacher education programs that aim to prepare graduates for new school environments. In particular, our decision to investigate curriculum integration for Junior secondary mathematics and history was prompted by our own mutual interest in cross-disciplinary dialogue, and made feasible by a BEd timetable that scheduled concurrent meeting times for our respective mathematics and history classes.

First steps

Towards the end of 1999 we planned a two hour workshop for our mathematics and history curriculum students, to find out whether pre-service teachers who would normally have little professional contact with each other could profit from working together on curriculum design tasks. After providing a brief introduction to the New Basics context, we asked students to form mixed curriculum groups and construct an outline for an integrated assessment task and four supporting lessons which required the use of mathematics and social science skills and knowledge at junior secondary

level. After one hour, groups reported back to the class on their assessment task and lessons. Topics addressed by the groups included:

- anti-Semitic propaganda during World War II
- the East Timor independence referendum
- comparison of the spread of AIDS in Australian vs African society
- casualty rates in wars in which Australia has fought.

We were sufficiently encouraged by the response to this workshop to build into our respective curriculum programs for the following year a series of similar classes leading to a group assessment task in which mathematics and history students were to produce an integrated curriculum unit.

Integrated curriculum in middle schooling project

Joint meetings of the mathematics and history classes took place during a nine week block at the start of the year, before students began their first practicum placement. (Both curriculum groups continued to meet separately throughout this period for subject-specific workshops.) These classes, which lasted one hour, were relatively unstructured and designed to provide students with time together to work on their curriculum units; however, we did address topics relevant to this task as shown in Figure 1.

Course Week	Topic/ Activity
1	The context: Education Queensland New Basics Project <ul style="list-style-type: none"> • Guest speaker (Deputy Director General, Education Queensland) • Browse New Basics website http://www.education.qld.gov.au/corporate/newbasics/
2	School Reform Longitudinal Study: <ul style="list-style-type: none"> • Productive Pedagogies observation categories and summary of findings Introduction to integrated curriculum unit assignment: <ul style="list-style-type: none"> • form groups, brainstorm possible topics
3	Groups evaluate topics, investigate teaching resources
4	Productive Pedagogies workshop (jigsaw technique)
6	Explanation of task specifications and assessment criteria and standards
8	Groups finalise structure of curriculum units, assign writing tasks
9	Due date for handing in assignment

Figure 1. Chronology of class meetings

An important feature of this project was our desire to model the kind of cross-disciplinary dialogue and intellectual risk taking that we hoped our students would embrace. This was manifested in several ways. First, we were explicit in stating the purpose of the integrated curriculum assignment as:

- to engage with school reform initiatives in Queensland

- to develop skills in planning an integrated curriculum for junior secondary schooling
- to promote professional dialogue with colleagues across curriculum areas.

Second, our planning for these class meetings was very flexible and evolved from our observations of the students at work. Even setting a unit planning task so early in the course was a risk, since students had only just been introduced to lesson planning principles and were some weeks away from beginning practice teaching.

In addition, we used questions and feedback from the students to help us design the assignment task, which was comprised of group, individual, and oral components. The unit plan was originally conceived as a group task; however, we soon recognised students' need to evaluate their individual contributions and to comment on the process of working as a cross-curricular group. To this end, we asked individuals to write a brief reflective analysis of the benefits and difficulties they experienced and how problems were dealt with in their group, and to identify implications for collaboration between teachers across different curriculum areas. The students also requested that unit plans prepared by groups other than their own be made available for sharing. We formalised this process by scheduling a two hour session during which each group was to present a short oral summary of their curriculum unit to their mathematics and history teaching colleagues. These presentations, which took place after students returned from the seven week practicum, gained an even more authentic purpose and audience when mathematics teachers from one of the New Basics trial schools were invited to participate and provide feedback on the students' work.

The curriculum units

The full list of curriculum units developed by the students is shown in Figure 2, together with an outline of the history/SOSE and mathematics subject matter dealt with by each.

The sequence of lessons in the curriculum unit was to lead to an assessment task with real world value and use that would allow junior secondary students to demonstrate the mathematics and SOSE knowledge and skills they had developed. Some of the more imaginative assessment tasks are shown in Figure 3.

A full description and evaluation of all the curriculum units is beyond the scope of this paper; however, some sense of the students' enterprise and creativity can be gained from the sample learning activities drawn from the Pyramids of Egypt unit, provided in Appendix 1.

Evaluation

We draw on three sources of information in evaluating this project — the pre-service students, practising teachers from whom we have sought feedback, and reflection on our own experience in working together while maintaining a commitment to our separate subject areas.

History first, maths second?

One of the major obstacles faced by the mathematics pre-service students was the realisation that the mathematical aspects of the curriculum unit played a secondary role to the history material. One described this as ‘feeling as if we had to let the history people come up with ideas first, so that we could build from them. It didn’t feel as if we were able to suggest maths ideas first ...’. Another reasoned nonetheless that ‘the maths needs to be a logical progression from the history’, yet it is still important ‘to give (school) students the message that both subject areas are as significant as each other’. Amongst practising teachers, these reservations surface as fears that the mathematics curriculum will be watered down, or that specialist mathematics teachers will no longer be required if integrated programs are introduced. Both of these fears rest on the assumption that generalist teachers, working as individuals, will be expected to teach cross-disciplinary units such as those prepared by our pre-service students. On the contrary, however, an integrated curriculum should *not* be taught by one person (Wallace et al., 2000) — this is a prime opportunity for *team* teaching by disciplinary specialists. In fact, subject specific expertise becomes more, not less, important, if potentially rich connections are to be made between curriculum areas.

Topic	History/SOSE content	Mathematics content
Pyramids of Egypt	When were the pyramids built? Political/social structure of ancient Egypt Geography of Egypt Hieroglyphics Mathematics of ancient Egyptians Religious/burial practices & beliefs Pyramid construction methods	Mass Ratio & proportion Plane & 3D shapes Measurement (length, area, volume, angle, time) Number study & operations Statistics
Australian post-war immigration policies	Background and motives for post-war immigration (including population analysis) Industrialisation: Snowy Mountain Scheme case study 1960s immigration and the end to the 'White Australia' policy The ethnic composition of Australia: Case study of North Queensland Cultural diversity	Percentage Ratio & proportion Statistics
Australian federal elections & opinion polling	Australian system of government How elections work The constitution and parliament Election campaigns Manipulation of statistics by media and political bias in newspapers The Liberal Party and GST policies Comparing the 1993 and 1998 elections Voting procedures	Data collection (sampling, surveys) Graphical representations of data
The medieval plagues	English society 1348–1500 (agrarian society & economy, politics & social hierarchy, medicine) Case study: the Black Death Social impact & aftermath	Data collection Data representation Analysis & prediction (mean, median, mode)
Archaeology: Investigation of a 15 th century Cossack site	Archaeological terms & concepts Excavation of a mock site Determining physical characteristics of the Cossack e.g. height Determining the age of Cossack remains	Algebra & functions Ratio & proportion Measurement
The Space Race	Early space exploration Apollo 13 mission Politics & the Strategic Defence Initiative	Trigonometry Geometry on a sphere
China	Geography (cities, rivers, climate) Politics (rise of communism, changes since death of Mao, economic growth) Culture & population (religions, population trends, one child policy) History (archaeology, Great Wall)	Percent, fractions, decimals Scale drawings Time lines Interpretation & graphical representations of statistical data

Figure 2. Integrated curriculum units — Topics and subject matter

Topic	Assessment Task
Pyramids of Egypt	You have been declared Pharaoh of Egypt! As a monument to your reign, you choose to build a pyramid in your honour. Determine resources required, list environmental impacts, forecast problems that may occur, and construct a scale model of your pyramid. Conduct a feasibility study and report on your findings.
Australian Federal elections & opinion polling	Write an article to be published in the 'Australian Government Weekly'. Analyse the issues in a specific election, including the use of statistics & opinion polls.
Australian post-war immigration policies	A new Minister for Immigration has plans to reinstate the White Australia Policy. Your advisory committee is to prepare him a briefing report on this decision.
The medieval plagues	Create a 20 minute TV current affairs or documentary program on the impact of the plagues on English society, OR 'What would we do if it happened again?'

Figure 3. Integrated curriculum units — Assessment tasks

Is full integration possible?

Figure 2 demonstrates that a wide range of mathematical concepts and skills can be brought to bear on the study of society and environment. Not surprisingly, however, it seemed that statistics provided the most relevant mathematical tools for understanding and analysing social issues. A comprehensive curriculum mapping exercise would no doubt identify fruitful connections between other mathematics topics and each of the remaining Key Learning Areas. Whether it is possible, necessary, or desirable for all the outcomes specified in current junior secondary mathematics syllabuses to be achieved through an integrated curriculum approach is an issue that has been raised by our pre-service students and by practising teachers with whom we have shared this work.

The question of the extent of integration also arises in the context of individual lessons; for example, must *every* lesson combine mathematics and SOSE? Should integrated lessons be used to *develop* new mathematical concepts, or to simply apply what students have previously learned? Our view is that an integrated curriculum should be built around rich tasks because they provide a powerful motivation for new learning — that is, a 'felt need to know'. For example, in the Pyramids of Egypt unit, students may reach the point where they need to know how to calculate the surface area and volume of the pyramid they are to build. Now the time is ripe to step back from the pyramid building project for a series of lessons focussing specifically on these mathematical requirements of the task.

Dealing with organisational constraints

One of the first questions asked by our students was how a secondary school timetable could accommodate the kind of integrated curriculum activities they were planning. How many lessons per week would be devoted to this unit — would each subject lose half of its allotted time? How long should each lesson be? Would the

history and mathematics classes be combined? What about teaching loads — how could two teachers be assigned to a single class? These are real obstacles in secondary schools, where the allocation of teachers to subjects, class times, and even staffrooms, makes cross-curricular collaboration very difficult. For the purpose of their assignment we allowed our students to assume that adequate time, resources and personnel would be available to implement their teaching plan. This was not wishful thinking on our part, but a deliberate effort to challenge assumptions about existing school structures (and parallel assumptions about the organisation of secondary teacher education). We wanted to emphasise that curriculum decisions had priority and that new organisational structures needed to be created in order for innovative teaching and learning approaches to flourish.

Cross-curricular collaboration and reflection

Each member of the mixed mathematics/history groups was forced to examine their professional values and disciplinary beliefs. Some groups, defeated by the logistics of collaboration, 'atomised the task and worked substantially in isolation'. Other individuals reported learning 'valuable lessons about diplomacy, compromise and exchange of ideas between teachers'. Many found that the initial excitement in brainstorming topics and lesson ideas turned to frustration, and then compromise, as the scope of the task became apparent. Yet what emerged was mutual respect for each other's professional abilities, and a clearer appreciation of the value of each subject. For example, one of the mathematics pre-service students commented that she 'finally began to appreciate that mathematics is instrumental in explaining and extending concepts in other areas and real world contexts. Rather than making it a lesser subject, this characteristic of mathematics is one of its greatest virtues'. A new regard for mathematics was also evident amongst the history students, one of whom noted that integration with mathematics increased the analytical focus of historical inquiry.

Conclusion — building a teacher professional learning community

Integration of two apparently disparate subject domains within a pre-service education context seemed to be successful in achieving reflective discussion, collaboration, and a focus on student learning — characteristics of a learning community that included not only our students, but also ourselves. (We note that all the comments that follow apply equally well to us as teacher educators who specialise in different curriculum areas.) The pre-service teachers simply had to talk to each other and work collaboratively, since neither mathematics teachers nor history teachers alone would have been capable of planning and teaching the integrated units. For this reason the actual teaching of the units would also have led to the deprivatisation of practice, another feature of the Seashore Louis et al. (1996) teacher professional learning community.

Discussions about curriculum, pedagogy and school organisation necessarily focussed on student learning since discipline specific assumptions and beliefs were constantly articulated and questioned. Consequently, an integrated approach works

to reconfigure conceptions about the role of subject specific knowledge within the curriculum. This seemed to have particular salience for the mathematics teachers, as comments reported in the previous section attest — although ultimately this need not diminish the intellectual commitment to the discipline as a legitimate area of study.

In addition, curriculum integration works to undermine traditional concepts of ‘teacher’, as Cumming (1998, p. 11) has noted that within such an environment ‘a teacher was seen more as a life long learner, and less as a font of all wisdom’. Within the pre-service project discussed here this recasting of the teacher worked alongside our intention to stress the importance of connecting the curriculum to the lives of students and of demonstrating to students the interconnectedness of knowledge. An integrated approach to the curriculum works towards all of these ends, as Brennan and Sachs (1998, p. 19) observe:

Rather than using a separate subject approach, the integrated curriculum introduces questions, problems and activities that will best serve as relevant learning experiences. In working in these learning situations both students and teachers are required to access both knowledge and learning strategies drawn from various disciplines and subject areas in order to discover the relevant information. In this way both the teacher and the student become challenged by and integrated into the learning process.

One measure of the success of a project such as this is the extent to which the learning community it seeks to build can be widened to include teachers and students working in authentic school contexts. The work of our pre-service students has so far met with an enthusiastic response from teachers, many of whom have obtained copies of the mathematics/history units for implementation into their own middle school programs. Clearly there is much for us to think about here in relation to ‘authentic assessment’ tasks for our pre-service teachers (Darling-Hammond & Snyder, 2000). We see continuing potential for developing teacher professional learning communities which feature collaboration between teachers and students in both school and pre-service teacher education settings.

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Appendix: Sample Learning Activities from Pyramids of Egypt Curriculum Unit

Activity #1: Where were the pyramids built?

Discuss students' impressions about the climate of Egypt (hot, dry, dusty, desert).

Elicit students' knowledge about the role, importance, and annual characteristics of the Nile River in Ancient Egypt.

Present information on annual rainfall in Cairo:

Month	Jan	Feb	Mar–Dec
mm	4.0	2.4	0.0

How does the Nile flood, when virtually no rain falls?

Investigate rainfall in the region of the Nile's sources (Addis Ababa, Ethiopia, Eritrea):

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
mm	17.1	38.2	67.5	85.8	85.5	131.5	267.8	281.1	185.6	28.4	11.3	9.7

Have students present both sets of tabulated data in histogram form, and calculate average rainfall for each season (January–March, April–June, July–September, October–December).

Does the period of maximum rainfall for the Nile's sources correspond to the period of inundation (June–September)? If not, how long does it take for the water to travel the length of the Nile?

It was during the inundation season that the peasants undertook most of the Pharaoh's building projects.

Activity #2: Size of the Pyramids of Giza

Pyramid	Side (m)	Height (m)	Base Area (m ²)
Khufu	230	146.5	
Khafre	216	140.5	
Menkaure	108	66.5	

How many Olympic sized swimming pools would fit into the base of Khufu's pyramid?

How many football fields would fit into the base of Khufu's pyramid?

If Khafre's pyramid were as tall as this classroom, how tall would you be?

Activity #3: Construction of the Pyramids of Giza

If the density of limestone is 2280 kg/m^3 , what is the total weight of Khufu's pyramid?

If the average weight of a limestone block is 2.5 tons, how many blocks comprise Khufu's pyramid? (1 ton = 1016 kg)

Khufu reigned for a minimum of 23 years. How many blocks of limestone needed to be delivered to the pyramid every hour for the pyramid to be completed within Khufu's lifetime ...

- if work continued all year round?
- if work took place only during the 3 months inundation?

About the presenters

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Mathematics and Visual Literacy in the Early Years

Rachel Griffiths

What is visual literacy? How does mathematics content and learning relate to visual literacy?

This paper explores issues relating to mathematical learning and visual literacy, and literacy, and proposes strategies for improving children's understanding of visual texts such as labelled and scale drawings, diagrams, graphs, tables, time lines, flow charts, maps.

What is visual literacy?

By visual literacy we mean the ability to interpret and to create the visual elements of texts, just as 'ordinary' literacy is the ability to read, write and understand the words in texts. These visual elements, or visual texts, include, for example, drawings and photographs, tables, diagrams, graphs, maps, time lines, flow charts. Some of these visual texts are particularly relevant to mathematics learning, for example scale drawings, graphs, tables, Venn diagrams, time lines and maps.

Literacy includes, at a basic level, the ability to read a sentence, visual literacy the ability, for example, to locate particular information in a cell of a table. But both kinds of literacy go beyond these basics, for example to include the ability to interpret, compare, make inferences and ask questions about the text, the intention of the author, the provenance of the data; and to create texts for particular audiences or for particular purposes.

The problem with graphs

There is evidence that, even at a basic level, students do have difficulty in interpreting visual texts. It is well documented, for example, that students in upper primary and secondary schools have difficulty with graphs. One difficulty is in interpretation, with many students seeing a graph as a 'picture'. Kerslake (1979), Swan (1988) and others have found that students frequently interpret a distance-time line graph as 'climbing a mountain', or 'going around corners'. Swan also found that a scattergram in which age was plotted on the vertical axis and height on the horizontal axis was widely misinterpreted as the natural tendency was to assume that 'a high point corresponds to a tall person'.

Another difficulty appears to be the limited range of graphs that students employ. When asked to sketch a graph to illustrate 'an almost perfect relationship between the increase in heart deaths and the increase in use of motor vehicles' over the past twenty years, 70% of grade 6 students drew a bar graph, and only 4% a line graph. Even in grades 9 and 10 over 30% of students chose to draw bar graphs. (Moritz, 1999)

Why these difficulties? And what can be done to remedy the situation?

Young children (and older ones too) don't often see graphs in their reading materials. Teachers can look for examples in magazines, nonfiction books, and posters. However, be wary. Many graphs, even in reputable newspapers and journals, have significant shortcomings, for example axes may not be labelled, or more attention has been paid to decoration than to clarity. While such graphs are valuable as learning opportunities ('How could you improve this graph?' 'Which part of this graph is difficult to understand?'), they do not provide good models for children who are starting out.

The choice of topic for graphs in the early years is often limited. Birthdays and eye colour seem to be a universal graphing topic for the first few years of schooling, and there is nothing wrong with these in themselves. However, teachers can widen the range, using topics from different curriculum areas such as Society and Environment, Science, Personal Development, Measurement, as well as building on the particular interests of children in the class.

While bar and column graphs provide an excellent and simple introduction to graphs, if these are the only kinds that children experience over the first few years of schooling, this may limit their choices. Teachers can provide a range of examples, such as picture-graphs, bar and column graphs, line graphs (usually measurement), even simple pie charts (to show proportions/fractions).

The method for constructing graphs in the classroom is usually highly teacher directed. Demonstrating and modelling is very important, and making a class graph to which every child contributes is valid. However, children need to construct graphs from their own data, and they need to decide for themselves how to draw axes, what scale to use, how to label the axes and so on. You can provide the structures they need to help them at their particular level of understanding (unifix or other concrete materials, squared paper, axes, tally sheets and so on), and you can also decide when they should attempt to construct a graph without these structures.

The difficulties students experience with graphs is evident in other areas of visual literacy. For example, Doig and Masters (1992) found that 42% of year 6 students in NSW were unable to identify the correct answer in a question involving reading a 3 x 2 table, relating cost of delivery to distance and weight.

Where to from here?

Just as providing good models is regarded as important in the teaching of literacy, so it is in visual literacy and in early mathematics. We need therefore to look for texts, or to create texts, which include the visual texts that we want children to become familiar with, and in which the visual texts are an important element of the reading and interpretation of the content. This is not easy. There are few published texts for young readers that include visual texts beyond illustrations. Even labelled and scale diagrams are rarely seen at this level. One program that addresses this issue is the InfoActive series (Drew, 1997, 1998, 2000; Clyne & Griffiths, 1997, 1998, 2000), that

sets out to include a wide range of visual texts and to make explicit the teaching and learning of visual literacy.

As well as exposing young children to visual texts in print materials, teachers can use opportunities that arise in both mathematics and other curriculum areas to model visual texts, and their use, explain these to children, and have children create their own visual texts. For example, children can:

- measure and graph the growth of seedlings over days or weeks;
- make time lines of their own lives, or of their activity through a day;
- draw plans and maps, of the classroom, their house, their neighbourhood, or based on a story they have read;
- record information in tables, both simple column tables and more complex row and column tables;
- create tables and diagrams to assist in the solution of problems in mathematics and in other curriculum areas.

Most important, we need to engage students at all levels in interpretation, creating and discussing visual texts, and in questioning both the content and the form of these texts. (See, for example, Moline (1995) and Griffiths & Clyne (1994) Chapter 5 for more on dealing with visual information.) Perhaps we can then look forward to a generation to whom visual literacy is a natural part of life.

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About the presenter

Rachel Griffiths has taught mathematics at every level from Prep to Year 12, as well as lecturing to tertiary education students. She has worked as a consultant in the Victorian Ministry of Education in literacy and numeracy and in mathematics, and as

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Information Texts: a Road to Numeracy and Literacy

Rachel Griffiths

Non-fiction books, magazines, newspapers, encyclopaedias and reference books all provide opportunities for children to learn about the world, learn to read and write, and learn to operate mathematically. Examples of texts and related numeracy and literacy activities are discussed, suitable for early childhood to lower secondary levels.

Numeracy, mathematics and context

Numeracy and mathematics are not identical, but obviously are intertwined. Numeracy implies using mathematics in everyday situations at home and at work, and is thus linked to mathematics in context.

We often present mathematical exercises and problems to students in a rarefied form — we give them exactly the information they need to reach an answer, and we ignore the context and purpose for the calculation. For example:

6 birds were on a tree. 3 more came. How many birds were there then?

At a higher level, you may remember the ‘stories’ about A, B and C emptying and filling baths at different rates. We never know why they were obsessed with bathwater!

Of course there are times when children need to concentrate on the mathematical processes without distraction by the context. However, we need to present mathematics more often in ‘real’ contexts, in which students need to think about the purpose of the exercise, what information is needed to solve a problem, whether the answer they reach is reasonable, or has practical value. It is the ability to use mathematics in context that defines a numerate person.

One source for these contexts is the wide range and variety of information texts that are now available, and that capture the interest of students either because of the topics they address, or because they are presented in exciting ways, or, in many cases, both. Another reason for using information texts in this way is the opportunity to integrate numeracy and literacy learning, which makes for a more efficient classroom, as well as giving a purpose to both the literacy and numeracy activities.

Information texts and literacy and numeracy skills

Learning to read or reading to learn? With information texts, the two processes are integrated. Narratives are traditionally the mainstay of reading programs, but increasing importance is being given to information texts within the literacy curriculum. The implications of this shift have not generally been worked out or

made explicit. For example, information texts require different literacy skills from narratives. This is because information texts often:

- have different language structures;
- have different layouts;
- include unfamiliar vocabulary;
- use longer words;
- include unfamiliar concepts and ideas;
- need not be read sequentially or completely;
- include book elements such as index, glossary, bibliography.

Information texts may contain mathematical data:

- in the body of the text;
- in tables, diagrams, graphs, maps, captions.

They may:

- be focused on mathematical concepts; or
- use mathematical data to support or illustrate the main ideas.

Information texts provide:

- contexts for illustrating and developing mathematical concepts;
- contexts for developing and applying mathematical skills and knowledge;
- opportunities for solving and posing problems;
- opportunities for investigations and projects.

How can all this be implemented? Some examples follow.

Information texts in the early years

InfoActive, an innovative early literacy program of information texts, provides many opportunities for integrating literacy and numeracy learning (Drew, 1997, 1998, 2000; Clyne & Griffiths, 1997, 1998, 2000).

On a first class reading of *Our Plant Diary*, which details the growth of a pea plant from seed to fruiting, children predict what may happen next, match words across each double-page spread, read the time line, interpret the scale that appears for the first time on page 8, and discuss the different ways the information is presented. This is not a process to be hurried; there is much to observe and talk about.

Further readings of the text, in a whole class or group situation, can focus on diary writing, on measurement, on plant parts, and on word wheels. Then children can grow plants from seed in the classroom, observe their growth, measure them, and record what happens using the same format as *Our Plant Diary*.

As they work with this text children are developing numeracy skills through:

- interpreting and making scale drawings;
- interpreting and making a time line;
- estimating and measuring lengths.

They will be acquiring literacy skills through:

- matching words in headings, body text, diagrams, time line, and word wheel;
- comparing information in headings, body text, diagrams, time line;
- writing a report with the same structure;
- creating word wheels;
- writing a diary.

Other titles in the *InfoActive* series provide opportunities for exploring measurement, shape and location, number, and chance & data at the same time as literacy learning.

As well as the *InfoActive* series, other examples of information texts for the early years that provide opportunities for numeracy learning can be found in the *Informazing* (Nelson), *Realization* (Rigby), and *Reading Discovery* (Scholastic) series. *Comet* magazine is another useful source, and junk mail such as catalogues also provides information that can be used for numeracy and literacy learning.

Information texts in the middle primary years

Learning about animals always interests children. The *Informazing* titles *The Book of Animal Records*, *What did you eat today?* and *Animal Acrobats* present information imaginatively and creatively. (Drew 1987, 1988, 1990; Clyne & Griffiths 1991; Griffiths & Clyne 1996a). As children work with these books, they will be:

- interpreting measurements in text and diagrams;
- estimating, measuring and comparing length, mass and time;
- making scale and life-size drawings;
- solving and posing measurement problems;
- interpreting a map.

They will be acquiring literacy skills by:

- comparing information in headings, body text and diagrams;
- using a glossary;
- using an index;
- researching and writing other records.
- researching and presenting information on other 'amazing animals'.

Other examples of information texts for the middle primary years that provide opportunities for numeracy learning can be found in the *Informazing* (Nelson), *Realization* (Rigby), and *Mathshelf Middle Level* (Scholastic) series, as well as in simple reference books, *Explore* magazine, junk mail and other environmental print.

Information texts in the upper primary and lower secondary years

At this stage, understanding topics in science, technology and studies of society and environment requires understanding of and skills in number, measurement, space, and data handling. The range of information texts that can be used expands, as newspapers, magazines, and reference books (for example *The Guinness Book of Records*, atlases and encyclopaedias) become more accessible to children whose literacy skills are now more developed. Series such as *Look Inside Cross sections* (Penguin), *Mathshelf Upper Level* (Scholastic), *Magic School Bus* (Scholastic), *Realization* (Rigby) and *Informazing* (Nelson) will stimulate children's thinking.

The article 'A High Old Time' by Stewart Wild, originally published in *Silver Kris*, the Singapore Airlines in-flight magazine, describes Big Ben, the famous London landmark, and its history (Griffiths & Clyne, 1996b). Mathematical information in the article includes number, time, length, mass, angle, shape and location. Children can be involved in:

- reading and writing Roman numerals;
- investigating the motion of a pendulum;
- investigating gears;
- estimating and comparing heights;
- measuring and comparing mass;
- interpreting maps;
- making a time line;
- investigating time around the world;
- making life-size and scale drawings of the clock face.

At the same time, children are developing literacy skills such as:

- summarising information;
- locating and extracting information;
- recording and reporting an experiment;
- using a dictionary;
- making a glossary.

Summary

Numeracy and literacy can be defined as the ability to apply mathematics and to read and write in a range of everyday contexts and practical situations. Using information texts as described in this paper will assist students to develop literacy and numeracy skills, while at the same time learning about the world around them.

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About the author

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Enrichment In Mathematics: Catering for Able Students in Years 4 to 6

Anne Joshua

In this presentation, practical ways to enrich a mathematics program will be discussed. Many examples of enrichment through challenging problems, broadening the curriculum with topics like number theory, 'mathemagic' and curiosities, experiments and constructions, investigations and open-ended problems, creative and recreational topics will be given. Throughout the session, problem solving strategies which enable students to solve non-routine problems will be emphasised. Through these problems it is hoped that children's curiosity will be aroused so that they will enjoy exploring new ideas and will be motivated to continue learning on their own teachers will be given some help to teach mathematics in a captivating way so that the creative and intellectual potential of more able students may be fully realised.

Our challenge as teachers is how we can best cater for mathematically able students and mathematically promising students in our class.

Curriculum compacting is very important when teaching talented students. It is the process of *identifying learning objectives, pre-testing students for prior mastery* of these objectives, *and eliminating needless teaching* of those aspects of the course where mastery can be documented. Students should have the opportunity to delete already mastered material from existing curriculum and to select to do only some of a set of exercises. These students need much less drill and reinforcement.

Differentiate the maths curriculum

With respect to the content

Facts, knowledge, concepts that the students must learn, as well as principles and generalisations they must develop. So the maths curriculum:

1. should provide challenging problems on all topics studied in that year, so that topics are taken far beyond the syllabus requirements. Problems should develop mathematical insight, ability and logical thought.
2. should contain non-standard problems, varied in difficulty and type, but based on work done in class. Problems with a twist or problems with multiple solutions.
3. should contain mathematics 'competition style' problems.

- 4 should include many topics that are not usually covered in the maths syllabus, creative and recreational topics, mathematical curiosities and topics that highlight the aesthetic aspect of mathematics such as triangular numbers and their applications, Knight's tour, Spirolaterals, the Fibonacci sequence, pigeon hole principle, bases, congruences, number theory, tessellations, probability, counting techniques, combinations and permutations. Games such as backgammon, hex, Dimensions and Nine-men Morris and drawing curves using straight lines only, are an excellent way to motivate students.

With respect to the process

1. The teaching methods used, the thinking skills required.
2. The creative thinking and inquiry processes that are encouraged.
3. The diversity of approaches that are encouraged and discussed when solving a problem. In this case we say the process is *open*.
4. Teamwork and cooperative learning is to be encouraged. The teacher's attitude is vital. An excellent way to start a lesson, for example is to say: 'Let's find out...'. Open-ended questions are excellent.

The spread of ability in mathematics is more marked than in many other subjects. Just to stress the difficulty faced by a teacher, the Crockroft Report suggests that for the new entry in a secondary school there is likely to be a seven year spread of ability in mathematics. In many schools, able and mathematically promising students could be up to 15% of all students. If the conventional syllabus and the one textbook is followed in the classroom, these students would be bored a great deal of the time. It is this group, this range of able students, that I would like to discuss.

It is a tremendous challenge to meet the needs of the talented maths student. There is no clear-cut line or path, which can be followed. Teachers need to adapt a variety of approaches to form a unique program that fits their particular situation.

Three conditions are absolutely essential in the development of talent among gifted students:

1. substantial time working with other gifted students;
2. challenging, advanced modified curriculum;
3. teaching methods appropriate for their level of ability.

For talented students to be identified, their talents must emerge. For this to happen, we need a combination of:

1. high motivation and a positive self esteem from the student;
2. rigorous and intellectually demanding work given in optimum environment;
3. enthusiasm and flexibility by the teacher who is ready to modify conditions to meet needs of student;
4. programs organised by the school, which cultivate and develop talent.

In purely practical terms, the most valuable resource for every teacher is a good stock of appropriate challenging problems on syllabus and non-syllabus topics. An open approach to teaching mathematics is recommended in which:

- problems to which there are a diversity of approaches (so the process is open);
- problems to which there are multiple correct answers (so the end product is open); and
- students are encouraged to formulate new problems (so the ways to create problems are open).

This *open approach* is an excellent way that teachers can cater for the able students in their class and will challenge the creative thinking of mathematically promising students.

Open-ended questions are extremely important, as the same problem can be a challenge to students of a wide range of ability. They can generate a variety of responses, all of which may be mathematically valid, differing only in the quality of understanding displayed.

Investigative work presents an open-ended situation in which students can work at their own developmental levels using learning styles that suit them. The important aspects of mathematical investigations are:

- there are no known outcomes at the beginning for the student and often for the teacher either;
- that students get to formulate their own questions and explore different possibilities. They have to ask, ‘What would happen if...?’;
- most problems can be investigated with a varying degree of sophistication and students can explore a question to the depth of their ability.

Examples of how traditional problems can be turned into open-ended ones are given below.

1. What is my question?

Write down some possible questions to which the answer is 8 and you have to

- a) add three numbers;
- b) find the difference of two numbers;
- c) use multiplication only;
- d) use any mathematical operation;

2. Make a true number sentence

- a) Write a number sentence using any three of the numbers 2, 3, 4, 6, 8 and 9 and any of the symbols +, −, × and ÷.

- b) Use four 4s and mathematical symbols to write a number sentence for the numbers from 1 to 50.

Example: $10 = 4! \div 4 + \sqrt{4}^{\sqrt{4}}$

For most of these numbers there are several solutions.

Hint: the following may be useful:

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$\sqrt{4} = 2$$

$$4^{\sqrt{4}} = 16$$

$$.4 = \frac{4}{10}$$

$$\frac{4}{.4} = \frac{4}{1} \times \frac{10}{4} = 10$$

3. Problem solving

- A bag of lollies can be divided in equal shares among 2 friends, 3 friends or 4 friends. How many lollies are in the bag?
- A bag of lollies can be divided in equal shares among 2 friends and 5 friends. How many lollies are in the bag?
- How many ways can you put five mice in two cages? Three cages?
- I am thinking of a two-digit number. It is less than 50. The sum of its digits is 10. What number is it?
- I am thinking of three numbers. Their product is 24. What three numbers are they?
- I am thinking of a two digit square number. This number is divisible by 4. What number is it?
- I am thinking of a two-digit prime number. Its unit digit is 1 (it ends in a 1). What number is it?
- I am thinking of 2 two-digit numbers. Both numbers are made up of the same two digits reversed (as in 14 and 41). They differ by 9. What two numbers are they?
- I am thinking of an even palindromic two-digit number. What number is it?
- Write at least two fractions greater than $\frac{3}{4}$.
- Write down some similarities between 49 and 121
- You have an inexhaustible supply of 5 cent and 8 cent stamps. Make a complete list of the amounts between 1c and 99c, which cannot be made.

- m) Suppose you have only 5 cent and 6 cent stamps. Make a list of which amounts can be made and which cannot be made.
- n) Write a number in the box – to make a fraction whose value is between 1 and 2.

4. Half and one quarter

Draw at least 8 identical squares and rectangles divided into grids. Then ask students to

- a) Shade one half for each shape in as many different ways that they can.
- b) Shade one quarter for each shape in as many different ways that they can.

5. Colouring in flags

How many different ways can you colour in flags with three stripes using red, blue and yellow if you can repeat colours?

6. Divisibility of numbers

Fill in the gaps so that

- a) the five-digit number 275_4 can be divided by 4 without a remainder;
- b) the five-digit number $53_ _ 7$ can be divided by nine without a remainder.

7. Continue the pattern

1, 2, 4, ...

Note that your answer can be

1, 2, 4, 8, 16, 32 ($\times 2$)

or 1, 2, 4, 7, 11, 16 (+1, +2, +3... or differences increasing by 1)

or 1, 2, 4, 5, 7, 8 (differences 1 and 2 or 2 patterns: 1, 4, 7, and 2, 5, 8..)

or 1, 2, 4, 8, 10, 20 ($\times 2, +2$)

or 1, 2, 4, 5, 10, 11 (+1, $\times 2$)

Note that three numbers may not determine a unique pattern. Other examples are

a) 1, 3, 9, ...

b) 1, 3, 6, ...

c) 2, 3, 5, ...

8. Measuring containers

- a) Jeremy has a 3 litre and a 5 litre measuring container with no measurements on it. To measure out 2 litres, he would fill the 5 litre container and pour 3 litres into the 3 litre container, writing this as $2 = 5 - 3$. How many different amounts can he measure out?
- b) If this time Jeremy has a 3 litre and 7 litre container, explain the different amounts he can measure out.

9. Nets

- a) Draw as many different nets for a cube that you can. Use large grid paper and check carefully by rotating and reflecting your shapes that they are different.
- b) Repeat the above for a rectangular prism.

10. Area and perimeter investigations

- a) The perimeter of a rectangle is 24 cm. Investigate the shape of all such rectangles if the sides are integral. Find the dimensions with largest area.
- b) The area of a rectangle is 24 square metres and its length and width each measure a whole number of metres. How many different rectangles can you draw? Which shape gives you the smallest perimeter?
- c) Draw some shapes, which have an area of 4 square units
 - i) using horizontal and vertical lines only;
 - ii) where diagonal lines are permitted.
- d) What is the maximum number of cards measuring 4 cm by 3 cm that can be cut from a piece of cardboard 12 cm by 9 cm? Do this question in two different ways.

11. Space

- a) A block of cheese 3 cm by 4 cm by 5 cm is covered with wax. If the cheese is cut into one centimetre cubes, how many cubes will
 - i) not have wax on them?
 - ii) have wax on 3 faces?

Investigate by considering different size cheese blocks.

Form a generalisation.

- b) In how many different ways can you divide a 4 by 4 square into four congruent parts?
- c) With 24 cubes, build a rectangular prism. How many different prisms can you make?

- d) Place 6 stars in a 3 by 3 grid and 8 stars in a 4 by 4 grid so that there will be 2 stars in each row and each column.
- e) Draw or name some solids which when cut will always give you a circular face.
- f) What solids can you have if all cuts need not produce the same size circle?

12 Axis of symmetry

Draw some shapes which have

- a) 1 axis of symmetry.
- b) 2 axes of symmetry.

13. Find my rule

a)

x	6	11	1	8	17	24	3	29	0	15
y	24	19	29	22						

b)

x	1	4	9	20	7	11	5	7	10	0
y	2	20	90	420						

c)

x	8	5	2	11	1	3	10	1	7	9
y	29	17	5	41						

d)

x	8	14	5	1	4	11	7	12	20	2
y	5	8	$3\frac{1}{2}$	$1\frac{1}{2}$						

14. Counting techniques

Mrs Kelly has four cards each with one of the numbers 3, 5, 7, and 8 on them.

3	5	7	8
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- a) How many different 2 digit numbers can she make?
- b) How many different even 2 digit numbers can she make?
- c) How many different 3 digit numbers can she make?
- d) How many different even 3 digit numbers can she make?

- e) How many different odd 3 digit numbers can she make?
- f) How many different 2 digit numbers which are divisible by 5 can she make?
- g) How many different 3 digit numbers which are divisible by 5 can she make?
- h) How many different 2 digit numbers which are divisible by 3 can she make?

15. Mathemagic

Think of a number

Add 3

Multiply by 2

Add 4

Divide by 2

Subtract the number you first thought of

What is your answer?

Repeat with different starting numbers.

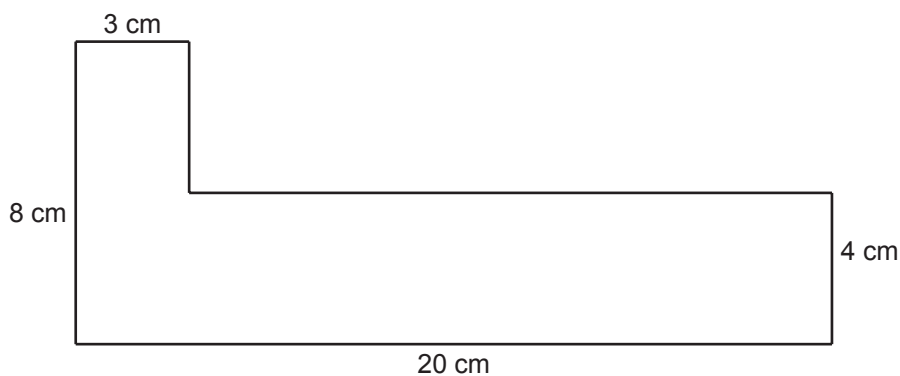
What do you find?

Make up some Mathemagic problems yourself.

16. Open process

Most problems can be solved in a number of ways, and teachers should encourage this open process.

- a) Find the sum of the first 1000 odd integers.
- b) Find the perimeter and area of the figure below.



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About the presenter

Anne Joshua is a mathematics teacher at Moriah College, Sydney, where she has a special interest in both remedial and gifted and talented students. She teaches able students at primary and high school level and is the coordinator of the talented program in mathematics. Anne has conducted numerous workshops for able maths students, and has given a number of inservice courses at both Maths Association Conferences and Gifted and Talented Conferences.

Anne is the author of the series of books: *Introducing Enrich-e-matics* (Books 1 to 3), *Enrich-e-matics* (Books A to E, Teacher's Books with each series) and *Enrichment Maths for Secondary School Students*. These books provide a wide range of extension mathematics material for the age group 5 to 15.

Anne obtained her M.A. Dip.Ed. from Sydney University and her M.Sc. from Oxford University.

Open-ended Questions and Investigations for Years 7 to 10

Anne Joshua

Open ended or free response questions provide the opportunity for students to demonstrate their level of understanding of a particular topic. They are excellent questions as they cater for able students as well as students with learning difficulty. In this session many examples of useful questions on all topics for junior years will be discussed.

To foster students' mathematical thinking, it is extremely important to give them freedom to use their own mathematical ways of thinking.

An open approach to teaching mathematics involves

- solving a problem to which there are a diversity of approaches, so the process is open;
- solving problems to which there are multiple correct answers to so the end product is open; and
- encouraging students to formulate new problems, so the ways to create problems are open.

This *open approach* will challenge the creative thinking of mathematically promising students. It can provide an excellent way that teachers can cater for the able students in their class.

Open-ended questions are extremely important, as the same problem can be a challenge to students of a wide range of ability. They can generate a variety of responses, all of which may be mathematically valid, differing only in the quality of understanding displayed. Teachers can encourage the most elegant solutions and can challenge students to find the solutions that require the deepest thinking, patterns and generalisations.

Investigative work presents an open-ended situation in which students can work at their own developmental levels using learning styles that suit them. The important aspects of mathematical investigations are:

- there are no known outcomes at the beginning for the student and often for the teacher either;
- that students get to formulate their own questions and explore different possibilities — they have to ask, "What would happen if...?";
- most problems can be investigated with a varying degree of sophistication and students can explore a question to the depth of their ability.

Examples of how traditional problems can be turned into an open-ended one are given below.

Number

1. Find at least two positive integer values for q for which
 - a) $588q$ is a perfect square.
 - b) $540q$ is a perfect cube.
2.
 - a) The average of 3 numbers is 7. Write down some possible numbers.
 - b) The sum of three consecutive odd numbers is 123. Find the numbers in as many ways as you can.
3.
 - a) The average of 3 consecutive numbers is 7. What are the numbers?
 - b) The average of 5 consecutive numbers is 7. What are the numbers?
 - c) The average of 7 consecutive numbers is 7. What are the numbers?
 - d) The average of 9 consecutive numbers is 7. What are the numbers?
 - e) Investigate and then form some generalisations using the above results and prove them using algebra.
 - f) Will the sum of 5 consecutive integers always be a multiple of 5?

Fractions

1. Find two fractions whose
 - a) sum is 1
 - b) sum is $\frac{5}{2}$
 - c) difference is $\frac{3}{4}$
 - d) product is 1
 - e) product is $\frac{2}{9}$

Solve these problems in several different ways.

2. Find two numbers given

	sum	product
a)	$\frac{5}{2}$	1
b)	$\frac{10}{3}$	1
c)	1	$\frac{2}{9}$
3. Find two numbers whose sum is 1 and whose difference is $\frac{1}{3}$.
4. Which is larger, $\frac{3}{5}$ or $\frac{2}{3}$?

5. The answer is $\frac{5}{8}$. What is my question if I am
- adding 2 fractions?
 - subtracting 2 fractions?
 - multiplying 2 fractions?
 - dividing 2 fractions?
6. Find 10 fractions between and.
7. Select two of the numbers from 4, 5, 6, 7, 8, 12, 11, 13, 14, 15, 18, 19, 21 and place them in the boxes

$$\frac{\square}{\square}$$

to make a fraction that simplifies to $\frac{1}{3}$.

8. A and B are two different numbers selected from the first ten counting numbers from 1 to 10 inclusive. What values can $\frac{A+B}{A-B}$ have? What is the largest value for this fraction?

Order of operations

1. Write the numbers from 1 to 10 using only 3 threes and any mathematical symbol you like.

Example: $0 = 3 - \sqrt{3} \times \sqrt{3}$

Note $3! = 3 \times 2 \times 1$
 $= 6$

and $.3 = \frac{3}{10}$ and $\frac{3}{.3} = 3 \times \frac{10}{3} = 10$.

These are some possibilities

$$1 = \left(\frac{3}{3}\right)^3 \qquad 1 = 3! \div (3 + 3)$$

$$2 = 3 - \frac{3}{3} \qquad 2 = (3 + 3) \div 3 \qquad 2 = 3! \div (\sqrt{3} \times \sqrt{3})$$

$$3 = 3 + 3 - 3 \qquad 3 = 3 \times 3 \div 3$$

$$3 = \frac{3}{.3} \times .3 \qquad 3 = (3! + 3) \div 3 \qquad 3 = 3! - (\sqrt{3} \times \sqrt{3})$$

$$4 = 3 + \frac{3}{3} \qquad 4 = 3! - \frac{3!}{3}$$

$$5 = 3! - \frac{3}{3} \qquad 5 = (3! \div 3) + 3$$

$$6 = 3 \times 3 - 3 \qquad 6 = 3! + 3 - 3$$

$$7 = \frac{3}{.3} - 3 \qquad 7 = 3! + 3 \div 3$$

$$8 = 3! + \frac{3!}{3} \qquad 8 = \left(\frac{3!}{3}\right)^3$$

$$9 = \sqrt{3} \times \sqrt{3} \times 3 \qquad 9 = 3^3 \div 3$$

$$9 = 3 + 3 + 3 \qquad 9 = 3! + \sqrt{3} \times \sqrt{3}$$

$$10 = \frac{\sqrt{3} \times \sqrt{3}}{.3}$$

2. Using four 7s and mathematical symbols write as many numbers as you can.

Some examples are:

$$2 = (7 + \sqrt{7} \times \sqrt{7}) \div 7$$

$$20 = \frac{7}{.7} + \frac{7}{.7}$$

$$4 = \frac{77}{7} - 7$$

Decimals

- Write down 5 numbers that round to. Describe all these numbers.
- Find two numbers whose product is.
- What is the 80th digit to the right of the decimal point in the decimal form of?
Can you generalise?
- In the decimal number system, 0.1 means one-tenth.
What do you think 0.1 means in base 2? base 3? base 4? etc.
What do you think 0.01 means in the above bases?

Highest common factor and lowest common multiple

- The highest common factor of 28 and another number is 4.
Write down some possibilities for the other number.
- Given the HCF and the LCM of two terms, try to predict the two expressions in each case.
 - HCF = 6 LCM = 90
 - HCF = $6ab$ LCM = $36a^2b^2$
- When a number is divided by 2, 3, 4, 5 and 6, there is a remainder of 1.

Write down 5 numbers that satisfy this condition, including the smallest.

4. I am thinking of a number.

The least common multiple of my number and 9 is 45.

What could my number be? (There are three possibilities.)

5. a) Investigate the conditions so that the LCM of two numbers is:
- one of the numbers?
 - the product of the two numbers?
- b) Investigate the conditions so that the HCF of two numbers is one of the numbers.

Factors

1. The number 6 has 4 factors: 1, 2, 3, 6; while the number 11 has only 2 factors: 1 and 11.

The factors of 24 are 1, 2, 4, 8, 3, 6, 12, 24.

To determine the number of factors of a number, we must express the number as the product of its prime factors and then write it in index form. If the prime factorisation of a number is $p_1^a p_2^b p_3^c \dots$ where p_i are all prime, then the number of factors is $(a + 1)(b + 1)(c + 1) \dots$

$$24 = 2^3 \times 3 \text{ so has } 4 \times 2 = 8 \text{ factors}$$

$$75 = 3 \times 5^2 \text{ so has } 2 \times 3 = 6 \text{ factors}$$

$$900 = 2^2 \times 3^2 \times 5^2 \text{ has } 3 \times 3 \times 3 = 27 \text{ factors}$$

Investigate the truth of the following statement " all perfect squares have an odd number of factors, whereas all other positive integers have an even number of them".

2. Find numbers that have
- 3 factors
 - 5 factors
 - 7 factors
 - 4 factors
 - 6 factors

Algebra

1. Write down some possible questions if :
- my answer is $-5x$
 - my answer is $2x^2$

2. The numbers 4 and $1\frac{1}{3}$ have the property that their sum is equal to their product. How many others can you find?
3. Investigate each of the following and prove your findings algebraically.
 - a) Difference between the squares of two odd numbers.
 - b) Difference between the squares of consecutive even numbers.
 - c) Difference between the squares of consecutive odd numbers.
4. The number $x = \frac{5}{3}$ is algebraic because it satisfies the algebraic equation $3x - 5 = 0$. Show that each of the following is algebraic by writing down a suitable equation with integral coefficients.
 - a) 7
 - b) $-\frac{1}{2}$
 - c) $\sqrt{3}$ or $-\sqrt{3}$
 - d) $\sqrt[3]{5}$

5. The squares of two consecutive integers differ by 99. What is the sum of these integers? What are the integers? Make up two similar examples.
6. The sum of two numbers is 10 and their product is 20. Find the sum of their reciprocals.

(Hint: let the two numbers be x and y . Write down what you are given and what you are looking for.) Make up a similar problem.

7. Mathematics is amazing!
 - Start with any number
 - Double the number
 - Multiply the number by itself
 - Divide the number by itself
 - Add the last three of these answers together.
 - What have you found?

Investigate different starting numbers, express in words your findings.

Prove your result algebraically.

Make up some Mathemagic problems yourself.

$$8. \quad \left(2\frac{1}{2}\right)^2 = 6\frac{1}{4} = 2 \times 3 + \frac{1}{4}$$

$$\left(3\frac{1}{2}\right)^2 = 12\frac{1}{4} = 3 \times 4 + \frac{1}{4}$$

$$\left(4\frac{1}{2}\right)^2 = 20\frac{1}{4} = 4 \times 5 + \frac{1}{4}$$

Write down the next three terms in this pattern, noting the mathematical shortcut and then prove the generalisation using algebra.

9. Find the missing term so that each of the following expressions will factorise.

$$a) \quad x^2 - 3x - \square \qquad b) \quad x^2 - x - \square \qquad c) \quad x^2 - \square x - 36$$

Indices

- Express $4^4 + 4^4 + 4^4 + 4^4$ in the form of. Write down the sum of other expressions that can be simplified to the above form.
- Which is larger:
 - $2^8 + 2^9$ or 2^{10} ?
 - 2^{15} or 3^{10} ?

Watch the different ways that students use to solve each of these problems and then ask them to make up similar ones.

Measurement

Ratio and rates

- What could the following rates be used to measure?

a) L/min	e) \$/m	i) kg/m^2
b) mm/day	f) \$/hectare (ha)	j) \$/day
c) \$/kg	g) Kg/month	
d) $\text{\$/m}^2$	h) m/min	
- What scale would you chose if you wanted to make scale models of
 - your room?
 - your school?
- The ratio of the number of boys to the number of girls is 5:4.
 - Make up five questions using this ratio for a test for your class.
 - If each boy is given 3 stickers while each girl is given 5 stickers, a total of 210 stickers are needed. How many children are there? Do this question in at least two different ways.
- It is not valid to find the average of two Speeds. This can be demonstrated with an extreme example. Suppose the Lee family travelled from Cooma to Canberra at 90 km/h and that their overall average speed there and back was 45 km/h. What was their speed on the return journey?

It is tempting to say it must have been 0 km/h, since $(90 + 0) \div 2 = 45$.

But if they travelled at 0 km/h they would never have left Cooma.

Note that the average speed for the round trip is the harmonic mean of the two average speeds. So the average speed for the whole journey is $\frac{2ab}{a+b}$, if the average speed from Town A to Town B is a km/h and the average speed from Town B to Town A is b km/h.

Area

1. A rectangular bathroom floor is tiled with grey square tiles with a single border of black square tiles along the edges. If on a floor there are 14 black tiles along the length and 8 tiles along the width, find the total number of grey tiles.
Investigate different shapes and form a generalisation for p black tiles along the length and q black tiles along the width.
2. Using square paper,
 - a) draw at least 5 triangles whose area is 12 cm^2 .
 - b) draw at least 2 parallelograms whose area is 12 cm^2 .
 - c) draw at least 2 trapeziums whose area is 12 cm^2 .
 - d) draw at least 2 L shapes whose area is 12 cm^2 .
 - e) draw some other shapes whose area is 12 cm^2 .
3. Show that the triangle whose sides are 5, 5, 6 has the same area as the triangle whose sides are 5, 5, 8. Find other pairs of isosceles triangles with integral sides whose areas are equal.

Volume

1. Name some shapes and their dimensions whose volume is 24 cm^3 .
2. The frame of a rectangular box of volume 24 m^3 is to be constructed using interlocking 1 m tubes. How many tubes are required?
3. What is the volume of a rectangular prism in which the length and breadth and height add to 20 cm and the sides are integral?
 - a) Which shape has the least volume?
 - b) Which shape has the maximum volume?
4. Design a container with
 - a) a capacity of 375 ml
 - b) a volume of 600 cm^3
5. Using interlocking red centicubes a 5 cm by 3 cm by 2 cm solid block was made. This block was then entirely covered with yellow cubes, so that a new solid was formed.
 - a) Find the dimensions of the new yellow solid.
 - b) Find the number of yellow cubes use to cover the red block.
 - c) If the original red solid is p cm by q cm by r cm, write down an expression for the new yellow solid and the number of yellow cubes used.

Surface area investigations

1. For this activity you will need a large handful of centicubes.
Form as many different solids using four cubes attached as you can. Check that your solids are different, i.e. if you rotate or reflect them, they are not the same.
Investigate which solid has the
 - a) the greatest surface area;
 - b) the least surface area and find its value in each case.
2. Repeat the above exercise using
 - a) 6 cubes
 - b) 8 cubes
3. Investigate what happens to the surface area of
 - a) cubes when the length of each side is :
 - b) rectangular prisms when the length of each side is:
 - i) doubled
 - ii) trebled
 - iii) multiplied by a factor of k
4. Using 27 centicubes, build a cube with 3 cm sides.
 - a) Find the surface area of this cube.
 - b) Remove one centicube from a corner.
Find the surface area of this solid now.
 - c) Remove one centicube from a face but not a corner.
Find the surface area of this solid now.

Space

1. Find possible dimensions for each rectangle if the diagonal is 13 units.
2. A rectangular prism is 10 cm long, 5 cm wide and 8 cm high.
Describe and draw the lines, which have a length of $\sqrt{10^2 + 5^2 + 8^2}$.

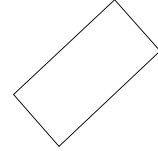
Pythagoras

- 1
 - a) Prove that a triangle with sides $2n$, $n^2 - 1$ and $n^2 + 1$ is right angled.
 - b) We can generate Pythagorean triplets by substituting various values of n in the above triplet. Draw a table for values of n from 2 to 12.
- 2
 - a) Prove that a triangle with sides $m^2 - n^2$, $2mn$, $m^2 + n^2$ is always right-angled.

- b) We can generate further Pythagorean triplets by substituting various values of m and n in the above triplet. Draw a table for these values of m and n .

How many different relations can you find?

3. An isosceles right triangle is removed from each corner of a square piece of paper so that a rectangle remains. What is a length of a diagonal of the rectangle if the sum of the areas of the cut-off pieces is 50 cm^2 . Solve this problem in two different ways.



4. Using the sides of a right triangle as bases, draw 3 similar figures. Is it still true that the area of the largest figure equals the sum of the area of the two smaller figures?

Co-ordinate geometry

1. How many solutions (x,y) , where x and y are integers can you find satisfying the following equations

- a) $x + y = 5$ b) $2x - y = 4$
 c) $2x + 3y \leq 6$ d) $x^2 + y^2 \leq 16$

2. If $A (0,0)$ and $B (6,0)$ are the base points of an isosceles triangle,

- a) Find the co-ordinates of the third vertex.
 b) How many answers can you find?

If the area of this triangle is 12 units squared, find the co-ordinates of the third vertex.

3. If $(0, 0)$, $(1, 5)$ and $(5, 5)$ are three points of a parallelogram, find the co-ordinates of the fourth point. How many different solutions can you find?

4. If $ABCD$ is a rhombus and $A (0,1)$, $B (6, 5)$ and $C (2, -1)$, find the coordinates of D using at least 2 different methods.

5. Give several examples of pairs of equations that would have

- a) no point of intersection
 b) 1 point of intersection
 c) 2 points of intersection
 d) 3 points of intersection
 e) an infinite number of points of intersection.

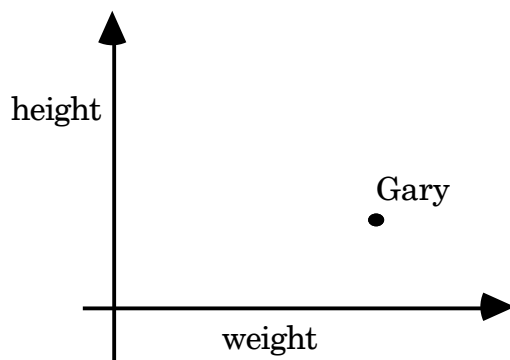
Chance and data

- Describe a problem for which the probability of the event occurring is
 - 1
 - $\frac{1}{2}$
 - $\frac{1}{13}$
 - 0
 - $\frac{1}{1000}$
- Describe a problem for which the answer is
 - $6!$
 - $2 \times 2 \times 2$
 - $5 \times 4 \times 3$
 - 4^3

Statistics

- The mean of a group of 4 numbers is 5. What might the numbers be?
 - The range of a group of numbers is 8 and the mode is 5. What might the numbers be?
- Describe a sample for which the median is
 - 2 more than the mean;
 - \$4000 more than the mean.

Graphs



Gary, Sam and Alex are friends.

Alex is taller but weighs less than Gary.

Sam weighs the same as Gary but is heavier.

On the graph mark a point A and a point S which could represent Alex and Sam.

Find the fallacy

Each of the following starts with a true statement. Then at some stage, it becomes false. Find the flaw in the following mathematical fallacies.

- If $a = b$

$a^2 = ab$	multiplying by a
$a^2 - b^2 = ab - b^2$	subtracting b^2
$(a - b)(a + b) = b(a - b)$	
$a + b = b$	dividing by $a - b$
$b + b = b$	substituting $a = b$
$2b = b$	
$2 = 1$	dividing by b

2. $11 > 8$
 $-4 > -7$ Subtract 15 from both sides
 $16 > 49$ Square both sides
3. If $f(n) = n^2 - n + 41$
 $f(1) = 1 - 1 + 41 = 41$ a prime number
 $f(2) = 4 - 2 + 41 = 43$ a prime number
 $f(3) = 9 - 3 + 41 = 47$ a prime number
 $f(4) = 16 - 4 + 41 = 53$ a prime number
 $f(5) = 25 - 5 + 41 = 61$ a prime number
 $f(6) = 36 - 6 + 41 = 71$ a prime number
 $f(7) = 49 - 7 + 41 = 83$ a prime number, and so on...
 therefore
 $f(n) = n^2 - n + 41$ is always prime
4. If $x = 4$
 $x^2 = 16$ squaring both sides
 $x^2 - 4x = 16 - 4x$ subtracting $4x$ from both sides
 $x(x - 4) = 4(4 - x)$
 $x(x - 4) = -4(x - 4)$
 $x = -4$ dividing by $x - 4$
5. If $x < y$
 $x(x - y) < y(x - y)$ multiplying by $x - y$
 $x^2 - xy < yx - y^2$
 $x^2 - 2xy + y^2 < 0$ bringing all terms to the LHS
 $(x - y)^2 < 0$

Extension

Counting techniques

It may be a help to use tree diagrams in solving these problems.

1. A home safe has a four digit combination lock.
 - a) How many different combinations are possible?
 - b) If the combination must have 4 different digits, how many combinations are there?
 - c) If a combination never begins or ends with 0 or a 9, how many combinations are there?
2. Work out how many two-digit numbers can be formed from the digits 3, 4 and 5:
 - a) if repetition of the digits is not allowed;
 - b) if repetition is allowed.

Different bases

Studying other bases provides an excellent stimulus for enrichment as it contributes immeasurably to the internalisation of our own numeration system.

1.
 - a) Convert 25_{10} to base 5
 - b) Convert 625_{10} to base 5
 - c) Convert 36_{10} to base 6
 - d) Convert 729_{10} to base 9
 - e) Convert 10000_{10} to base 100
2.
 - a)
 - i) Convert 121_{10} to base 11, 9 and 8
 - ii) Convert 121_{11} to base 10, 12, 9 and 8
 - iii) Convert 144_{12} to base 10, 11, 13 and 8
 - b)
 - i) Continue the pattern

$$121_3 = 9 + 6 + 1 = 16$$

$$121_4 = 16 + 8 + 1 = 25$$
3.
 - a) Investigate the truth of the following statement:
The number 121_b is the square of an integer for all b (where b is the base).
 - b) What statement can you make about 144_b ?
 - c) Write down any number in base b , that you are certain is a square for some values of b .
 - d) Show that 1331_b is a cube for $b > 3$.

4. a) Show that for all $b > 5$, $121_b = 100_{b+1} = 144_{b-1}$
 b) Show that for all $b > 4$, $144_b = 121_{b+1} = 100_{b+2}$
5. a) Can you make up some general rules for each of the above patterns?
 b) Can you find other similar patterns?

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About the presenter

Anne Joshua is a mathematics teacher at Moriah College, Sydney, where she has a special interest in both remedial and gifted and talented students. She teaches able students at primary and high school level and is the coordinator of the talented program in mathematics. Anne has conducted numerous workshops for able maths students, and has given a number of inservice courses at both Maths Association Conferences and Gifted and Talented Conferences.

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Parent Perceptions of the Teaching and Learning of Primary School Mathematics

Paulene Kibble

This paper reports on a study in which the writer aims to gain some insight into the nature of parent perceptions of the teaching and learning of primary school mathematics and what these insights can suggest in improving parent involvement in the mathematics learning of their children.

Data from the parents of five families, each with at least one child attending a public primary school in the Australian Capital Territory (ACT), were used.

The writer argues that parents believe they have a role to play in the mathematics education of their children. Parents recognise that there has been a change in the nature of school mathematics since they went to school and that they need assistance from teachers to enable them to contribute in a positive way to maximise mathematics learning outcomes for children.

Background

Who succeeds and who fails at school is being decided outside the school, primarily by family factors. Family factors outweigh school factors in determining educational success (Eastman, 1989, p. 19).

Parents can be a vital resource in the education of their children. Before this resource can be effectively utilised it is necessary to understand parent perceptions of schooling, and how these perceptions influence children's learning.

A generation has passed between the time of the parents' schooling and that of their children. In that time there have been considerable changes in the philosophy of mathematics education, mathematics content knowledge, mathematics pedagogy, the role of parents in education and the relationship between children and teachers.

Studies undertaken, particularly during the last twenty years, have led to a greater understanding of how children learn mathematics. Research suggests that the vast majority of children are capable of achieving at mathematics (Ginsburg & Allardice, 1984).

Concerns that a large number of students were not achieving success in mathematics led to the development, in Australia, of a national position statement on mathematics education, The National Statement on Mathematics for Australian Schools. This Statement acknowledges that,

Mathematics pervades all aspects of our lives... we need to aim for improvement in both access and success in mathematics for all Australians (Australian Education Council, 1991, p. 1).

The policy of the ACT Department of Education and Training is for school based curriculum development. The Mathematics Curriculum Framework of the ACT Department of Education and Training (ACT Department of Education and Training, 1994) is used as a guide by schools for curriculum documents. A National Statement on Mathematics for Australian Schools provides a basis for the Mathematics Curriculum Framework of the ACT.

The nature of schooling has changed since the Second World War and is reflected in a change from the traditional model to a situation today where greater school based management requires negotiation between the school and its community on school structures and curriculum. The traditional structures of schooling and curriculum were non-negotiable, with parents delivering their children into the care of the state (Kallantzis, Cope, Noble & Poynting, 1990, p. 242).

The greater democratisation of the schooling process requires, in theory, parents to be conversant about what is happening in schools so that they can participate in all those aspects of schooling which contribute to children's improved learning outcomes.

The ACT Strategic Education Plan 1995–1998 (ACT Department of Education and Training, 1995) has, as a planned outcome from the goal, to promote dynamic learning communities,

...parents participating in the education of their children and involved in their children's progress and welfare.

This reflects recognition of the need to develop a shared purpose between the child, parents and the school.

Dramatic changes in conceptions of mathematics content, and models of teaching and learning, have occurred since many parents were at school (Australian Education Council, 1991; NCTM, 1989). The mathematics that most parents experienced was characterised by fixed sets of facts and procedures with the teacher as expert and the answers to all problems available in textbooks. Students learnt by listening to the teacher and practising examples. That experience is not what would be considered appropriate in today's schools.

It is now acknowledged that students possess prior knowledge, which shapes new learning, and that learning takes place through active involvement of the student. In *Everybody Counts* (National Research Council, 1989 p.58) it is claimed that,

No one can teach mathematics. Effective teachers are those who can stimulate students to learn mathematics. ...Students learn mathematics well only when they construct their own mathematical understanding.

This construction of knowledge takes place in a social context (Clements, 1997).

Although considerable research has taken place which supports current beliefs about the nature of mathematics and mathematics learning (Grouws, 1992), it remains that conflicting views exist among professionals and practitioners as to the teaching and learning of mathematics. Dossey (1992, p. 42) argues that the conception of mathematics held by the teacher is critical to teaching practice. Raymond (1997)

shows that teaching practice can be at odds with beliefs about the nature of mathematics. That,

...deeply held, traditional beliefs about the nature of mathematics have the potential to perpetuate mathematics teaching that is more traditional, even when teachers hold non traditional beliefs about mathematics pedagogy.

Ernest identifies three conceptions of mathematics:

- as dynamic problem solving
- a static body of knowledge
- a bag of tools

which can be included in any one teacher's concept of mathematics but which, at the same time can be contradictory (Thompson, 1992, p. 132).

There is no reason to doubt that parents, equally, hold such conflicting views. Preliminary discussions with parents would support this view.

Several factors work together to constitute teacher knowledge in the mathematics classroom. Knowledge of subject matter, pedagogy, student thinking and teacher beliefs all work together to form a complex and dynamic learning environment.

Teachers have to take their complex knowledge and somehow change it so that their students are able to interact with the material and learn. This transformation...is continuous and must change as students who are being taught change (Fernema & Franke, 1992, p. 162).

Parents may have varying knowledge of subject matter and pedagogy but they have a great deal of knowledge of the child.

Parent involvement in education

There are many activities that constitute parent involvement and their common characteristic is that they in some way

bring together the separate domains of home and school (Jowett & Baginsky, 1988, p. 37).

The right to a quality education for children is the principle, which underlies the argument for parent involvement in education. Home related factors account for the greatest proportion of variability that occurs in student achievement (Cairney, Ruge, Buchanan, Lowe, and Munsie, 1995; Bastiani, 1993; Eastman, 1989).

Over 85% of a child's waking hours, up to the child's age of 16, is spent outside a formal school environment where parents are the most significant adult influence. Parents have been educating their children for many years before these children attend formal educational institutions. Schools offer specialist facilities and expertise to support families but do not replace the family once children reach school age. Through their continued influence parents often determine the effectiveness of teachers in schools (Macbeth, 1993).

The language used to describe the most effective role parents have in relation to their children's schooling is that of partnership. This relationship between families and schools carries different connotations dependent on the perspective of the user, be they parent, education professional, public administrator or politician. Characteristics of partnership (Bastiani, 1993) include:

- sharing of power, responsibility and ownership — though not necessarily equally;
- a degree of mutuality, which begins with a process of listening to each other and incorporates responsive dialogue and 'give and take' on both sides;
- shared aims and goals, based on common ground, but which also acknowledges important differences;
- a commitment to joint action, in which parents, pupils and professionals work together to get something done.

Russell (1991) contends that while, at the preschool level, there has been an emphasis on strategies to achieve parental involvement, less attention has been given to characteristics of parents, such as attitudes and beliefs or values, which may determine such involvement. His hypothesis, that parental beliefs about their role in the education of their children influence the amount and type of actual involvement, was confirmed. Parent's lack of awareness of the potential or significance of their role as an educator of their child needs to be overcome and this would occur if beliefs, similar to those held by Group 1 parents (see following), were achieved. The categories Russell used are:

- Group 1: parent-teacher partnership in education — parents believe that parents play a major role in their children's education. Education is defined broadly, giving equal status to learning, which takes place at home, preschool and school, with parents and teachers holding similar status.
- Group 2: parent role as subordinate and to support teachers. A difference in importance given to parents compared to teachers. Focus is on parents' role with regard to discipline and behaviour.
- Group 3: no educational role for parents but to support teachers. Education is defined by these parents as what took place at preschool and school. Children's educational progress is seen as the responsibility of teachers.
- Group 4: no educational role for parents. Education is seen as the job of trained professionals. The parents' role is to select a good institution and the parents do not have a part in children's learning.

While Russell warns that results from this preschool study cannot be assumed to apply to other areas of school education, his four categories for classifying parent beliefs about their role in children's education are useful for this study.

Barriers to parent involvement

Greenwood & Hickman (1991) identified several barriers to parent involvement in education. These included:

- the attitudes and abilities of parents — some parents do not value education and do not see schools as *'a place of hope'* while others feel powerless to influence schools. Still other parents believe education should be left to the experts;
- work and poor health of parents;
- attitudes, knowledge and skills of teachers and administrators — some professionals don't know how to involve parents while others prefer to see parents in traditional activities such as fundraising rather than as decision makers.

The researchers concluded, that parents need to feel welcome and be treated respectfully.

Parents and mathematics education reform

Problems, which can develop if parents aren't adequately informed and consulted, are detailed by Dillon (1993). In a study involving the implementation of teaching practices consistent with reform recommendations (NCTM 1989), at the second grade level of schooling, parents almost brought the project to a halt because of their concerns that their children were not getting *'the best education possible'*. Some parents, who had not attended the initial information session about the project, became concerned when their children brought home school-work that was different from what their children had traditionally done. Children also spoke about working with partners in class to do much of their mathematics work and these parents believed that children should work on their own. Parents who had attended the information session and had seen the children at work in the classroom were supportive of the project and also reported enthusiasm for mathematics in their children. While positive comments outweighed negative ones many parent expectations about what they believed second grade mathematics should look like were not fully realised by the project so doubts developed in some parents. Pressure was exerted on the school Board to investigate the project and the school administration became acutely aware of the influence and power a minority of parents can have with regard to curriculum change. This study concluded the need to take into account beliefs about teaching and learning held by participants in the school community. Further development of the project took into consideration parents' and educators' concerns, with a working relationship between the parties informed by current research knowledge. A conclusion reached was that parents could play a powerful but often unpredictable role in influencing educational change. While parents' focus on the needs of their own child can hinder a broader view of school reform efforts, it remains that change will be closely scrutinised to ensure *'what is best'* for students.

If we want people to behave differently, we must create the conditions under which it will be both easy and attractive for them to do so (Mackay, 1994, p. 224).

Parents may better support school programs if they are given the opportunity to understand them. Simon (1993) argues that it is necessary for educators who want to change paradigms to speak the language of those who are following the traditional paradigm. This requires reformers in mathematics education to structure programs so that participants do not feel threatened or uncomfortable and that some aspects of the reforms are familiar. There is a need for educators to recognize that parents have not had the sets of experiences in mathematics education which have led to educators' understandings about the need for reform and parents need to be given a variety of experiences to assist in their understandings.

Leaders in mathematics education reform who express concern and frustration at the less than anticipated acceptance of the need for such reform acknowledge the problems associated in not adequately including parents in the reform process. Price (1996) reflects that efforts to inform parents about the Standards in USA have fallen short and this has led to some parents calling for a '*return to basics*' while other parents question the rigour of current mathematics education practice. Burrill (1997, p. 60) says,

our challenge is to help children and their parents understand that mathematics is about thinking and listening — a very basic skill.

Many scholars have not examined the cultural level as one composed of beliefs and attitudes held by community members that influence what these individuals believe should (and will allow to) occur in classrooms (Dillon, 1993, p. 72).

This study provides such an examination of beliefs and attitudes of parents of primary school aged children concerning the teaching and learning of mathematics.

Research overview

Data from the parents of five families, each with at least one child attending a public primary school in the Australian Capital Territory (ACT), were used. Each participant met with the researcher once, for approximately one hour, to present their views and experiences on a range of questions related to their own and their children's experiences of school mathematics. These views were expressed through interviews and questionnaires.

Interpretation

This study identifies many aspects of parents' beliefs about the teaching and learning of primary school mathematics, an understanding of which can contribute to improved learning outcomes for children. The findings from this study are similar to those found by Epstein and Dauber (1991) in that more similarities exist between parents and teachers than many realise.

Parents are aware that their children's experiences of school mathematics are different from their own with understanding being a desired outcome. Parents characterised their own primary school experiences of mathematics as rote learning

and memorisation of facts. Understanding was not a primary focus. Parents struggle with the dichotomy between what they learnt and values derived from their mathematics education and what they understand of the learning experiences of their children. One example is that, while several of the parents use calculators and computers at work instead of pencil and paper for mathematics calculations, they still value rote learning of computations and are uncertain about their children becoming too dependant on calculators and computers.

Parents are involved with their children in home activities associated with mathematics. This involvement is unrelated to mathematics tasks set as homework. Through such activities parents pass on to their children a view about mathematics, which may or may not be consistent with that being taught at school. Parental views about what mathematics consists of are influenced by the extent to which they (parents) perceive they use mathematics in their own lives.

There is a wide range of views, held by parents, about the nature of mathematics and this could lead to fundamental conflicts for children between what they learn at home and what they learn at school.

Even though half the parents in the study had unsuccessful experiences of school mathematics they believe that mathematics is not only for an elite group and it is important for all children to succeed in mathematics.

Parents in this study experienced varying degrees of satisfaction from interactions with teachers and feelings of dissatisfaction arose when parents felt that teachers did not give due regard to parent concerns. Parents want their role in children's education to be acknowledged and valued by teachers.

Parents recognised that they had insufficient knowledge of practices in mathematics education and want information on curriculum and pedagogy to enable them to more effectively assist their children with mathematics.

All these factors suggest that a more detailed understanding of current mathematics learning theory and practice would help parents with the mathematics learning of their children. As parents believe they have a role to play in helping their children learn mathematics, and they see their role as supporting the school, these parents may be receptive to positive initiatives taken by teachers to assist them to help with their children's learning of mathematics

Implications

The majority of children's time is spent outside the school environment and in a myriad ways parents are influencing their children's understandings and attitudes towards mathematics.

This study shows that parents care deeply about their children's learning in mathematics and that parents want their children to achieve in mathematics. Parents indicated that they worked with their children at home in mathematics activities and were guided by their beliefs about mathematics. These beliefs were based on their own school and work experiences as well as what they believed was 'best for their children'. To achieve the best learning outcomes for children there needs to be

consistency between what is taught at home and what is taught at school. The choice is not about whether to consider parents as partners but rather to develop strategies to maximise the value of efforts parents already make.

Parents identified some preferred ways in which they would like to support their children's learning in mathematics. They saw their relationship with the school and communication with teachers as critical to their capacity to support their children's mathematics learning. While there is, among parents, a diversity of preferred strategies they are all based on the premise that teachers need to engage in communication practices which recognise equal status between themselves and parents and that the parents' role is to support the school.

To ensure that both parents and teachers are conveying similar messages about what it means to do mathematics schools must develop on going programs to inform parents about current mathematics teaching practices and to make explicit to parents the significance of their role in their children's mathematics education. A variety of approaches need to be taken in recognition of the diverse ways in which parents want, and are able, to be involved in the mathematics education of their children.

Schools should become the centres of parent education (Edgar, 1997, p. 14).

Communication must take place in both directions. Teachers need to find out about parent perceptions and understandings as well as informing parents about current practice in mathematics education. Parents are best able to identify what their needs are and teachers can gain this information through such activities as interviews, surveys or workshops. Homework is a powerful communicator of teacher and parent values. In recognition of this any work sent home for children to complete should reflect best teaching and learning practice, as it is from this work that parents develop views about school mathematics. This work is most effective if it engages the parents as learning partners with their children rather than merely as supervisors. Teachers should also seek parents' written comments on their child's efforts and respond to these as a way of showing parents that their input is valued.

All those involved in mathematics education need to see the development of quality school-parent partnerships as not another role for schools but a fundamental ingredient of an effective school and as such provide resources to facilitate such partnership.

Conclusions

This study clearly shows that:

- parents want to be involved with children's mathematics learning;
- there is a need for regular communication by teachers, appropriate to parents' understanding of how children learn mathematics;
- teachers need to develop effective strategies for communicating with parents about mathematics learning;
- teachers need to fully inform parents of policies, procedures and practices being implemented in classrooms;

- teachers need to demonstrate through a variety of means, e.g. homework, parent–teacher discussions, workshops and good practice ways to help educate parents on best practice in mathematics education.

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About the presenter

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By this stage she had three school-aged children and was experiencing, first hand, challenges linked with mathematics education. At the same time Paulene became involved in parent organisations, namely school P&Cs and the ACT Council of P&C Associations.

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Algebra and Technology: Emerging Issues

Barry Kissane

Although there are a number of technologies related to school algebra, it is only the personal technology of the graphics calculator that seems likely to be available widely enough to influence curriculum design and implementation on a large scale. The algebra curriculum of the past is overburdened with symbolic manipulation at the expense of understanding for most students. But algebra is much more than just symbolic manipulation. Connections between some aspects of algebra (expressing generality, functions and equations) and some graphics calculator capabilities are described. Some capabilities of algebraic calculators for symbolic manipulation are illustrated. Together, these suggest a fresh look at school algebra is needed.

The main purpose of this paper is to highlight some of the useful connections and emerging issues between algebra in the secondary school and currently available technologies, particularly graphics calculators. The focus is on elementary algebra, usually the province of the secondary school years in Australia, formally starting in either Year 7 or Year 8 (dependent on the state concerned). In fact, the study of algebra starts much earlier, in the primary school, with a focus on important mathematical ideas associated with patterns and regularities, and continues into the early undergraduate years for some students.

A number of current technologies are related to algebra in a variety of ways. Spreadsheets, which have been available for more than twenty years now, offer mechanisms to represent relationships both numerically and graphically, using symbolic representations to do so. Graphics calculators, which are about fifteen years old, are related more directly to algebra, and come in both unsophisticated and more sophisticated versions. For the past few years, algebraic calculators, sophisticated graphics calculators which deal directly with symbolic manipulation, have also been available. Computer algebra systems (CAS) of various kinds have been around for more than twenty years, with very sophisticated versions such as *Mathematica* and *Maple* produced for microcomputers over the past decade. These are widely used in professional mathematics, scientific and engineering circles, although are less evident in schools, partly because of their expense and partly because they greatly exceed the requirements of school mathematics. Recently, palmtop computers have been bundled with smaller version of such CAS software.

There are other kinds of technologies produced for school algebra, consisting of specific computer software of various kinds. Some of this is directly of a drill-and-practice variety, essentially drilling students with formal school algebra. Some of it is a little more ambitious, claiming to 'teach' students algebra, usually in the form of symbolic manipulation algorithms. In my view, neither of these offer much to either

pupils or teachers beyond an automated version of what is already available, and thus are of little interest here.

Although spreadsheets were first developed for, and still find their most use in, the commercial world, they can be used to advantage in some aspects of algebraic work. Their significance derives mainly from the fact that almost all households with a personal computer probably have a spreadsheet somewhere, since such software is often provided with the computer purchase. Spreadsheets offer pupils the opportunity to evaluate functions for many different values of a variable, thus giving some meaning to both the function itself and to the idea of a variable. Recently, these numerical evaluations can be graphed to provide a graphical representation of a function, also of value to pupils. Spreadsheets also handle iterative procedures quite well, lending themselves to exemplifying and exploring recursive situations. A disadvantage of spreadsheets is their use of non-standard notation.

Graphics calculators are small, hand-held calculators about the same size as a scientific calculator. The most obvious difference between a graphics calculator and a scientific calculator is the small graphics screen on the former. One of the several uses of the graphics display screen is to draw graphs of functions, so graphics calculators are sometimes called 'graphing' calculators, although this description is too restrictive in outlook. Technology of this kind has been around now since the mid 1980s. Graphics calculators are now widely used in parts of Australia, North America and Europe. Two distinctive differences between graphics calculators and other technologies for school mathematics, such as computers, is that they were produced mainly for educational use and they are much more portable. Indeed, graphics calculators are arguably the first examples of a genuinely personal technology for school mathematics.

Personal technology

As the phrase suggests, 'personal technology' refers to the technology available to individual pupils on a personal and unrestricted basis. Most so-called 'personal' computers are not examples of personal technology in schools, despite the use of the term 'personal' to describe them. (A possible exception is the case of laptop computers, but these are still much too expensive for the great majority of pupils, and so are not dealt with here.) For economic reasons, computers in most schools are available at the collective level, such as in a computer laboratory, rather than at the individual level. In such situations, their use is controlled by the teacher, rather than by the pupil. They are not permitted for use in examinations, particularly high-stakes examinations external to schools, because of difficulties of assuring equity is preserved between students. Although some pupils have individual access to a personal computer at home, many others do not. They are still too expensive for any curriculum authority to produce curricula based on the assumption of individual ownership. Mostly for these reasons of access, the mathematics curriculum and common teaching practices of schools have been only slightly affected, if at all, by the increased availability of personal computers in schools.

In contrast, there have been significant changes in school mathematics in a fairly

short period of time as a consequence of the availability of personal technology. In some Australian states (presently Western Australia and Victoria), students are permitted — in fact, expected — to use a graphics calculator in high stakes external examinations. A consequence of this is that the classroom experience is affected, with students needing to learn how and when and why to use a graphics calculator to help them to think about or to do mathematics.

At present, and for the last twelve years or so, the most mathematically powerful examples of personal technology are graphics calculators. These come in various forms, but at least three are distinguishable when thinking about algebra. The least powerful models are 'low-end' graphics calculators. They appear to have been produced mainly with younger pupils in mind and are manufactured by Casio, Sharp and Texas Instruments. Importantly for the notion of personal technology, they are relatively inexpensive, with some costing not much more than scientific calculators of the kind that have been routinely purchased by many, if not most, Australian secondary school students for almost two decades. While still not cheap, low-end graphics calculators are comparable in price with other adolescent purchases in affluent countries, such as a pair of shoes or two or three modern CDs, and thus are already affordable to the great majority of Australian families. In fact, the majority of Australian pupils do not require a much more powerful calculator than a low-end graphics calculator to meet their mathematical needs. The basic functionality provided by these calculators concerns the representation of functions and handling of elementary data analysis.

The next set, 'high-end' graphics calculators, are designed to accommodate the needs of students in the later years of schooling and the early undergraduate years. All four manufacturers, Casio, Hewlett-Packard, Sharp and Texas Instruments make good examples of these, which are deservedly becoming quite popular in many senior secondary schools in Australia. Algebraically speaking, they are distinguished by having various automated equation solving capabilities and a wider range of function representations (rectangular, parametric, polar and recursive) than the low-end graphics calculators. They are probably the most popular graphics calculators, since they span a wide range of uses over the spectrum of secondary and lower undergraduate education. Students who acquire a modern graphics calculator of this kind in the secondary school will still find it of use some years later in the early undergraduate years.

The most powerful graphics calculators are 'algebraic' calculators, containing Computer Algebra Systems (CAS). Given their capabilities, including extensive symbolic manipulation in algebra and calculus, algebraic calculators raise a number of significant issues for algebra teaching and learning, some of which are discussed in Kissane (1999). These sorts of calculators can readily perform all of the symbolic manipulation expected of secondary school mathematics students. This observation alone suggests that such technologies are worth a closer look.

Yet another form of personal technology is the palmtop computer, such as Casio's PC-Extender. These have now developed to the point where sophisticated software (such as versions of *Maple* and of *Geometer's Sketchpad*) are available as plug-in ROMs. Although such an idea might be promising in the long run, at present the

technology is too expensive for many students to acquire, and is probably most appropriate for those who already have access to a personal computer and a graphics calculator. (Such devices are designed to interface smoothly with desktop computers.) In addition, for reasons mentioned above, it is unlikely that such devices will be acceptable to Australian examination authorities in the near future, thus limiting their attractiveness to schools and pupils.

There is an urgent need to reconsider the secondary school algebra curriculum in the light of what technology is potentially available, either through ownership or long-term personal loan, to every single pupil. For at least the next few years, it seems likely that only graphics calculators will fit this description. This paper describes some of the relationships between this kind of technology and the algebra curriculum.

Algebra

Evidence from many sources, over many years, from the anecdotal to the more carefully researched, suggests that algebra in secondary schools has often been characterised by limited success and even dread (on the part of pupils and teachers alike). The algebra offered by schools, until very recently, appears to most students to have been preoccupied with routines for symbolic manipulation, of dubious utility and devoid of much meaning beyond the confines of the mathematics classroom. These routines have included the 'collection of like terms', 'expanding', 'simplifying', factorising expressions and solving (a remarkably small repertoire of) equations. Even today, many students seem to interpret algebra in such a procedural way. Although we have managed to produce a small subset of pupils with technical competence at such manipulations, very few of these have gained much insight into what algebra is (and is not), what it is for or why it is important. For most students, much of the time, algebra mainly comprises a collection of symbolic manipulation procedures, rather than also including a richly intertwined collection of concepts and strategies. The noted mathematics educator, Robert Davis, was less than complimentary about such an emphasis:

At one extreme, we have the most familiar type of course, where the student is asked to master rituals for manipulating symbols written on paper. The topics in such a course have names like 'removing parentheses,' 'changing signs,' 'collecting like terms,' 'simplifying,' and so on. It should be immediately clear that a course of this type, focussing mainly on meaningless notation, would be entirely inappropriate for elementary school children; many of us would argue that this type of course, although exceedingly common, is in fact inappropriate for all students. (1989, p 268)

A recent attempt to try to inject meaning into the algebra curriculum and to focus more carefully on the important ideas of algebra was provided by Lowe et al. (1993–4). In part, this work was informed by the seminal work in both *A National Statement on Mathematics for Australian Schools* and the *National Mathematics Profile*, which identified three broad dimensions of algebra and indicated how these might develop over the early years of the algebra curriculum. The three 'substrands' of the 'Algebra' strand were labelled 'Expressing generality', 'Functions' and 'Equations'.

Later refinements of these documents built upon the same structure, identifying for example that the study of functions involves both relationships and graphs, and that inequalities and equations ought to be considered together. These texts were developed under the assumption that students had access to suitable technologies, including graphics calculators and spreadsheets, although this was not the main feature of their development (Kissane, Grace & Johnston, 1995).

Connections

The most important connection between personal technologies such as graphics calculators and the algebra curriculum is that the technology provides fresh opportunities for pupils to learn about algebra. The key to these is the capacity of the calculator to enable *exploration* of key concepts — related to the metaphor of the calculator as a laboratory (Kissane, 1995). Space precludes an exhaustive listing of the kinds of explorations made possible, many of which are contained in publications such as Kissane (1997). However, a few of the connections are described briefly below, to offer a glimpse of the changed environment for learning and making sense of algebra for pupils who have ready access to technology.

Expressing generality

As noted above, spreadsheets allow for some expression of generality. A disadvantage for beginners is that they use unconventional symbols to do so (such as a cell reference, A1, to represent the value in the top left cell) and also require multiplication signs to be written (requiring $A1 \times B1$ instead of $A1B1$ or $A1(B1)$.) In contrast, graphics calculators use the standard conventions of algebraic representation; for example, $2AB^2$ means $2 \times A \times (B \times B)$, both on the calculator screens, on whiteboards and in texts. Regular use of a calculator with alphabetic memories storing numbers seems likely to help students come to terms with the notion of variables as place holders. In the same way, $2(A + 1)$ and $2A + 2$ will give the same numerical value on a calculator, regardless of the value of A , while $2(A + 1)$ and $2A + 1$ will (usually) give different values. Some of these characteristics are shown in Figure 1, in which A has been given the value 4 and B the value 5. (These and other screens in this paper were produced on a Casio cfx-9850GB+ calculator, a good example of a 'high-end' graphics calculator.)

4→A	4	2(A+1)	10
5→B	5	2A+2	10
2AB ²	200	2A+1	9

Figure 1. Using symbols on a calculator.

Equivalence transformations such as factorising, collecting like terms and expanding

can be represented either numerically and graphically to enhance meaning. For example, graphs of $y = x^2 - 1$ and of $y = (x + 1)(x - 1)$ are identical, as are their associated numerical tables of values. The critical concept of algebraic identity is representable both numerically and graphically; in contrast, student work with identities has often been restricted to the symbolic in the past. The screens in Figure 2 show some of these kinds of connections. (Only one graph is showing — the one drawn second, as the two graphs are of course the same.)

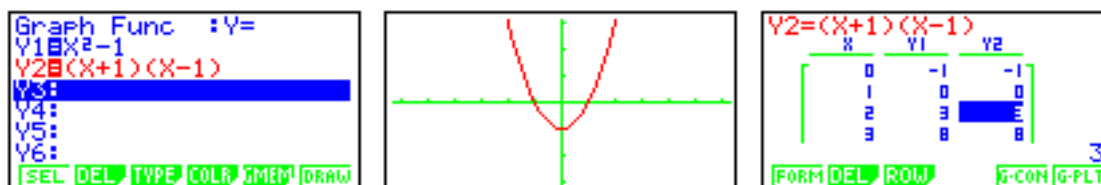


Figure 2. Showing the numerical and graphical equivalence of $x^2 - 1$ and $(x + 1)(x - 1)$.

Equivalence can also be shown or tested on a graphics calculator using logical statements, as described by Kissane & Harradine (2000). The calculator will give a correct equivalence the value of one and an incorrect equivalence the value zero, allowing pupils to check either their expansions or their factorisations readily, as shown in Figure 3.

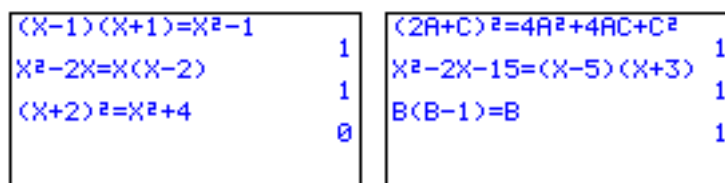


Figure 3. Checking equivalence with a graphics calculator.

This procedure will almost always work for the intended purpose, although it needs to be noted that it will appear to suggest that two statements are equivalent even when they are equivalent for only a single value of the independent variable. The final example above shows this erroneous case (in the particular case of $B = 2$, the value currently stored in the calculator). To avoid such potential pitfalls, it is a good idea to set the value of the variable to be a three digit integer — less likely to be the solution of the equations usually encountered at school.

Functions

Aspects of the study of functions are also positively affected by the capabilities of graphics calculators to represent relationships symbolically, numerically and

graphically — the so-called ‘rule of three’. Explorations of functions represented graphically or numerically (in tables) can be readily undertaken by pupils on graphics calculators with minimal prior experience. While movements among representations were available before personal technologies like this were invented, they were frequently hindered by the time and error-prone complexity of producing them (by hand graphing or numerical substitution, or both). Thus, students can now readily produce the family of graphs shown in Figure 4 and focus on how and why the graphs differ.

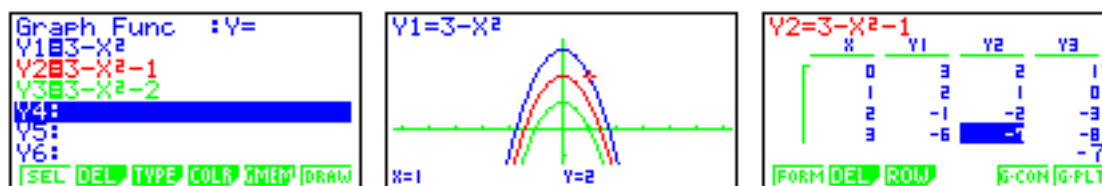


Figure 4. Different representations of a family of functions.

As well as movement among representations, graphics calculators permit pupils to readily explore families of functions and thus the crucial notion of a transformation. The screens below show an example of this with a function transformer (Kissane, 1997), to explore the transformation related to addition of a constant.

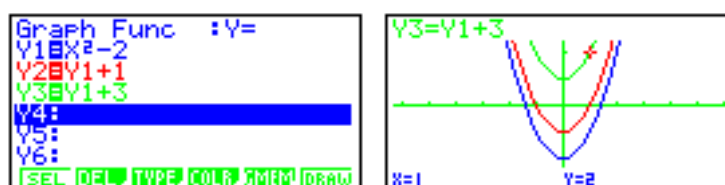


Figure 5. Using a function transformer to explore vertical shifts.

A graphics calculator allows the algebra curriculum to focus on classes of relationships of obvious importance that were traditionally neglected until later algebra study (such as exponential functions). Although exponential functions are very important because of their usefulness to model growth situations, they are usually not dealt with in introductory algebra courses, because students find it difficult to deal with them without first coming to grips with logarithms. However, using a graphics calculator, exponential functions are no more difficult to graph or to tabulate than are quadratic functions, and so they are an integral part of the *Access to Algebra* series.

Tabulation of functions is also possible using spreadsheets, which leads in turn to some ways of solving equations numerically. (These too are described in *Access to Algebra*). Graphing of functions is similarly available on a spreadsheet, although it is

more difficult for pupils to manipulate the functions and their corresponding graphs than it is for a graphics calculator.

Automatic graphical exploration capabilities of calculators allow students to deal numerically with questions which were previously not accessible until the calculus had been studied. For example, the screens below show how a calculator can locate a relative minimum point of a function graphically. With a graphics calculator available, students might be expected to learn to *use* graphs of functions rather than merely to draw them; we might even expect that students will decide for themselves when and why a graph would be appropriate, rather than relying on teachers and textbooks to tell them. The difference is of considerable practical importance to the algebra curriculum.

Equations and inequalities

Elementary equations and inequalities can be explored profitably by pupils making use of calculator capabilities for graphing and numerical tabulation. Unlike the conventional equation-solving algorithms using symbolic manipulation, such explorations are not restricted to the linear and the quadratic. (In fact, it is not commonly recognised that there are only two algorithms, 'Do the same thing to both sides' and using the multiplication property of zero after factorising.) Armed with graphics calculators, students might be expected to explore in new ways relationships between functions, equations and graphs and to develop a repertoire of ways of dealing with equations and inequalities, rather than the 'one best way' characteristic of the past. (See Kissane (1995) for an extended example of this.) We might expect that pupils will be able to solve a particular equation in several ways, and will develop the acumen to choose the most appropriate method for a particular circumstance. To give an elementary example, Figure 6 show two ways in which (numerical) solutions of the cubic equation $x^3 - 2x = 1$ can be obtained.

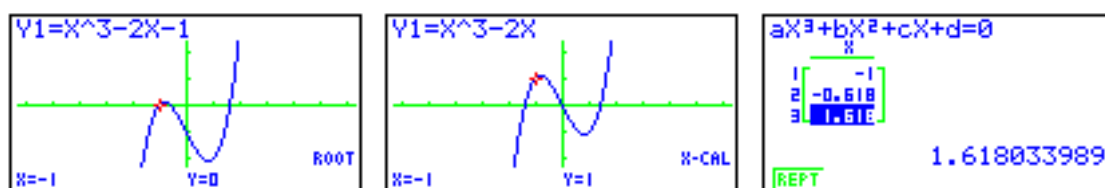


Figure 6. Three different solutions to $x^3 - 2x = 1$.

Numerical solutions to equations through refining a table of values can also be obtained on spreadsheets as well as graphics calculators, of course.

It is clear that the significance of factorising quadratic expressions and of the quadratic formula is altered by the availability of technology of these kinds.

Symbolic manipulation

The second major relationship between technologies and algebra is that some of the routines of algebraic work can now be performed routinely by devices like algebraic calculators and their larger versions of CAS on computers. Space precludes a complete treatment of this topic here, but the following examples (taken from Kissane (1999)) together suggest that school algebra curricula are likely to be affected considerably. Figure 7 shows some examples of some equivalence transformations, which are handled efficiently by Casio's Algebra fx 2.0, using conventional syntax.

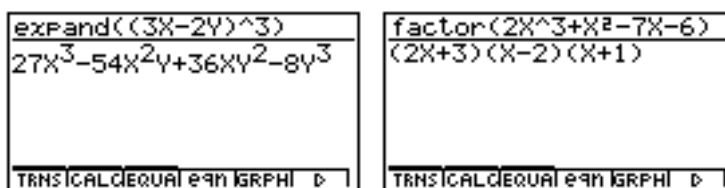


Figure 7. Expanding and factorising expressions on a Casio Algebra fx 2.0.

Similarly, Figure 8 shows examples of calculator solutions to equations and inequalities

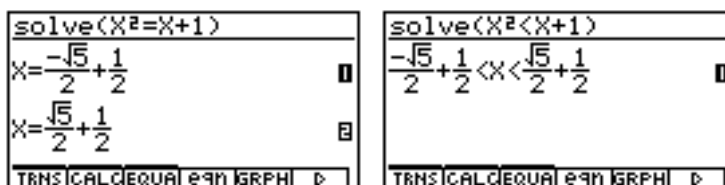


Figure 8. Solutions to an equation and a related inequality.

More sophisticated — and more complex — commands are accessible using technology of this kind, as shown by the examples in Figure 9.

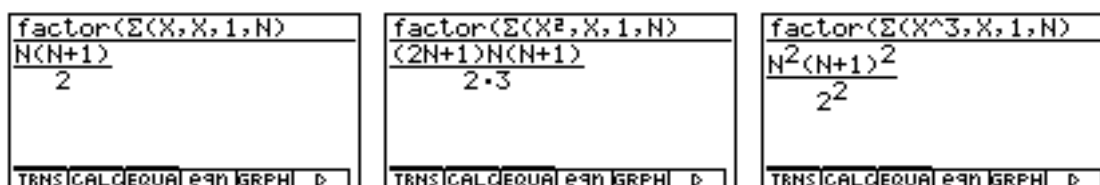


Figure 9. Exploring sums of powers on a Casio Algebra fx 2.0.

It seems important to re-evaluate the significance of different aspects of school algebra, particularly the focus on symbolic manipulation, in the light of these kinds of capabilities. It might be argued, for example, that we should concentrate more than in the past on helping pupils to *express* relationships algebraically rather than on manipulating the expressions themselves; similarly, we may focus more attention on helping pupils to *formulate* equations and *interpret* solutions, rather than only on the algebraic manipulations required to solve equations. As a final example, Figure 10 shows how this algebraic calculator can be used to carry out the standard steps of ‘doing the same things to both sides’ in order to solve a linear equation.

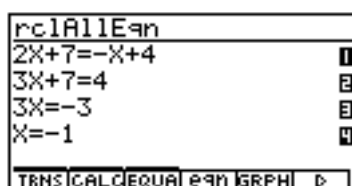


Figure 10. Solving an equation by doing the same thing to both sides.

In this case, the user of the calculator does the thinking (ie what to do to each side of the equation each time) and the calculator carries out the associated symbolic manipulations, remembering the result each time.

Conclusion

Connections of the kinds illustrated above between elementary algebra and graphics calculators deserve attention in secondary school curriculum development, as they fundamentally change the environment in which algebra is learned. A graphics calculator has the potential to richly exemplify many aspects of elementary algebra, and help pupils to see that it does make sense after all. Although computers are much more powerful forms of technology, graphics calculators offer powerful new ways of dealing with the problems traditionally addressed by secondary school algebra.

The development of the graphics calculator demands that we take a fresh look at the existing algebra curriculum, how it is taught and how it is learned, under an assumption of continuing and self-directed personal access to technology. Similarly, the development of the algebraic calculator suggests that we look closely at the content of our algebra curriculum and consider carefully a new role for symbolic manipulation, both by hand and by machine.

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About the presenter

Barry Kissane is a Senior Lecturer in Education at the Australian Institute of Education at Murdoch University in WA. He is interested in the relevance of technologies to mathematics education at various levels; he is especially interested in the place of graphics calculators as examples of technologies that are more likely to be available on a wide scale. Barry has experience as a textbook author, is a former President of the Mathematical Association of WA, is the present Editor of the *The Australian Mathematics Teacher* and is an experienced conference presenter. As well as technologies, Barry is interested in curriculum development, assessment, outcomes-based education, numeracy and mathematical thinking processes.

Assessing the Impact of CAS Calculators on Mathematics Examinations

Barry McCrae and Peter Flynn

The mathematics curriculum of the future will be reshaped by the increasing accessibility of CAS. This is immediately evident by looking at the impact CAS availability would have on current year 12 examinations. In this paper, we investigate the impact of CAS availability on a recent VCE Mathematical Methods examination paper. A variety of classification schemes are applied to determine whether individual questions are CAS-sensitive or CAS-resistant and what makes them so. Specific examples are discussed.

CAS-active examinations

The University of Melbourne's Computer Algebra Systems in Schools — Curriculum, Assessment and Teaching (CAS-CAT) project aims to investigate the changes that regular access to CAS calculators may have on senior secondary mathematics. The study is funded from 2000–2002 by the Australian Research Grant Strategic Partnerships with Industry Scheme. The industry partners are the Victorian Board of Studies and three calculator suppliers and manufacturers: Hewlett-Packard, Shriro (Casio) and Texas Instruments. Further details can be found at the project web site <http://www.edfac.unimelb.edu.au/DSME/CAS-CAT/>, and in a series of papers (Stacey, McCrae, Chick, Asp, & Leigh-Lancaster, 2000; Stacey, Asp, & McCrae, in press; Stacey, Ball, Asp, McCrae & Leigh-Lancaster, in press) that address preliminary issues.

It is planned that the culmination of the CAS-CAT project will be the trial in 2002 of a Victorian Certificate of Education (VCE) mathematics subject, as an alternative to the current Mathematical Methods 3/4 subject (Board of Studies, 1999), that assumes students have access to an approved CAS calculator. Mathematical Methods 3/4 is essentially a functions and calculus based subject, with some study of probability distributions (about 15% of the content), and is intended to provide an appropriate background for further study in, for example, science, economics or medicine.

Since the use of CAS will be an integral part of the learning and teaching of the proposed new subject, it is essential that this be reflected in its assessment regime — that is, a substantial part of the assessment should be CAS-active (Stephens & Leigh-Lancaster, 1997). However, various authors (Taylor, 1995; Drijvers, 1998) note that examining mathematics with students having access to CAS will initially cause considerable difficulties due to its impact on the structure and types of questions that can be asked. Hong and Thomas (1999) observe that the main requirement of many questions included on tertiary entrance examinations in mathematics at present is the performance of standard algebra and calculus algorithms that can be easily carried out on a CAS calculator.

Other concerns include that the removal of examination questions testing routine skills and standard algorithms, or their placement in a worded context, could increase the difficulty of such tasks (Monaghan, 2000). Of course, the availability of CAS will necessitate some redefining of what constitutes routine skills and fundamental algebraic competence (Heugl, 1999; Herget et al, 2000; Stacey, 1997).

Classification schemes for examination questions

The CAS calculators supplied to students participating in the CAS-CAT project are the Casio FX 2.0, the HP 40G and the TI-89. Each of these calculators has graphics capabilities, *but* (non-CAS) graphics calculators have been allowed in VCE mathematics examinations since 1997. Accordingly, schemes originally devised to assess the impact of graphics calculator availability on examination questions (Jones, 1995; Jones & McCrae, 1996), can be adapted to assess CAS by defining impact to mean that *a CAS user would have an advantage over a graphics calculator user*. In particular, the Jones and McCrae classifications become:

CNI The availability of CAS would have no impact on the question.

CIU The availability of CAS would have an impact, but the question could remain unchanged.

CIO The availability of CAS would have an impact and the question would need to be omitted in its current form.

Schemes have been devised specifically to classify the impact of CAS on a question (Kokol-Voljc, 2000; MacAogáin, 2000). Kokol-Voljc suggests that each question should be analysed according to the extent it tests basic abilities such as concept knowledge, modelling of real world situations and reflecting about mathematical content, and the extent to which it requires algorithmic and calculation skills. In her scheme, traditional examination questions are classified as:

CAS-insensitive questions (CASI)

CAS plays little or no role in the actual calculation and the focus is predominantly on conceptual understanding.

Questions changing with technology (QCWT)

Questions for which there is a shift of focus from technical/mechanical/routine work to mathematical/semantic/conceptual/application work. Such questions may need to be omitted or modified.

Questions devalued with CAS (QDWC)

Questions that exclusively test skills and become solely a test of the technical ability to use a CAS. They are worthless for providing feedback on a student's mathematical ability.

Questions testing basic abilities and skills (QTBS)

Questions that become trivial with CAS, but an underlying connected goal, such as testing knowledge of the syntactical structure of a mathematical expression, means they can continue to be used.

Kutzler, cited in Kokol-Voljc (2000), proposes classifying questions according to the role that CAS plays in answering them. His two-way classification scheme first looks at how significant the use of CAS is (*primary* versus *secondary*), then at how well the student needs to know how to use it (*routine* versus *advanced*) — see Table 1.

Table 1.

Kutzler's 2-dimensional classification scheme

	Routine CAS use	Advanced CAS use
Primary CAS use	Primary Routine (PR)	Primary Advanced (PA)
Secondary CAS use	Secondary Routine (SR)	Secondary Advanced (SA)
No CAS use	(NC)	

Thus applying Kutzler's classification scheme produces the following five categories of questions:

Primary Routine CAS use questions (PR)

Problems for which CAS use is the major activity although only superficial knowledge of the tool suffices.

Primary Advanced CAS use questions (PA)

Problems for which CAS use is the major activity but in-depth knowledge of the tool is required.

Secondary Routine CAS use questions (SR)

CAS use plays only a minor role in solving the problem and only superficial knowledge of the tool is required.

Secondary Advanced CAS use questions (SA)

CAS use plays only a minor role but advanced knowledge of the tool is required.

No CAS use (NC)

CAS is of no help in answering the question.

Kokol-Voljc ranks the categories in the following descending order in terms of their value for testing mathematical abilities: SR and NC (equal), SA, PA, PR.

MacAogáin's (2000) classification scheme is less elaborate than those of Kokol-Voljc and Kutzler, and can be more easily applied to determine whether CAS users would have an advantage over graphics calculator users. MacAogáin's categories are as follows:

CAS trivial (CT)

Questions that reduce down to two or three steps, such as enter the expression and differentiate, with CAS and so are no longer suitable.

CAS easy (CE)

Using CAS significantly reduces the difficulty of these questions, although some substantive mathematical knowledge is still required to answer them.

CAS difficult (CD)

These questions retain (most of) their level of difficulty although CAS is of some help.

CAS proof (CP)

CAS is of minimal or no use in these types of questions.

MacAogáin calculates a *CAS index*, ranging from 0 to 10, as a measure of the advantage of using CAS in an examination when it is not allowed. Letting $x\%$ = score obtained by correctly answering all CT and CE questions (only), the *CAS index* is calculated using the formula $(100 - x)/10$ and rounding to the nearest whole number. In his analysis of the 1999 Irish Leaving Certificate mathematics papers (graphics calculators not allowed), MacAogáin found that 81% of the Paper 1 questions were trivial or easy with CAS, which converts to a *CAS index* of 2, while paper 2 had a *CAS index* of 7. The main topics on Paper 1 are algebra and calculus; on Paper 2, the main topics are geometry, trigonometry and probability and statistics.

Analysis of a current examination

There are currently two end-of-year examinations for Mathematical Methods 3/4. In 2000, Examination 1 (Facts, skills and applications) consisted of 27 multiple-choice questions each worth 1 mark (Part I) and eight short-answer questions worth a total of 23 marks (Part II). Examination 2 (Analysis task) consisted of four extended-answer questions worth a total of 55 marks.

As discussed in Stacey, Ball, Asp, McCrae & Leigh-Lancaster (in press), one of the issues being researched by the CAS-CAT project is whether Examination 1 for the proposed new CAS-active subject should itself be CAS-free — that is, should Examination 1 be CAS-neutral (Stephens & Leigh-Lancaster, 1997)? With this in mind, each question on the 2000 Mathematical Methods 3/4 Examination 1 (Board of Studies, 2000) was classified by the authors according to the four schemes described in the previous section, *but assuming that students would continue to have access to graphics calculators*.

First, both authors classified each question on the examination working independently of each other. We then met and vigorously debated the merits of each question to come to a consensus classification in each scheme. This was much easier for some questions than for others. As Kokol-Voljc (2000, p. 70) notes:

... with any classification scheme, there is no clear-cut dividing line between the categories, because the reality is continuous — not discrete. Hence, for some exam questions it may appear arbitrary to put them into one or the other category.

We had most difficulty in applying Kokol-Voljc's (2000) scheme. In particular, we are not sure that we properly understand what she means by 'Questions testing basic

abilities and skills'. Her explanation seems to give this category a very limited scope and led to us not classifying any question in this way. By contrast, the 'Questions changing with technology' category would benefit from the creation of subcategories that give a more precise indication of the impact of CAS availability on a question.

Results and discussion

Table 2 summarises the results of applying the Jones and McCrae, MacAogáin, and Kokol-Voljc classification schemes to the 2000 Mathematical Methods 3/4 Examination 1. It also shows the relationships that emerged between the different categories used in the three schemes.

Table 2.

Classification of the 2000 VCE Mathematical Methods 3/4 Examination 1

Jones & McCrae (1996)	MacAogáin (2000)	Kokol-Voljc (2000)	Multiple-choice (%)	Short-answer (%)	Total (%)
CAS No Impact (CNI)	CAS Proof (CP)	CAS Insensitive (CASI)	59.3	47.8	54.0
CAS Impacts: leave Unchanged (CIU)	CAS Difficult (CD)	Questions Changing With Technology (QCWT)	7.4	0.0	4.0
	CAS Easy (CE)		3.7	0.0	2.0
CAS Impacts: Omit (CIO)	CAS Trivial (CT)		Questions devalued With CAS (QDWC)	7.4	21.7
			22.2	30.4	26.0

According to these results, CAS availability would not impact on 54% of the total marks allocated — that is, 54% of the marks are CAS proof or CAS insensitive (though not necessarily graphics calculator neutral). An example of a question from this category is shown in Figure 1. (Note that the five optional answers are not shown for any of the multiple-choice items reproduced in this paper.)

The graph whose equation is $y = \sqrt{x}$ is reflected in the x -axis and then translated 2 units to the right and 1 unit down. The equation of the new graph is ...

Figure 1. VCE 2000 Mathematical Methods 3/4 Examination 1, Part I, Question 2.

At the other extreme, questions worth a total of 26% of the marks would be trivialised or devalued by the availability of CAS and so would need to be omitted from a CAS-neutral (and probably even from a CAS-active) examination. As with the Irish papers, questions from the algebra and calculus areas of study were the most affected (52%). The questions in Figures 2 and 3 are typical examples.

If $x = 4$ is a solution of the equation $\log_e(ax + 2) = 3$, then the exact value of a is ...

Figure 2. VCE 2000 Mathematical Methods 3/4 Examination 1, Part I, Question 9.

If $f(x) = e^{-x}(x^3 - 4)$ then $f'(x)$ is ...

Figure 3. VCE 2000 Mathematical Methods 3/4 Examination 1, Part I, Question 17.

Table 2 shows that, to make the examination CAS-neutral, we would omit questions worth a further 14% of the total marks (a further 20% of the algebra and calculus marks). These are questions that are not trivialised by CAS, but in our opinion are made much easier by its availability. An interesting example is given in Figure 4. Despite the routine nature of its solution by hand, this question still poses a problem for students using a CAS calculator because the calculators automatically produce the general solution, involving a parameter, to trigonometric equations. To determine all required values of x within the given interval, appropriate values for the parameter must be substituted into the general solution. For this reason, this question may still be suitable for a CAS-active examination.

Find the exact solutions of the equation $\sin(2x) = \sqrt{3} \cos(2x)$, $-\pi \leq x \leq \pi$.

Figure 4. VCE 2000 Mathematical Methods 3/4 Examination 1, Part II, Question 5.

Figure 5 shows the only question that we classified as being made easier by the availability of CAS, but not sufficiently so to warrant its omission. The correct answer amongst the five alternatives given is the unsimplified form obtained by direct application of the quotient rule. However, each of the calculators gives a (different) simplified version of the answer that results after a number of extra steps.

If $y = (\tan x) / x$ then dy/dx is ...

Figure 5. VCE 2000 Mathematical Methods 3/4 Examination 1, Part I, Question 18.

The probability distribution for the discrete random variable X is given by

x	0	1	2	3
$\Pr(X = x)$	k	$2k$	$4k$	$8k$

The value of k is ...

Figure 6. VCE 2000 Mathematical Methods 3/4 Examination 1, Part I, Question 23.

The question shown in Figure 6 is an example of an item that is only slightly impacted by the availability of CAS since the algebra involved is very simple. It retains its level of difficulty and so need not be omitted from a CAS-neutral examination. A similar question with more difficult algebra (arising from more complex probability terms) would need to be omitted, but could be part of a CAS-active examination with the availability of CAS changing the question's focus to mainly conceptual.

Kutzler classifications

By definition, Kutzler's 'No CAS use' category applies to all (and only those) questions classified as being CAS proof/insensitive by the other schemes. Also, it follows from the definitions that CAS trivial/devalued questions are Primary Routine questions according to Kutzler. In our analysis, however, we did not find

that, conversely, every PR question was devalued by the availability of CAS. The exception was Question 16 (Figure 7).

Rainwater is being collected in a water tank. The volume, $V \text{ m}^3$, of water in the tank after time, t hours, is given by $V = 2t^2 - 3t + 2$. The average rate of change of volume over the first ten hours in m^3 per hour is ...

Figure 7. VCE 2000 Mathematical Methods 3/4 Examination 1, Part I, Question 16.

Though impacted by the availability of CAS, this question retains its relative difficulty because it can be solved in CAS-style on a graphics calculator ($Y1 = V$, $Y2 = (Y1(10) - Y1(0)) / 10$). The primary objective of Question 16 is to test whether the student understands what is meant by ‘average rate of change’. It appears to be a weakness of Kutzler’s scheme that questions categorised as PR may range from being trivialised at one extreme to retaining their (conceptual) difficulty at the other extreme. Perhaps questions like Question 16 should be classified as Secondary Routine in Kutzler’s scheme? MacAogáin’s scheme copes well in this respect, but Kokol-Voljc’s scheme could benefit from a redefinition of her ‘Questions testing basic abilities and skills’ category to include such questions.

Apart from Question 16, the remaining questions classified as changing with technology according to Kokol-Voljc’s scheme, were classified as Secondary Routine according to Kutzler. This confirms Kokol-Voljc’s high ranking of SR questions. No questions were judged to require advanced knowledge of CAS. This is not surprising given the format of the examination (multiple-choice and short-answer).

Brand-neutral assessment

Another issue being researched in the CAS-CAT project is whether fair brand-neutral assessment is possible (Stacey, McCrae, Chick, Asp, & Leigh-Lancaster, 2000). Of concern here, is the extent to which the different CAS capabilities of the available calculators affect the difficulty of questions. Two examples arose in our analysis. One of these questions has already been discussed (Figure 5). The answer given by one of the calculators is much closer to the required form than the answers given by the other two calculators. The other question is given in Figure 8. This can be solved with one instruction ($\text{solve}(dy / dx > 0, x)$) by two of the calculators, whereas the other calculator appears to only solve linear inequalities.

For the curve with equation $y = -x^3 - x^2 + 2x + 2$, the subset of R for which the gradient of the curve is positive is closest to ...

Figure 8. VCE 2000 Mathematical Methods 3/4 Examination 1, Part I, Question 15.

Conclusion

About 40% of the VCE 2000 Mathematical Methods 3/4 Examination 1 would need to be replaced to ensure that a student using a CAS calculator would not have a potential advantage over a student using a (non-CAS) graphics calculator. About

three-quarters of the questions from the algebra and calculus areas of study would have to be replaced. Some of these questions, however, would be suitable for a CAS-active examination where access to a CAS calculator could be assumed. Kutzler's (Kokol-Voljc, 2000) scheme could be used to check that CAS use was appropriately tested in such an examination, but a typology akin to that developed by Kemp, Kissane and Bradley (1996) for the use of graphics calculators, is likely to be more useful to the examination designer. Variations in the CAS capabilities of calculators warrant close attention when setting CAS-active questions.

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About the presenters

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Peter Flynn taught in Victorian secondary schools for 10 years before taking up the position of research scholar with the CAS-CAT project during 2000. Peter has taught a variety of VCE mathematics subjects and immediately prior to joining the project he was Mathematics Coordinator at Woodleigh School.

Modelling Growth and Decay with the TI-83 and Excel

Frank Moya

Graphing calculators and dynamic spreadsheets provide powerful tools for investigating exponential functions and for modelling real-world data that is changing by a constant ratio at regular time intervals. The particular modelling tasks explored in this paper involve predicting the future population of all Australian states and territories based on various growth rate scenarios.

Introduction

The tasks introduced in this paper aim to make students aware that:

- data that is changing by a constant ratio at regular time intervals gives rise to a geometric sequence and to an exponential model;
- Euler's number, e , arises naturally from continuous exponential growth;
- the formula for exponential growth is $N = N_0 (1 + k)^t$. For *continuous* growth this becomes $N = N_0 e^{kt}$, where N is the number at time t and $k = \frac{r}{100}$ for growth at $r\%$ per unit time;
- the TI-83 can be used to investigate the properties of $y = e^x$;
- a project on Australia's population data provides a real context for modelling exponential growth using dynamic *Excel* spreadsheets.

A wide range of real-world contexts lend themselves to being modelled as geometric sequences and exponential functions. Some examples that have previously been published include car depreciation (Moya, 1999) and A-series paper sizes (Moya, 2000).

Investigating exponential functions

Introducing 'e' through growth

The textbook definition of Euler's number 'e' is

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

This esoteric definition can be 'brought to life' using an exponential growth example.

Problem

One million dollars is invested with *Loanshark Inc.* at 100 % pa. Table 1 shows the value, in millions, of the investment at the end of 1 year if the interest is calculated yearly, monthly, daily and every second (note: $e \approx 2.718281828$).

Time interval	Number of payments	Percentage rate per time interval (r)	Rate per time interval (k)	Value of investment $N = N_0 (1 + k)^n$
yearly	$n = 1$ (year)	$r = 100\%$	$k = 1$	$N = 1 \times (1 + 1)^1$ $= 2$
monthly	$n = 12$ (months)	$r = \frac{100}{12}$ (= 8.33%)	$k = \frac{1}{12}$ (0.0833)	$N = 1 \times \left(1 + \frac{1}{12}\right)^{12}$ $= 2.6130$
daily	$n = 365$ (days)	$r = \frac{100}{365}\%$	$k = \frac{1}{365}$	$N = 1 \times \left(1 + \frac{1}{365}\right)^{365}$ $= 2.7146$
every second	$n = 31536000$ (seconds)	$r = \frac{100}{31536000}\%$	$k = \frac{1}{31536000}$	$N = 1 \times \left(1 + \frac{1}{31536000}\right)^{31536000}$ $= 2.71828$

Table 1. Value of investment at 100% pa calculated yearly, monthly, daily and every second.

By carrying out the exercise in table 1, students can better appreciate that Euler's number arises naturally from continuous growth. As the number of payments, $n \rightarrow \infty$, students observe that $\left(1 + \frac{1}{n}\right)^n \rightarrow e$. This gives meaning to the definition

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

Two possible extensions to this exercise are:

Investigate growth rates other than 100% pa to show that for continuous growth,

$$N = N_0 (1 + k)^t = N_0 e^{kt}$$

Show, with examples, that as $k \rightarrow 0$, $e^k \rightarrow (1 + k)$

Investigating $y = e^x$ using the TI-83

In this exercise students compare the functions $y = 2^x$, $y = 3^x$ and $y = e^x$. The [DRAW:Tangent] function of the TI-83 is used to compare the gradient of the tangent at $x = 0$ for the three functions. The instructions given to students are shown in Table 2.

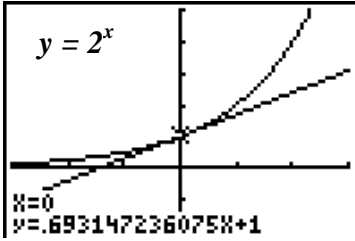
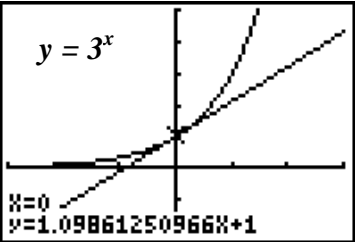
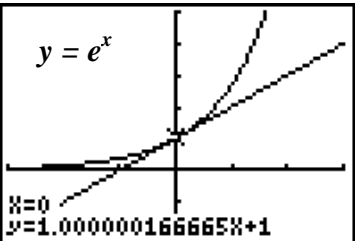
Function to be investigated	Keystrokes	Screen
<p>1. Graph $y = 2^x$. Find the gradient of the tangent to $y = 2^x$ at $x = 0$ Gradient at $x = 0$ $m = \dots\dots\dots$</p>	<p>$\boxed{Y=}$ enter $Y1 = 2^x$ (ie. $\boxed{2}\boxed{\wedge}\boxed{X,T,\Theta,n}$) \boxed{WINDOW} enter $X_{min} = -3, X_{max} = 3, X_{scl} = 1,$ $Y_{min} = -2, Y_{max} = 5, X_{scl} = 1$ \boxed{GRAPH} Use the 'DRAW:Tangent' function $\boxed{2nd}\boxed{PRGM}\boxed{DRAW}\boxed{5}:\text{Tangent} \boxed{0}\boxed{ENTER}$</p>	
<p>2. Graph $y = 3^x$ Find the gradient of the tangent to $y = 3^x$ at $x = 0$ Gradient at $x = 0$ $m = \dots\dots\dots$</p>	<p>$\boxed{Y=}$ enter $Y2 = 3^x$ (ie. $\boxed{3}\boxed{\wedge}\boxed{X,T,\Theta,n}$) (Turn off $Y1 = 2^x$) \boxed{GRAPH} Use the 'DRAW:Tangent' function $\boxed{2nd}\boxed{PRGM}\boxed{DRAW}\boxed{5}:\text{Tangent} \boxed{0}\boxed{ENTER}$</p>	
<p>Given these results, there must be number (called 'e' by Euler), just below 3, such that the gradient of the exponential function equals one ($m = 1$) at $x = 0$.</p>		
<p>3. Graph $y = e^x$ Find the gradient of the tangent to $y = e^x$ at $x = 0$ Gradient at $x = 0$ $m = \dots\dots\dots$</p>	<p>$\boxed{Y=}$ enter $Y3 = e^x$ (i.e. $\boxed{2nd}\boxed{LN}\boxed{X,T,\Theta,n}$) (Turn off $Y2 = 3^x$) \boxed{GRAPH} Use the 'DRAW:Tangent' function $\boxed{2nd}\boxed{PRGM}\boxed{DRAW}\boxed{5}:\text{Tangent} \boxed{0}\boxed{ENTER}$</p>	

Table 2. Instructions for finding the gradient of the tangent at $x = 0$ to $y = 2^x$, $y = 3^x$ and $y = e^x$.

Investigating the derivative of e^x using the TI-83

In tables 3 and 4 the investigation is extended to illustrate that $\frac{d(e^x)}{dx} = e^x$.

Investigation	Keystrokes	Screen
Find the gradient of the tangent to $y = e^x$ at $x = -1, 0, 1, 2, 3$ using the DRAW menu.	$\boxed{\text{Y=}}$ enter $Y3 = e^x$ (Turn off all other graphs) $\boxed{\text{GRAPH}}$ $\boxed{2\text{nd}} \boxed{\text{PRGM}} \boxed{\text{DRAW}} \boxed{5}$:Tangent $\boxed{\leftarrow}$ $\boxed{1}$ $\boxed{\text{ENTER}}$	
Record the gradient of the tangent at each of these points.	$\boxed{2\text{nd}} \boxed{\text{PRGM}} \boxed{\text{DRAW}} \boxed{5}$:Tangent $\boxed{0}$ $\boxed{\text{ENTER}}$ Draws tangent at $x = -1$	
	Draws tangent at $x = 0$ Continue for $x = 1, 2, 3$.	

Table 3. Instructions for finding the gradient to the tangent at $x = -1, 0, 1, 2, 3$.

A table of values for $y = e^x$ may be obtained using the 'TABLE' function of the TI-83, using the keystrokes $\boxed{2\text{nd}} \boxed{\text{GRAPH}} \boxed{\text{TABLE}}$. These values can then be compared with the gradient to the tangent found by completing the exercise in table 3.

The advantage of using the DRAW menu for this task is that it reinforces that the derivative is the gradient of the tangent. However, an alternative approach to illustrating that $\frac{d(e^x)}{dx} = e^x$ is to use the 'nDeriv(' function on the $\boxed{\text{MATH}}$ menu, as shown in table 4.

Investigation	Keystrokes	Screen																					
Enter $y = e^x$ and $y = \frac{d(e^x)}{dx}$.	Enter $y = e^x$. Turn off all other graphs. $\boxed{\text{Y=}}$ enter $Y3 = e^x$ Enter the derivative of e^x in $Y4$ $\boxed{\text{Y=}}$ enter $Y4 = \text{nDeriv}(Y3, X, X)$ $\boxed{\text{MATH}} \boxed{8}$ nDeriv($\boxed{\text{VAR}} \boxed{\rightarrow}$ $\boxed{1}$ Y-Vars $\boxed{4}$ $Y4$ $\boxed{,}$ $\boxed{\text{X}}$ $\boxed{,}$ $\boxed{\text{T}}$ $\boxed{\ominus}$ $\boxed{,}$ $\boxed{\text{X}}$ $\boxed{,}$ $\boxed{\text{T}}$ $\boxed{\ominus}$ $\boxed{,}$ $\boxed{}$																						
Show a table of values for $y = e^x$ and $y = \frac{d(e^x)}{dx}$ in juxtaposition.	$\boxed{2\text{nd}} \boxed{\text{GRAPH}} \boxed{\text{TABLE}}$ Use 'TBLSET' to set the table parameters. $\boxed{2\text{nd}} \boxed{\text{WINDOW}} \boxed{\text{TBLSET}}$ 	<table border="1"> <thead> <tr> <th>X</th> <th>Y3</th> <th>Y4</th> </tr> </thead> <tbody> <tr><td>0</td><td>.36788</td><td>.36788</td></tr> <tr><td>1</td><td>2.7183</td><td>2.7183</td></tr> <tr><td>2</td><td>7.3891</td><td>7.3891</td></tr> <tr><td>3</td><td>20.086</td><td>20.086</td></tr> <tr><td>4</td><td>54.598</td><td>54.598</td></tr> <tr><td>5</td><td>148.41</td><td>148.41</td></tr> </tbody> </table>	X	Y3	Y4	0	.36788	.36788	1	2.7183	2.7183	2	7.3891	7.3891	3	20.086	20.086	4	54.598	54.598	5	148.41	148.41
X	Y3	Y4																					
0	.36788	.36788																					
1	2.7183	2.7183																					
2	7.3891	7.3891																					
3	20.086	20.086																					
4	54.598	54.598																					
5	148.41	148.41																					

Table 4. Instructions for creating tables of values for $y = e^x$ and $y = \frac{d(e^x)}{dx}$.

The exercise in table 4 clearly illustrates the unique property of the derivative of e^x .

Shaping Australia's population

Population projections to 2021

In this task students projected Australia's future populations for different annual growth rates. For the purpose of this exercise, it was assumed that the population will change continuously, by a constant annual rate, between the years 2000 and 2021. Australia's past and current population statistics were obtained from the Australian Bureau of Statistics (ABS) website (HREF1).

The ABS population clock showed an estimated population of 19.07 million on 1st January 2000. The ABS provides three projected population scenarios, with average annual growth rates, k , between 2000 and 2021, of $k = 0.9\%$, $k = 0.8\%$ and $k = 0.7\%$.

The population, t years after 1st January 2000, can be predicted using the rule investigated earlier $N = N_0e^{kt}$, where $N_0 = 19.07$ million and k is the growth rate.

A dynamic spreadsheet was created in Excel by assigning a scroll bar to the growth rate, k , in the formula $N = N_0e^{kt}$. This is shown in figures 1 and 2. By sliding the scroll bar the graphs and population projections change. This is a very powerful tool to instantly observe the effect of changing a variable on the projected outcome.

Dynamic spreadsheets are easily created through the Forms toolbar. The electronic version of this paper allows the embedded spreadsheet in figure 1 to be manipulated.

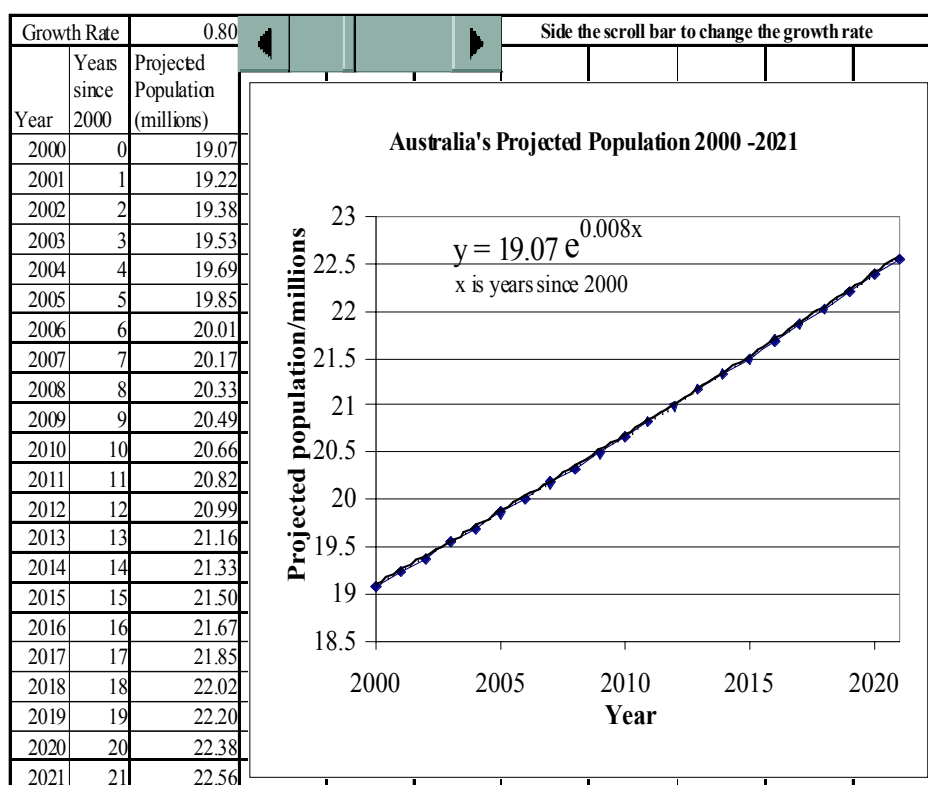


Figure 1. Dynamic Excel spreadsheet for predicting Australia's population.

Slider set at a growth rate of 0.8% p.a.

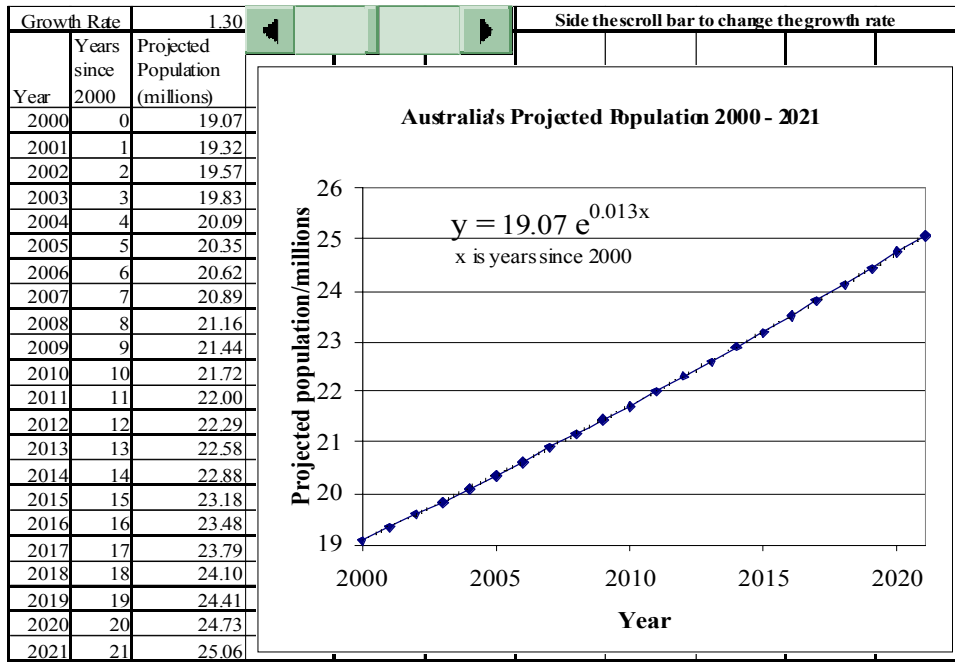


Figure 2. Excel spreadsheet for predicting Australia's population, set at 1.3 % growth.

In the year June 1995 to June 1996, Australia's population grew by 1.3% (HREF 1). Figure 2 shows that if this growth rate continued, by 2021 the population would be just over 25 million. This is well above any of the projections made by ABS, with their middle prediction being a population of 22.5 million in 2021, representing an average annual growth rate of approximately 0.8%, as shown in figure 1. However, the ABS modelling is more complex than the exponential models shown in these spreadsheets.

ABS predictions assume that the rate of growth in Australia's population will vary at different times between 2000 and 2021, with a clear long-term declining trend. The slowing in growth is already evident in some other OECD countries, including UK (0.2% pa), Japan (0.2% pa) and Germany (0.7% pa) (HREF 2). The main reason for the projected decline in Australia's growth rate is the decline in the natural increase (births minus deaths) of the population. This is largely a result of the increasing number of deaths occurring in an ageing population, coupled with low and declining fertility.

Shaping the Federation

For this task students modelled population growth for each state, as shown in figure 3.

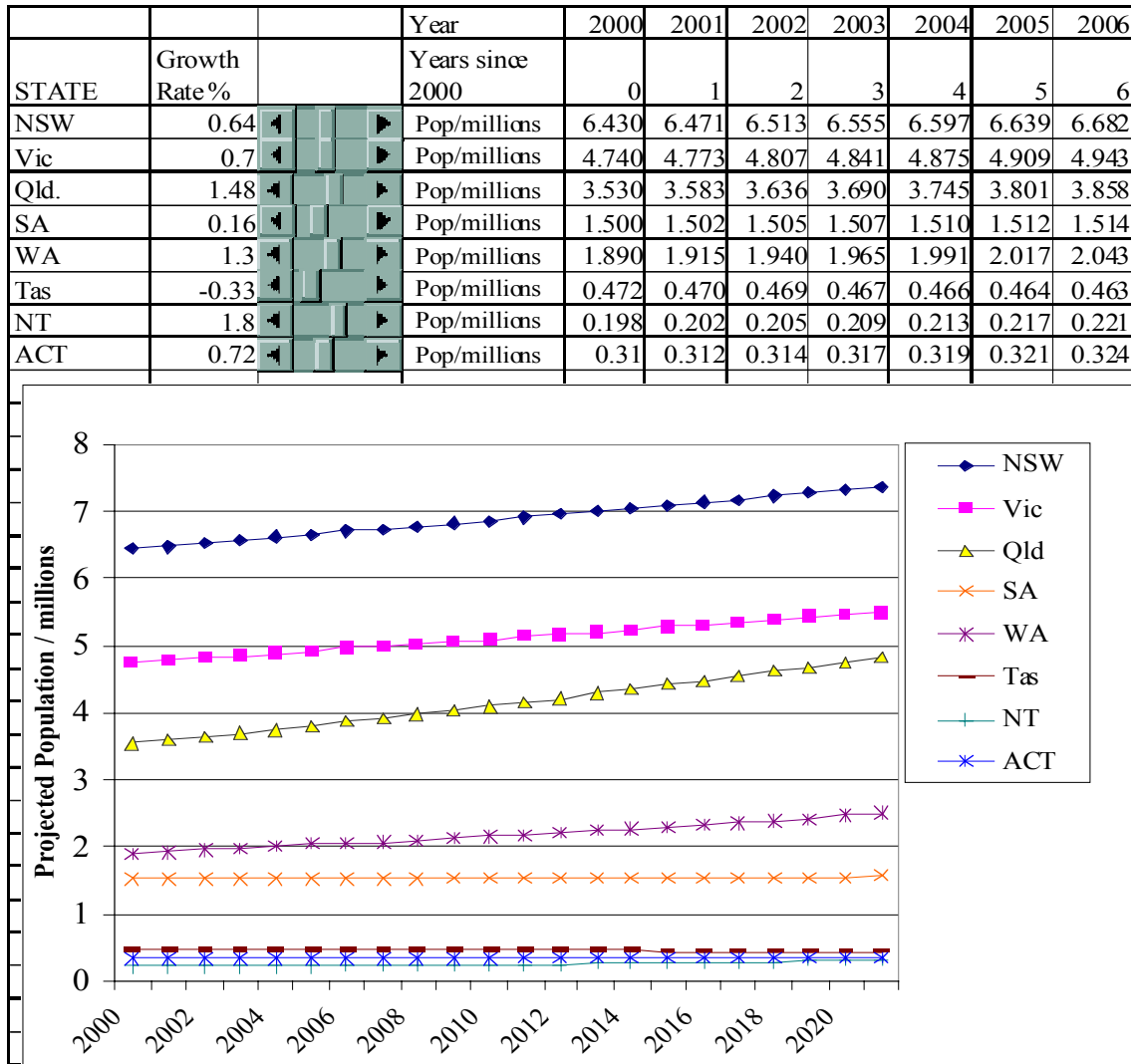


Figure 3. Dynamic spreadsheet for projecting the population of each state to 2021.

The spreadsheet in figure 3 has a scroll bar assigned to the growth rate, k (in the formula $N = N_0e^{kt}$) for each state or territory. The effect on the projected population, between 2000 and 2021, of a state or territory can instantly be observed by sliding the scroll bar. In figure 3 the growth rates have been set to the average rate in the middle-range prediction, as gleaned from the ABS website.

In predicting these growth rates the ABS is assuming net gains and losses through migration between states. The Northern Territory, Queensland and Western Australia are predicted to be the high growth states, with average annual increases of 1.8%, 1.48% and 1.3% respectively. These states and Victoria (which has reversed the

trend of the early 1990s) are the ones that have experienced net positive interstate migration in recent years. The middle growth states are predicted to be Victoria (0.7%), NSW (0.64%) and the ACT (0.72%). However, another scenario by the ABS actually shows the ACT decreasing in population. The states predicted to experience low or even negative population growth are South Australia and Tasmania.

The graphs, obtained from the spreadsheet in figure 3, clearly show the effect of these predicted average growth rates on the projected populations over the next 21 years. In particular, the narrowing gap in population between Queensland and Victoria is evident. Students were required to extend the range of the spreadsheet or use a graphing calculator to find the year in which the population of Queensland would equal that of Victoria, assuming that these growth rates continued. This is shown in figure 4 as a TI-83 screen dump. The result indicates that if Queensland continues to grow at 1.48% pa, and Victoria at 0.7% pa, both states will have a population of about 6.2 million by 2038.

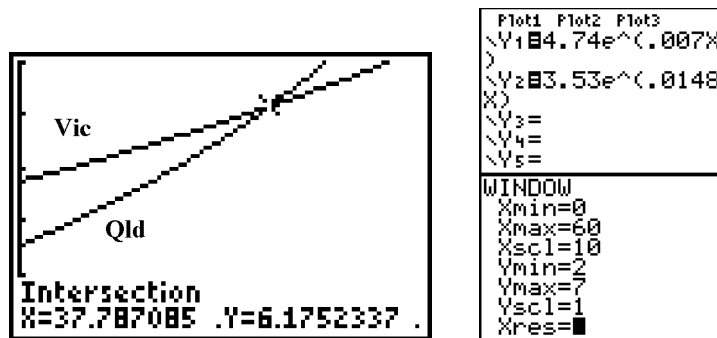


Figure 4. Equivalence point for the populations of Victoria and Queensland.

The graph in figure 3 also shows the widening gap in the populations of Western Australia and South Australia under this scenario. The growth in the population of the Northern Territory is not fully appreciated on the scale of the graph. Students were required to use the spreadsheet to produce a second graph for the 3 smallest states/territories using a more suitable scale.

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Internet references

- HREF1: <http://www.abs.gov.au> Australian Bureau of Statistics
- HREF2: <http://www.census.gov> US Bureau of Census

About the presenter

Frank Moya is currently the Head of Mathematics at Frankston High School, a school of over 1500 students located by the bay, 43 km from the Melbourne CBD. A special feature of the school is a laptop program, with about 60% of year 7 students bringing their own computer to class. A focus of Frank's teaching has been the use of technology to model real-world problems. He has presented several papers on this theme at the conferences of the Mathematical Association of Victoria (MAV) and the AAMT. Frank and his wife, Anne, are part-time cattle farmers and keen gardeners.

Literally Teaching Literacy In Maths: The Thebarton Senior College Experience

Derek Nash

The history of Australia has been, and is being shaped by people from other cultures who choose to make this country their home. Many of these recent arrivals are returning to school for a variety of reasons, and mathematics is very important to them in shaping their future.

Thebarton Senior College is an Adult Re-entry College with a large 'New Arrivals' program.

This session will be an anecdotal account of the joys and trials of teaching literacy in senior mathematics to adults from all parts of the world, with ages ranging from 16 to 80+ years, and maths backgrounds from zero to university degrees.

I want you to picture your favourite maths class.

Can you see it? Can you imagine the students sitting there? ...quietly... respectfully... agog for new knowledge... impatient for you to start the lesson?

NOW let's try to make a few changes to your pictured class:

1. Does your favourite class contain 30 students? If not, fill up all of the empty seats in your classroom.
2. Adjust your image so that the students now come from a selection of the following 58 countries:

Afghanistan	Africa(?)	Albania
Argentina	Australia	Boznia-Herzegovina
Burma (Myanmar)	Cambodia	Chile
China (Not Taiwan Prov.)	Croatia	Cyprus
Ecuador	Egypt	El Salvador
England	Eritrea	Estonia
Ethiopia	Fiji	(Former) Macedonia
Yugoslavia	France	Germany
Greece	Hong Kong	Hungary
Indonesia	Iran	Iraq
Italy	Japan	Kenya
Korea (South)	Lebanon	Malaysia
New Zealand	Pakistan	Papua New Guinea
Peru	Philippines	Poland
Portugal	Romania	Russian Federation
Serbia	Somalia	Spain
Sri Lanka	Sudan	Syria

Taiwan	Thailand	Tibet
Turkey	Ukraine	Uzbekistan
Vietnam		

3. All but 10% or so will now probably (but not certainly) have one of the following 49 languages as their first language:

Afrikaans	Albanian	Amharic
Arabic	Armenian	Bosnian
Bulgarian	Burmese	Cantonese
Chinese	Croatian	Dari
Farsi	Filipino	French
German	Greek	Hebrew
Hindi	Indonesian	Japanese
Khmer	Korean	Lao
Macedonian	Malay	Maltese
Mandarin	Maori	Persian
Polish	Portuguese	Punjabi
Romanian	Russian	Serbian
Sinhalese	Slovak	Somali
Spanish	Swahili	Tagalog (Filipino)
Thai	Tigrinya	Turkish
Ukrainian	Urdu	Vietnamese
Italian		

The remainder have ENGLISH as their first language.

4. Now adjust the ages of the students in your imaginary class, so that their ages vary across the whole range from 17 years to 77 years.
5. Finally, in case your imagined class is not sufficiently different from how you originally pictured it a few minutes ago, lets add in a few interesting personalities; at any time, you may have in your class any (but probably not all) of these people:
- a retired general of the army of the Peoples Republic of China
 - the self-styled 'defence minister' of a government in exile of a group of islands occupied by a neighbouring country
 - a (retired) professor of mathematics from a Russian university
 - the bouncer/ chauffeur/bodyguard for a brothel
 - a woman who 'hears voices' and is constantly yelling out or turning around in loud argument with them
 - a 40 year old ex-union official from El Salvador, whose education did not progress past primary level.

If you think you have a reasonable idea of the make up of our fictitious class, then what you have is a fairly typical maths class at the Thebarton Senior College in Adelaide.

Now I would be quite happy if you went away at the end of the next hour telling yourself that you haven't learnt anything new, because you already do all of the things we will have discussed; this will be a good indication that we are all on the same (correct) path.

[Note, however, that this may well be a *cultural* reaction; the Australian reaction of 'what a waste of time; I knew all that' equates to the year 10 remedial maths class's reaction of 'we've done this before; why do we have to do it again?', whereas the Asian student's reaction is often 'Good, I have done all of this before, so it is good reinforcement, and improves my chances of achieving 100%']

So, what problems do you imagine you would face in teaching maths to this class?

- Discipline?
- Differing levels of prior maths learning?
- Cultural difficulties and differences?
- Language?

...Well *discipline* is not, by and large, an issue.

The students ARE sitting respectfully, expectantly agog for new knowledge, etc., and will usually continue in this way for the full 1 hour lesson (but, try to get the lady who turns around to yell at the 'voices' to sit in the back row!).

However, the other side of this coin is that you, the teacher, are expected to be organised, well prepared, and to know *everything!* The teacher is expected to be the font of all wisdom (see *cultural differences* below). If students suspect that the teacher does not know what he/she is doing, does not prepare well, is not doing a good enough job, etc., then they are likely to be down at a counsellor's office trying to change to another class.

Differing levels of prior learning

There really is not an easy answer to this problem, as far as I can see. If students have been enrolled at a school from year 8 to year 12, a certain amount of selection must automatically take place, so that learning levels in many classes are fairly similar.

This is NOT the case in a school which BEGINS its intake at year 11 level, and certainly not the case when the student cohort is adult. If students start at year 11 level, then there is a good chance that they will find their correct Maths level in their choice of Year 12 subjects, but those who step straight in to year 12 studies often have unreal expectations of themselves and their teachers.

At Thebarton, we have 4 weeks at the beginning of each year in which new students can enrol, and in which students can change from one subject to another. This means that:

- as a teacher trying to progress through the first topic, you are constantly having new students appear in your class;

- the pressure is on teachers to identify struggling students early, and to try to counsel them into a more suitable subject. However, many adult students have their own agenda, and will not accept this type of advice, preferring to go with the option of having a try, then if necessary repeating the subject again in the next year or semester.

In specific cases, it is possible to take advantage of a student's prior learning.

Take the previously mentioned Professor of Mathematics, for example. This gentleman had retired, come out to Australia from Russia to be with his family, and did not want to sit around home all day, unable to go out because he did not speak the language, waiting for his children and their family to come home.

Like many others, he decided to enrol into the New Arrivals program at Thebarton Senior College, with the intention of learning enough English to be able to help in the community (with projects such as Meals on Wheels, etc.), and so he was enrolled in a course called 'ESL for Living and Working'.

NOW, from a Maths teachers point of view, one could either be daunted by the prospect of having this person in the class, or see him as an excellent resource, a chance to get all of those REALLY hard problems solved!

And think of the opportunities to use this man as a peer tutor for others who are having difficulties!

Cultural difficulties and differences

The expectation that the teacher is the 'font of all wisdom' has previously been mentioned. This is an expectation in many cultures, especially Asian cultures, where class sizes are often much larger than is usual in Australia,

with teachers disseminating information to large numbers of students via monitors, using a pontifical lecture style, and not being open to questions or interruptions. One student from an Asian country addressed our staff on this issue of differing cultural expectations, and two points that she made which stick in the memory were:

- students from her country had no idea about research based learning, and so had great difficulty when told to go away and research something;
- when they were to write an essay, the teacher would do an outline essay plan, distribute it via the monitors to the 50 or more students in the classroom, and they would all basically write the same essay.

What does this mean for us as mathematics teachers?

Many, if not most non-English-speaking background students are very reticent about asking questions. Even if they know a mistake has been made, they are unlikely to point it out, as it is often culturally unacceptable to point out a teacher's error. This makes your job as a maths teacher much more difficult, unless you can somehow instil in your students the healthy view that in this classroom, asking questions is expected, encouraged and essential!

It will also help to quickly disabuse them of the idea that you know everything. As previously mentioned, it is essential to be well prepared, but NOT to be rigid in your approach to problems. Encourage discussion on alternative solutions, and be open and accepting of comments such as, 'In my country we were taught to do that in this way...'. Make use of your students' previous life experience; it will enrich your classroom life.

Be aware of 'cultural traps' in your maths teaching

There is an obvious need to be culturally inclusive in your notes and test questions; a question that once mentioned that 'Bob, Carol, Ted and Alice set off for a quiet weekend in a remote cabin. If they travelled at 45 km/h for the first 45 minutes, then...' etc., now needs to be written as 'Ivan, Ludmilla, Quan and Than Vu set off...'.
 However, this is trivial and obvious.

What may be less obvious is the cultural bias we often put into our questions without even realising it.

Take for example, this question which I put into a year 11 test on mensuration a few years back:

Take for example, this question which I put into a year 11 test on mensuration a few years back:

A running track is in the form of a rectangle, with a semicircle on either end...

The question then went on to ask various area questions. Imagine my surprise when one student handed up a solution based on Figure 1.

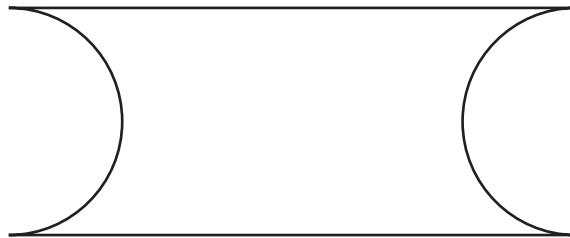


Figure 1

Now a colleague was fairly disbelieving of this recently, saying that EVERYONE knew what a running track looks like, especially as we have just held the Olympics!

What cultural assumptions was this colleague making?

Language

Our discussion of cultural differences above has (quite expectedly) drifted into the area of LANGUAGE, and the following points need to be recognised and acknowledged:

- *ALL* of the ‘non-English speaking background’ students are with you to learn the subject specific language of mathematics;
- *SOME*, but by no means all of them, are also there to learn the maths itself;
- many students from overseas have already completed the equivalent of our final year of secondary schooling prior to University, some have done some tertiary subjects, and some have already completed one (or more) degrees! This does not seem to worry them, because in many cultures (unlike our own, in my experience) it is considered a good thing to be covering topics already learned, as this increases the likelihood of scoring high marks. However, the issue of ‘RPL’, or ‘Recognition of Prior Learning’ is one which should be addressed when course requirements allow.

The following are issues which I have identified in my teaching as important literacy issues in maths

Defining terms

All mathematical terms *MUST* be defined the first time you use them — and probably several more times!

It has been my experience with people learning in English for the first time, that they do not necessarily remember a new term the first time it is introduced to them. It is probably also true, if obvious, that many words you might use as everyday usage are, in fact, new to some students.

The only one who will *know* if a word, term or phrase needs defining is the student himself or herself. This leads to our next obvious point.

Question technique

Students *MUST* be taught, encouraged, shamed, embarrassed, or otherwise forced into a classroom culture where asking questions is accepted, routine, and safe.

Questioning technique needs to be *taught*, and constantly *reinforced*; for example:

If I use a word you don’t know, you *must* ask me what it means.

can often be followed up by pointing to an unusual word on the board, and asking,

Who knows what this means? Can anyone tell me? Well then, why hasn’t anybody asked me?

However, what then usually happens is that you will use a word in general conversation — which you would almost certainly get away with in a class of teenage Australians and which has nothing to do with maths — and a hand will shoot up and someone will want to know what the word means. If care is not taken, you will be off into an English lesson sidetrack.

Well, you told them to ask, didn’t you!

A memorable example of this was in a year 11 class, where a particular student had been stirring a little. Since he usually left his cap on in class, I decided to pay him out about this, so I suggested that the polite thing to do was to remove his 'titfer' when he entered a room. BIG MISTAKE!

The hands shot up, and I found myself talking about Cockney Rhyming Slang, which of course necessitated definitions of Slang, Rhyming and, the big one, Cockney. Meanwhile, the elderly Vietnamese lady in the front row was busily writing down 'Tit for tat = hat', 'plates of meat = feet', etc., because she 'wanted to learn to speak English properly'!

(We normally do not bother about hats in class, or what students wear, because (1) it can be a culturally sensitive area and (2) it really is not important anyway!)

Diagrams must be provided

Unless producing the diagram is part of the desired outcome of the question, NOT providing a diagram often means that the students are not able to show that they know how to do a particular problem. (Recall the running track problem).

Worded questions are dreaded!

This ties in with the previous points; students are often not able to differentiate between key terms, and words which are not at all important to the problem.

This leads to excessive use of dictionaries, and much concern and anguish over completely irrelevant information.

This is possibly *THE* most concerning problem for non-English speaking maths students, and for their teachers.

I teach my students to identify KEY words and phrases; I also try to convince them that, if a word is NOT one which has been used in class during the year, then it is almost certainly NOT a key part of the problem.

Example:

An experimental battery produces a voltage that is dependent on the temperature t (in degrees Centigrade). The voltage, V , is modelled by the formula $V = 2t^3 + bt^2 + c$ for $-1 \leq t \leq 2.5$, where b and c are real numbers. Measurements show that V decreases to zero at 2 degrees C, and then begins to rise again as t increases.

Find the quotient when V is divided by $t-2$, etc. (PES Mathematics 1, SA, 1997)

Who would have thought that this was a Remainder Theorem problem?

Homonyms

The reason many people have given me for finding English a very difficult language is the fact that many words have more than one meaning; e.g. plane.

Recently a German student admitted in class that throughout the whole of the semester, whenever I talked of equations of planes, lines intersecting planes, etc., his

first thought was of planes in the sky, and he really did not know what I was talking about.

Differentiate

I used this word earlier in this paper to mean ‘tell the difference between’, but of course in mathematics it has a much more specific meaning.

These are just two which quickly came to mind. Problem words such as these can be dealt with by remembering to always, and frequently, define the terms you are using.

Closing comments

It is my opinion that one cannot be a maths teacher, without also being a teacher of language.

Many of the issues I have raised in this paper would apply equally to English speaking background students.

If one knows the subject matter, and enjoys teaching Mathematics, then there are no problems, only challenges, and teaching adults from other cultures, with the huge variety of life experience they can bring with them, simply enriches the teaching environment.

If anyone wishes to discuss this topic any further, I can be contacted by e-mail at nash@tsc.sa.edu.au.

About the presenter

Derek Nash has been teaching Maths in SA Education Department Schools for 28 years, and is currently an Advanced Skills Teacher. Early in his career he also dabbled in Junior Science and Health Education, but latterly has concentrated solely on Senior Maths. For the last 9 years he has been teaching Maths to adults, many from a non-English speaking background, at Thebarton Senior College.

At a previous school Derek taught blind and visually impaired students, and this connection led to him occasionally ‘babysitting’ working guides dogs. It is not unusual to find one of these dogs ‘helping’ Derek with his classroom teaching.

Indigenous Mathematics — A Rich Diversity

Kay Owens

Papua New Guinea, Australia, East Timor, West Papua, and Oceania have more than 2000 different languages, all with their own counting systems. A brief overview of these systems illustrates the cycles and patterns to be found in some of these. The relevance of this knowledge to our teaching is discussed.

Introduction

I once taught, and learnt, extensively in Health Education. One thing I learnt about Indigenous nutrition issues is that food taboos are very common. In order to cope with their impact on nutrition, it was common practice to encourage people to consider whether the food taboos fitted into the category of (a) good for nutrition, (b) bad for nutrition, or (c) it did not matter whether the food taboos were followed or not as far as good nutrition was concerned.

For example, I had a close teaching friend. She was pregnant and she looked awful. She was tired and very anaemic. We discussed her food taboos as I knew she came from an area in which eating fish and dark green leafy vegetables were taboo during pregnancy (although these provided the most common protein and iron and B12 needed to avoid anaemia). I suggested that she could try eating other protein foods like peanuts. Unfortunately, over the years, it seemed that the taboo on fish protein had spread, with a little nutrition knowledge, to all protein. We discussed why this taboo had developed. It was probably related to the problem of large babies and difficult village births for mothers who were well-nourished as adults but not so well-nourished and hence small bodied as children. This in fact was not a problem for this teacher and she had access to a hospital. The taboo on fish also related to totems.

I think we can view Indigenous mathematics in much the same way. Does the Indigenous mathematics reinforce the school mathematics, does it lead to conflicts (which may need resolving) with school mathematics, does it not matter. My personal view is that in the cases with which I am familiar, whether enhancing or different, it is most likely to be beneficial to mathematical learning. It does, in fact, mean both teacher and students' mathematics is developing.

I wish to present some aspects of Indigenous mathematics that is different to school mathematics but which can be used to enhance all teachers, and students of both the Indigenous culture and others.

Like food taboos, mathematics is not free of other, very significant aspects of culture. In recognition of this, I wish to say that I hope I do not offend or misrepresent the Indigenous mathematics.

I am drawing extensively on the work of the late Glendon Lean whose thesis *The Counting Systems of Papua New Guinea and Oceania* is a classical record and analysis of the counting systems. Glen recorded and analysed nearly 900 of these counting systems. His collated data from first contact in the 1800s and 1900s, as well as questionnaire, and field data is extraordinary. This material is now being catalogued and held in the Glen Lean Ethnomathematics Centre at the University of Goroka. A copy of his thesis is probably available at Monash and Deakin Universities and copies of his Papua New Guinea appendices are available in most secondary and tertiary institutions in Papua New Guinea and in numerous Australian libraries e.g. UWS Bankstown.

I was also drawn into considering the comments of Robin Williams at the 1999 MANSW conference on the Yupno body-part tally system (original source, Wassman & Dasen, 1997). I found no confirmation of this data (Lean, 1991, 1993; Smith, 1988; Wurm & Hattori, 1981) but my check took me through a number of issues such as the likelihood of counting systems changing, especially dying out. The system did, however, prove to be unusually unique in that it was unlike, in fact unsupported by other body-part tally systems (Owens, in press). It is therefore worth having Glen's records and his summary of common features although systems are unique. It is also worth knowing about other ways of determining quantity and the cyclic nature of counting systems.

The languages of the region are classified into the Austronesian languages, Oceanic languages, and the non-Austronesian or Papuan languages. The Austronesian languages are generally around the coast and represent a second-wave of languages. The non-Austronesian languages are considered to be older and have a huge variety of structures unlike the Austronesian languages.

Different cycles

A 5-cycle system will have the basic number words of 1, 2, 3, 4, 5 with other counting words being made up of combinations of these words. (5, 20) digit-tally systems use hands and feet with 15 being *two hands and one foot* and 20 being *one man*. Additional basic number words are now included, for example the word for 20 which is also *man*. Many counting systems have several cycles before settling down into one supercycle like the 20 cycle.

Table 1 illustrates the large diversity of counting systems that Glen had collated and analysed. He selected to group all systems with the lowest cycle. The table shows what is frequent.

Table 1

Summary of counting systems in Papua New Guinea and Oceania as recorded by Lean (1992).

Type of System	Alternatives	Distribution
10-cycle	<p>'pure' type 10, 100, 1000;</p> <p>'Manus' type $7=10-3$, $8=10-2$, $9=10-1$;</p> <p>'Motu' type $6=2x3$, $8=2x4$ (some with $7=2x3+1$, $9=2x4+1$);</p> <p>(10, 20);</p> <p>(10, 60)</p>	189 out of 217 AN; NAN influenced by AN neighbours
5-cycle	<p>(5, 10) or (5, 10, 100)</p> <p>(5, 10, 20)</p> <p>(5, 20)</p>	<p>Second most common AN</p> <p>All systems in New Caledonia, some AN and NAN in PNG and Irian Jaya</p> <p>AN and NAN</p>
2-cycle	<p>'pure' with 1 and 2</p> <p>(2, 5) or (2, 5, 20)</p> <p>(2', 4, 8)</p> <p>2 more types</p>	<p>201 NAN, 30 AN</p> <p>37 NAN, 2 AN; usually associated with body-part tally systems</p> <p>Most common, 18 AN; digit-tally system with 2 subordinate</p>
4-cycles	<p>Cycle from first 4</p> <p>Cycle from second 4</p> <p>Superordinate cycles 28; 48; or 60</p> <p>Special name for each cycle</p>	<p>Highlands of PNG</p> <p>Enga dialects</p>
3-cycle		Restricted areas
6-cycle		Restricted areas
Body-part tally system	Diversity	Highlands of PNG, possibly once among Indigenous Australian languages

Note. AN are Austronesian languages, NAN are non-Austronesian or Papuan languages. 2' are modified 2-cycle systems. Not all languages are recorded.

Body-part tally systems

These systems can have a range of different cycles depending on which body parts are included in the cycle — the most common is 27 but they range from 18 to 74. They occur now in PNG and Irian Jaya but seem to have occurred also in Torres Strait and Australian language groups. Tallying usually begins on the small finger of the left hand, to the wrist and then along the arm, shoulder, left ear and eye, nose or

central part, and then down the other side of the body. If vocalised they tend to use the body part (see Figure 1).

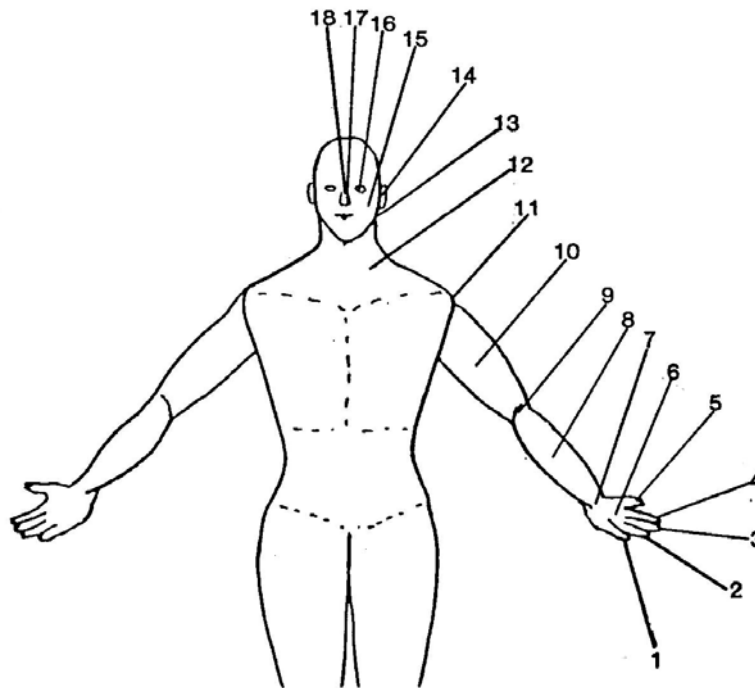


Figure 1. Body-part tally system of the Fasu, Southern Highlands Province.

Classifiers in counting systems

In some areas, particularly on Bougainville but also in New Ireland, New Britain, and Milne Bay and to a lesser extent in other Provinces, a morpheme may be used to distinguish different classes or groups of objects. They occur in both Austronesian and non-Austronesian languages. This morpheme is associated with number words providing a different set of counting words for different classes. In some cases, the counting cycle size and words change for the different classes of objects but this is likely to be a 'borrowed' idea from a neighbouring language or trading partner.

Non-counting systems

In a few different language groups, the larger amount of objects is compared by the amount of space taken up rather than by counting objects precisely. This is not an area or volume idea per se but a recognition that approximation and spatial abundance can be sufficient for a transaction.

Bases and cycles

There are few counting systems that have a regular base in which numerals are used for powers of the base. For example, only some island languages have a true base 10 system with numerals for $100 = 10 \times 10$, $1\ 000 = 10 \times 10 \times 10$ etc.

For this reason, the recognition of cycles and patterns within the counting system was more beneficial than bases for Lean to describe, collate, and analyse the data.

The patterns of the counting system

Lean (1991) has recorded and documented the patterns of the counting systems in his collection. He uses the term *operative pattern* for regular patterns. Operative patterns may include how the numbers between 6 and 9 are formed, the regular use of decades, (i.e., $20=2 \times 10$, $30=3 \times 10$), digit tally and body-part tally.

The digit-tally systems with (2, 5, 20) cycles have the following operative pattern which combines the frame words 1 and 2, $3 = 2+1$, $4=2+2$, then there is a frame word for 5, then 6 to 9 are combinations of the word for 5 (or another morpheme) and the words for 1 to 4. In a system like this, the counting frame is 1, 2, 5, and 20.

Geographical distribution of the counting systems from both Papuan and Austronesian language groups

The linguistic systems in Polynesia and Micronesia suggest that the 10-cycle systems of the Austronesian languages in the region retained the essential cyclic nature of their counting systems over a long time scale partly due to their relative isolation in Oceania. Glen's maps show the extent of 2-cycle systems. However, many of these systems also have (5, 10, 20) or (5, 20) cycles. These supercycles are more akin to the base of a system in practice.

Some uses of numbers

Case studies show that some non-Austronesian societies make extensive use of numbers in ceremonial contexts (e.g. Melpa). Others count a wide variety of objects using a 2-cycle variant system (e.g. Mountain Arapesh) while others adopted a 10-cycle system (e.g. Ekagi). Some societies place little importance on counting despite their 10-cycle systems and place an emphasis on the indivisible mass of a visual display. In these circumstances, some societies like the Loboda retained their 10-cycle system while others like the Adzera have numeral systems modified to a 2-cycle with digit-tallying like their non-Austronesian neighbours. The ability to amass large quantities of wealth items accords status and prestige to clans and individuals but the judgment of quantity, however, varies in each society from the visual impression to the precise counting. Lean notes that such diversity is also evident when other questions are considered, for example, (a) what are countable objects, (b) when are they counted, (c) which economies and exchanges use counting, (d) are different types of objects counted in different ways, (e) are some objects not counted, and (f) how are totals recorded. 'Counting does not exist in isolation. it quantifies and qualifies relations between people, objects and other entities' (Bowers & Lepi, 1975).

It is often heard that some of these systems are too 'primitive' to count large numbers. It is not possible to generalise about the use of large numbers in these languages although the systems can be used to generate large numbers and continue forever. Some further points are made by Lean: (a) some of the words for large numbers may mean *countless* or *indefinitely large*; (b) the same word is used in different languages for 1 000 or for 10 000; and (c) non-Austronesian languages tend not to have single terms for large numbers, like a million, unless borrowed from Austronesian influence.

Fractions are generally not used except a *half*. The Chuave make use of *half* to refer to a hand, that is *half of the hands of a man*. This word is repeated for the two hands. Usually *man* is used in the (2, 5, 20) systems for 20.

Multiples of two are common. Melpa and other groups seem to like twos and tend to count in twos and to give in twos, especially making 8 or 10 items (Strathern, 1977). Completing a pair seems important.

Enga is an unusually large language group with over 170 000 resident speakers (1982 census data) with 11 dialects and apparently some recent influence of 10-cycles from the English or Tok-Pisin systems. Some young informants thought that the non-decimal counting system was only used for large ceremonies. Some of the words for one dialect given in Table 4 was recorded in 1938. The 60-cycle system consists mainly of a sequence of 4-cycles beginning at 9 and each cycle typically has the construction: cycle unit+1, cycle unit+2, cycle unit+3, cycle unit+end where the cycle units, 13 in all, are not numerals as such but may be words or phrases with some non-numerical meaning like *dog*.

The origins of counting systems

Glen's thesis emphasised that counting systems did not spread around the world from the Middle East. The Indigenous counting systems belong to cultures that are much older than those of the Middle East. While the cycle of 2 is strong in remote places, nevertheless there is evidence to suggest that the parts of the body could influence the development of number. A key example is that of (5, 20) cycles and the 10 cycles. It is also obvious in the systems that have a 4 or 6 cycle in which the hand is then considered as having 4 or 6 parts.

Other mathematical concepts

Other mathematical concepts are also embedded in cultural activity. For example, *all groups measure*. In particular, comparisons are made by using existing lengths like a length of string. Equal lengths are particularly important in building in which shapes like rectangles are used with equal length sides and diagonals. Floor spaces are frequently divided up into equal parts and again equal lengths are used. A length of rope or a long stick is used to mark the points of a circle when circular houses are built.

When carving, men mark *symmetrical points* by marking off with equal-length sticks or string.

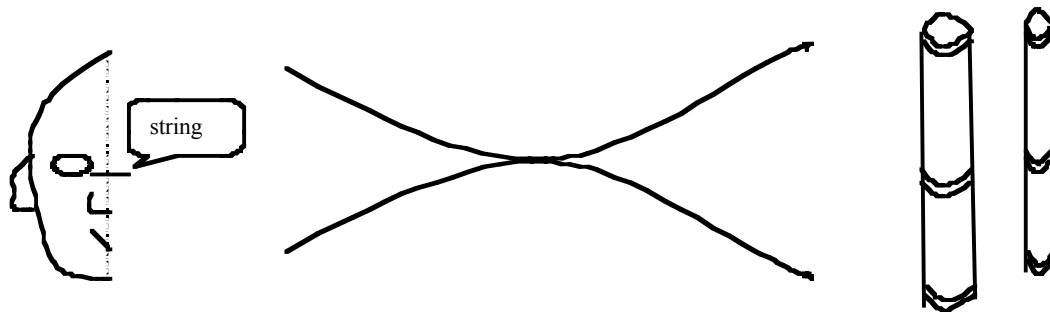


Figure 2. Symmetry, rope becomes diagonals of rectangle, bamboo volume units.

Water is regularly carried in containers, especially bamboo lengths. The *volumes* of thick and thin bamboos are associated with the amount of garden that can be watered. These proportional relationships are intuitively used. If the garden is three times the basic *area* that can be watered by one length, then the three length container will be chosen.

Spatial thinking is extensively employed in making decisions. For example, if a standard house is enlarged the increases in materials is known by the master builder. A good mental image seems to be held of the size of a house when a floor plan is enlarged.

Designs are regularly shared and modified. *Shared patterns* are usually illustrated by example but the number of strips under which a leave is tucked while weaving or the number of stitches made when weaving a string bag is noted. The overall design is dominant but how to get the design has known tactical procedures.

Interesting ideas are used by teachers naturally in primary school classrooms. For example, in joining papers together two techniques have been used, neither needing glue or paper fasteners. One is to make a hole in each piece of paper and join with a narrow cylindrical paper. The other is to slit two pieces so they can slide together.

Many activities require *organisation*. The order in which things are carried out is significant and emphasised. The collaboration of people's effort is also recognised. This takes the form of deciding on time for an activity and then allocating work and amounts of materials to be used to different people.

Time is also well developed as a concept. While marking in hours is not, until recently, well delineated, nevertheless the amount of time needed to undertake an activity was well developed. For this reason, people could rise in the middle of the night to get ready to set sail for a distant airport or to walk to the road head to catch transport to market to sell their garden produce. Lengths of time were intuitively compared. Time to walk to different places was also well established by experience and could be, to some extent used for deciding on other walking lengths.

Balance is another key idea that is well established. For example, many bridges are counter-balanced. Heavy rocks are used to counter the weight of a bridge swinging out across a fast-flowing stream. The number and thickness of posts needed for different types of houses and different parts of the house (walls or ceiling) is also

established. In a round house, the use of cross-beams in the ceiling and care with circularity has led to a recognition that the central pole can be cut away and the equal force of the ceiling on the top of the central pole is sufficient to keep it in place.

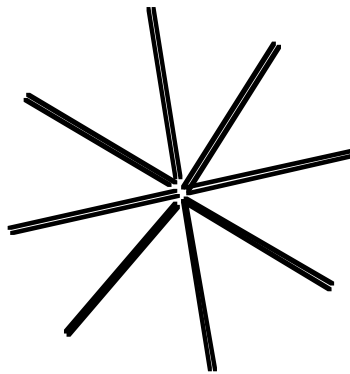
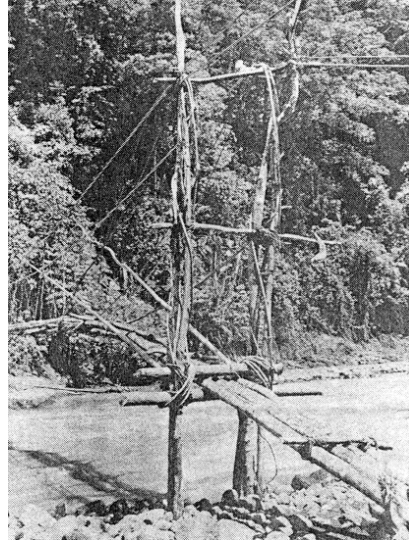


Figure 3. Partly counter-balanced bridge, circle formed by marking off points with a stick.

The seasons are a particularly important feature of hunter-gatherer societies that exist in Australia and parts of Papua New Guinea. People often move with the cycle of seasons that are described often in terms of natural phenomena such as flowerings, winds, and rains. These make interesting calendars.

Maps are also used. These generally feature spiritual connections to the land and to relationships between people. The connectedness of places frequently dominates the map with direction and distance being secondary.

Indigenous mathematics and teaching

There are several points to raise. Where the Indigenous culture is either strong or in need of preserving, students need to learn the mathematics of the culture. These conceptual, contextual mathematics have intuitive meaning for children. They form the foundation of learning.

Just as it was important to know when teaching nutrition that food taboos exist, so it is important for teachers to know about the existence of forms of mathematics other than the school mathematics with which we grew up.

It may be that some of our students come from cultural groups in which the mathematics has significant differences to those we are teaching. For example, many Indigenous cultures of the Pacific and the Americas have classifiers that are important in counting. It simply will not seem correct to have just one set of counting words for some children.

But the classifiers actually enhance another form of mathematics. Classification is a key mathematical process. It varies from culture to culture. Take for example the Greek and hence the Western form of classifying shapes. There are other ways of classifying, equally as valid, in other cultural groups. Doesn't this knowledge give us, as teachers, a bigger picture of mathematics itself. It should be shared with children at some stage.

Similarly, knowledge of different bases or rather different cycles, which is a much more useful term for describing different systems, enriches our understanding of mathematics. It certainly helps students to recognise a key feature of the base 10 place value system and this too should be shared with students at some stage, as it is currently in Year 7.

Of course, this brings us to another issue. Are there key processes in all mathematics. The closest we can probably come to that is the six mathematical activities suggested by Bishop (1988). These are measuring, designing, classifying, playing, making, and counting. All require mathematics of some form but they are activities not processes per se.

Finally, Indigenous mathematics indicates that some histories of number and some comments about Indigenous mathematics need to be questioned. The Middle East is not the only centre for the development of numerical mathematics nor is Greece the only centre of shape classification.

Furthermore, our teaching of geometry and shape often misses some critical issues. What do you think of when I mention the properties of rectangles. Many people cannot remember anything about diagonals but young men involved in rectangular house-building in Papua New Guinea rarely forget about the diagonal properties. They also happen to be very good at having an eye for equal, perpendicular, and horizontal lines. These are best decided when a number of posts are in position and can be lined up. That is, more than two points helps determine linearity and direction.

Now what of the issue of whether any Indigenous mathematics may be harmful. Just as it is frequently an issue that can be side-stepped with nutrition by substituting one food for another, so it is with mathematics. For example, village garden lands are frequently compared by using the sum of length and width. Now many gardens are not rectangular and even when they are, two different areas can have the same sum of length and width (or semi-perimeter). You do not have to be in a traditional culture to confuse perimeters and areas of rectangles etc. Try out how many students think that all rectangles with perimeters of 12 cm have equal areas. However, reference to traditional mathematics is a neat way in which confusions can be addressed. There is much more to deciding land issues than mere size.

Despite the number of times, that I have used my non-Indigenous mathematics to explain what I regard as Indigenous mathematics, I am deeply aware that I do not have the language or the experience to really have that 'sense' of Indigenous mathematics or that real mathematical understanding that comes with language and culture. However, that does not mean that we should not make links. It does mean that learning mathematics in a first language is very important. It also means that mathematics must be seen as socially constructed and where differences seem to appear, these must be addressed.

Traditional mathematics may remain the providence of that society, it may have links with Western society mathematics, and it may be basically equivalent.

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About the presenter

Kay Owens is a Senior Lecturer in Mathematics Education at the University of Western Sydney. She has taught students at primary, secondary and tertiary level. Her main subject interests are mathematics, mathematics education and health education. Her research has mainly focused on how students learn from being responsive during problem solving and Space mathematics. She also has an interest in ethnomathematics, especially in relation to Papua New Guinea cultures, and in evaluation of preservice teacher education. Kay joins in teaching for an hour each week in primary schools to try out ideas that she has developed from research and to keep her feet on the ground.

Mathematical Expectation in Gambling and Games of Chance

Robert Peard

The concept of mathematical expectation has a variety of practical applications and is central to the application of probability to decision making. The topic is now included specifically in many secondary school mathematics curriculums. Earlier research by the author demonstrated that relatively sophisticated applications can be performed by students with relatively little mathematical background. Consequently, the author has developed a mathematical content elective unit in probability for B.Ed. primary pre-service teachers at QUT based around the mathematics of games of chance and expectation. This paper describes some of the mathematical content of the unit. One application shows how professional gambling syndicates operate with a positive mathematical expectation. An analysis of the recently reported 'sting' of the Queensland TAB by such a syndicate is given as an illustration.

Introduction

The concept of the mathematical expectation of the outcome of an event as the product of its probability and the return or consequence is one that has a variety of practical applications and is the key concept in the application of probability to the decision making process. The decision of an airline to overbook flights, for example, involves computing the various probabilities of the numbers overbooked and forming the product of these and the associated cost of each eventuating. These are then compared with the probabilities and costs of numbers of empty seats. Other applications include insurance, warranties, restaurant overbooking, cloud seeding, and a variety of situations in gaming and betting. The topic is now included specifically in many secondary school mathematics curriculums (Queensland Senior Mathematics A and B for example). Earlier research by the author (Peard, 1995) concluded that people who engaged in the playing of games of chance and were familiar with betting at odds had an intuitive understanding of the basic nature of mathematical expectation. They understood its reciprocal nature: the smaller the probability, the greater should be the return. They also demonstrated that relatively sophisticated applications of the concept in these contexts can be performed by people with relatively little mathematical background. Building on this research, the author has developed a mathematical content elective unit for B. Ed. primary pre-service teachers at QUT developed around the concept of expectation. The mathematical prerequisite knowledge for this unit is requires only competence in basic arithmetic and a little algebra. It has proved to be a popular elective attracting about 80 students each year. The content and approach would also be suitable for the probability topics in many secondary school courses and an example is presented here in this light.

The gambling context

Concern with the general mathematical competencies of pre-service primary teacher education students has been expressed for some time now (See for example, Carroll, 1994). Often these students express negative attitudes towards mathematics as a result of their own school mathematics experiences. It is contended that without appropriate pre-service intervention, many of these students will transfer their own attitudes later to their own pupils thus repeating the cycle. Most pre-service primary teacher education programs allow only a limited time for the teaching of mathematical content and often what content is presented is seen by the students as being of limited value or relevance, often repeating what they disliked in school mathematics. The content elective at Queensland University of Technology, 'The mathematics of gambling and games of chance' has been developed with the aim of helping to break this cycle by offering the students an elective that develops the basic probability content in the motivational context of the study of gambling.

Probability in the curriculum

The importance of understanding probabilistic concepts in modern technological societies has been well established for some time now. It has been argued that it is essential that students be taught how to deal realistically with uncertainties so that they may respond to probabilistic situations without preconceived notions, emotive judgements or even a lack of awareness that chance effects are operating. Recent curriculum developments in primary school mathematics have seen a much greater emphasis on the role of probability in the classroom (Borovcnk & Peard, 1997). Chance and Data features as a strand in the *National Statement on Mathematics for Australian Schools*, and the Queensland curriculum includes both topics in all years from 4 to 12. Despite the recognition of the importance of probabilistic concepts by mathematics educators, the inclusion of probability into most mathematics curriculums is a relatively recent development and studies have shown that the content knowledge of topics in probability and statistics for both primary and secondary teachers is often deficient (Shaughnessy, 1992). Furthermore, many pre-service primary teacher education students may have had little formal school education in the topic.

Content and approach

The approach to the unit is informal and intuitive building on the students' interest and familiarity with the subject without assuming any prerequisite knowledge other than the ability to convert fractions to decimals and percents, fractional equivalence and basic operations. One of the major objectives of the unit is to ensure that the students do not hold any of the misconceptions about probability that are reported as common (See, for example, Peard, 1996a, 1996b). These misconceptions, including the 'gamblers' fallacy', are not confined to naive subjects and are prevalent among tertiary students (Shaughnessy, 1992) and pre-service teacher education students (Peard, 1996b). Key to the remediation of these misconceptions is the development of the concept of independence and mathematical expectation or expected return.

Expected return

This concept is introduced by an analysis of the playing of common casino games such as roulette. If you bet say \$10 on the 'red' on a roulette wheel, the probability of winning is $18/37$. If you win you will receive \$20 (your \$10 bet plus your \$10 win). We say that your expected return is the product of the probability of winning and the return from such a win.

$$ER = p A$$

where p is the probability of winning, and

A is the total amount of payment you will receive if you win.

In this case $ER = 18/37 \times \$20 = \9.73

What this tells you is that in the long run you can expect to get back \$9.73 for every \$10 bet. Although on any one particular play you will get either \$20 or nothing, the figure of \$9.73 is what you *expect* to get on the *average*. Sometimes we refer to this as the *Mean* or *Average return*. This figure is better expressed as a % of the outlay. In this case:

$$ER = 9.73/10 \times 100 \% = 97.3 \%$$

We then examine the playing of roulette and show that *all* bets on roulette have an expected return (ER) of 97.3%. This means that for every \$100 bet, the house pays out \$97.30 and keeps \$2.70. In a lottery with several prizes, the expected return can be computed by considering the product of the probability of each and the amount of each. Mathematically,

$$ER = \sum p_i A_i$$

Alternatively, the expected return can be computed more easily as:

$$ER = \frac{\text{the total amount paid out in winnings}}{\text{the total amount taken in.}}$$

Mathematical expectation

This is the probability of winning times the amount won less the probability of losing times the amount lost.

$$Ex = p W - q L$$

where p is the probability of winning and $q = 1 - p$ the probability of losing,

W is the amount won, and L is the amount lost.

Using the roulette example again:

$$p = 18/37, W, \text{ Amount won} = \$10$$

$$q = 19/37, L, \text{ Amount lost} = \$10$$

$$\text{So } Ex = 18/37 \times 10 - 19/37 \times 10$$

$$= - \$ 0.27$$

This means that on a \$10 bet you 'expect' to lose 27c.

Or on *any* roulette bet you expect to lose 2.7%

$$Ex = -0.027 \text{ or } -2.7\%$$

We can show that if $Ex = -2.7\%$, $ER = (100 - 2.7)\%$ and that in general:

$$ER - Ex = 1$$

Thus the two terms are measures of the same concept and either can be used. A negative expectation implies an expected return of less than 100%. The expected return of all Casino games (with the exception of blackjack) is determined. Since these are always less than 100%, (or negative expectation) there can be no long term system of winning. Thus, mathematically there can be no 'system' of winning and a number of common fallacies are analysed.

Mathematical fairness

Mathematical fairness and its relationship to gambling and games of chance in bookmaker betting, the setting of bookmaker odds and the determination of 'fair' odds is another central concept. In everyday or colloquial use, a 'fair' game can have different meanings. A football game is 'fair' if there is no foul play. A roulette wheel is 'fair' if each number has an equal chance of showing. A teacher is 'fair' if all children are treated equally. A horse race is 'fair' if there is no outside interference even though we know that each horse does not have an equal chance of winning. 'Fairness' does not necessarily imply 'equally likely' although when we use the term with reference to coins and dice, it does. In these situations it is better to speak of 'unbiased' coins rather than 'fair' coins. A mathematical definition of fairness can be formulated from expectation:

A game is fair if the total paid out by the losers is the same as the total collected by the winners. Each player (including the house, if any) has the same mathematical expectation. That is to say, the Expected Return for all concerned is 100%. In this sense none of the Casino games examined in the last chapter are 'fair', since the players' Expected Return is always less than 100%. On the other hand, the play of the games is always 'fair' in the non-mathematical sense in that it is free from interference and the roulette wheels, coins, dice etc. are 'unbiased'.

Totalisation systems and mathematical expectation

In situations such as horse racing where the probabilities of the various outcomes can be only estimates, the mathematical expectation of the bettor is best determined by considering:

Expected Return to the bettor is equal to the total amount paid / the total taken in.

We examine the operation of a totalisation system and see how it cannot lose since the % profit is subtracted before the winnings are distributed. For example, using Microsoft *Excel*:

#	win bets	dividend		place	dividend
1	\$ 5.00	\$ 3.40		\$ 5.00	\$ 1.13
2	\$ 1.00	\$ 17.00		\$ 1.00	\$ 5.67
3	\$ 2.00	\$ 8.50		\$ 2.00	\$ 2.83
4	\$ 1.00	\$ 17.00		\$ 1.00	\$ 5.67
5	\$ 2.00	\$ 8.50		\$ 2.00	\$ 2.83
6	\$ 1.00	\$ 17.00		\$ 1.00	\$ 5.67
7	\$ 5.00	\$ 17.00		\$ 5.00	\$ 5.67
8	\$ 3.00	\$ 5.67		\$ 3.00	\$ 1.89
total	\$ 20.00		total	\$ 20.00	
win pool	\$ 17.00		place pool	\$ 5.67	(each)

Table 1. An example of a totalisation system.

In the above example a total of \$20 is bet on the 8 horses, 15% profit deducted and the remaining \$17 is placed in the win pool to be paid to the winner according to the amounts in the adjacent dividend column. In the same way, in the place bet, the \$17 is divided by 3 then paid to the three places according to the amounts in the column. Thus, in both win and place betting, the mathematical expectation of the punter is approximately 85%.

Professional gambling syndicates

All casino betting situations operate with a mathematical expectation for the punter ranging from about 85 % to 97.3 %. Professional gamblers operate from two basic axioms:

1. the mathematical expectation is greater than 100%
2. short term losses can be covered.

As such, professional gamblers do not bet at casinos (except for blackjack, which is discussed separately), but confine their betting to on track racing and lottery situations where jackpots can occur and greatly increase the expectation (See, Peard, 1998, for an analysis of this situation). Professional syndicates also generally avoid totalisation betting. However the Brisbane Courier Mail, 3/10/00, reported 'Betting sting on ice after TAB hit'. The hit referred to was a set of bets placed by a Canadian syndicate on an obscure greyhound race that netted the syndicate a \$170 000 profit. This was a classic case of an organised syndicate betting legally with a positive mathematical expectation. Just how this could happen is a mystery to most members of the public. Unfortunately, the reporting of events such as this tend to reinforce public misconceptions about 'systems' of betting on situations such as roulette where no system can exist. However the mathematical analysis of this situation is relatively simple.

The syndicate bet \$730 000 in such a manner that there were three possible outcomes;

- a win of about \$170 000,
- a win of about \$ 30 000,
- or a loss of about \$ 120 000.

To understand how this situation arose, we need to consider the 'loophole' in the rules that was exploited.

TAB Queensland had a rule that if more than 50% of the pool was bet on any one outcome and if, after totalisation, the dividend was less than \$1, then it would be rounded up to \$1. That is you at least get your money back if you get a place. This can greatly reduce the TAB's % profit.

For example:

#	place bets		payout		
1	\$ 50.00	\$ 0.69	\$ 1.00		
2	\$ 10.00	\$ 3.44	\$ 3.44		
3	\$ 10.00	\$ 3.44	\$ 3.44		
4	\$ 10.00	\$ 3.44	\$ 3.44		
5	\$ 10.00	\$ 3.44	\$ 3.44		
6	\$ 10.00	\$ 3.44	\$ 3.44		
7	\$ 10.00	\$ 3.44	\$ 3.44		
8	\$ 10.00	\$ 3.44	\$ 3.44		
total	\$ 120.00				
place pool	\$ 34.40		\$ 118.80	total on	
			\$ 120.00	in	
			\$ 1.20	profit	1%

Table 2. A totalisation system with minimum return of \$1

In the above example \$50 is bet on the one outcome, resulting in a totalised return on this of \$0.69. Since the \$50 is less than 50% of the total bets (\$120) the 0.69 is rounded up to \$1 so that in the event of a place by #1, the \$1 outlay is returned. Should #1 place, the TAB will pay out a total of \$118.80 for a profit of only \$1.20 or 1%.

However, in reality this rarely happens because as betting proceeds and the payouts are displayed, no one will place bets on #1 showing a return of \$1 and other punters will come in and increases the bets on the other outcomes thus redistributing the various returns.

For example,

#	place bets		payout		
1	\$ 50.00	\$ 0.95	\$ 1.00		
2	\$ 20.00	\$ 2.37	\$ 2.37		
3	\$ 30.00	\$ 1.58	\$ 1.58		
4	\$ 25.00	\$ 1.89	\$ 1.89		
5	\$ 10.00	\$ 4.73	\$ 4.73		
6	\$ 10.00	\$ 4.73	\$ 4.73		
7	\$ 10.00	\$ 4.73	\$ 4.73		
8	\$ 10.00	\$ 4.73	\$ 4.73		
Total	\$ 165.00				
place pool	\$ 47.30		\$ 144.60	total	on 1,2, 3
			\$ 165.00	in	
			\$ 20.40	profit	12%

Table 3. Totalisation system with redistribution.

In the situation reported, the syndicated reasoned as follows:

- Eliminate the influence of other punters; choose an obscure event with a small pool (about \$2000) and ‘swamp’ it with a huge bet (\$730 000).
- Bet on the *two* favourites so that the more than 50% rule cannot apply.
- Place enough on the two favourites (\$350 000) to give a small dividend to them (which will then be rounded up to \$1) and large dividends to the other six. Bet (\$5000) on all the other six.
- Select an event where it is highly unlikely that neither of the two favourites will not place (a greyhound race was selected).
- Place the bets as close as possible to the last moment so that other punters will not have time to place enough other bets to greatly affect the payouts (All bets were placed in the last 6 minutes of betting).

This is what is most likely to happen:

#	place bets		rounded	payout		
1	\$350,000	0.59	\$ 1.00	\$ 350,000		
2	\$350,000	0.59	\$ 1.00	\$ 350,000		
3	\$ 5,000	41.37	\$ 41.35	\$ 206,750		
4	\$ 5,000	41.37	\$ 41.35	\$ 206,750		
5	\$ 5,000	41.37	\$ 41.35	\$ 206,750		
6	\$ 5,000	41.37	\$ 41.35	\$ 206,750		
7	\$ 5,000	41.37	\$ 41.35	\$ 206,750		
8	\$ 5,000	41.37	\$ 41.35	\$ 206,750		
	\$730,000					
place pool	\$206,833			\$ 906,750	total	on 1, 2, 3
				\$ 730,000	in	
				\$ 176,750	loss	for TAB

Table 4. Syndicate betting with two favourites placing.

So, if the two favourites both place, the TAB loses about \$177 000, nearly all of which goes to the syndicate.

If only one of the two favourites place then there will be a profit of about \$30000:

	#		
1st	1	1	\$ 350,000
2nd	8	41	\$ 206,750
3rd	7	41	\$ 206,750
		collect	\$ 763,500
		in	\$ 730,000
		profit	\$ 33,500

Table 5. Syndicate betting with one favourite placing.

If neither of the two favourites place, then a loss of about \$111 000 will occur.

	#		
1st	6	41	\$ 206,750
2nd	8	41	\$ 206,750
3rd	7	41	\$ 205,000
		collect	\$ 618,500
		in	\$ 730,000
		profit	\$(111,500)

Table 6. Syndicate betting with neither favourite placing.

The influence of other punters will affect these figures slightly. In actual fact, the two favourites placed 1st and 3rd with a payout of \$1 each, 2nd place paid \$41, and the syndicate won \$170 000.

The mathematical expectation of the syndicate in this situation

The determination of this illustrates a number of basic probability principles that have been developed. Some assumptions and estimations need to be made.

There are three outcomes:

- both favourites place: a win of about \$170 000,
- one favourite places: a win of about \$ 30 000,
- neither favourite places: or a loss of about \$ 120 000.

We can estimate the probabilities of each of these:

Starting with (c), the syndicate has used the frequentist approach to estimate this probability. They observe that at greyhound racing the number of times that neither of two short odds favourites have placed is very low. Let's assume a figure of 0.1.

This will give us an estimate that the probability of each favourite placing of about 0.7

Probability of not placing 0.3 (Probability of both not placing 0.3×0.3 or approximately 0.1)

(Of course these need not be equal, and are not strictly independent).

However, using these estimates we get:

$$p(c) = 0.1; \quad p(a) = 0.7 \times 0.7 = 0.5; \quad p(b) = 1 - 0.6 = 0.4$$

$$\begin{aligned} \text{So Expected Return} &= 0.5 \times 170\,000 + 0.4 \times 30\,000 - 0.1 \times 120\,000 \\ &= \$85\,000 + \$12\,000 - \$12\,000 \\ &= \$85\,000 \end{aligned}$$

This is on an outlay of \$730 000, or about 12 % or ER = 112 %

Students in the unit use Excel spreadsheet to explore the expectation under various probabilities.

Note: After this happened the Queensland TAB changed their rules so that no payout can exceed what is in the pool for that event. This was already in effect in other States.

Conclusion

Students in this unit have demonstrated that they are quite capable of performing the above analysis. One of the themes of this unit and our other Mathematical Foundations unit is that the study of mathematics empowers people in the understanding of the operation of many aspects of society. Knowledge of probability and how it is used in the decision making process is a critical component of this. The performance of the students in the unit and their motivation by examples such as the one illustrated here have been most encouraging in the endeavour to improve the mathematical content knowledge of pre-service primary school teachers.

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About the presenter

Dr Robert Peard is a member of the School of Mathematics Education at the Queensland University of Technology where he teaches pre-service teachers, both primary and secondary. His research interests are in the fields of the teaching and learning of probability and statistics with a special interest in misconceptions. He currently coordinates a unit in Mathematics Foundations and is encouraging pre-service primary teachers to select mathematics content electives in their B.Ed. program.

Counting On: An Evaluation of the Learning and Teaching of Mathematics in Year 7

Bob Perry and Peter Howard

Counting On is a numeracy program in New South Wales government schools that targets Year 7 students who have not achieved Stage 3 outcomes in mathematics upon entry to high school. During 2000, *Counting On* has been implemented in 40 high schools and the authors are evaluating this implementation. As part of this evaluation, the authors conducted case studies. This paper reports on two of these case studies with particular reference to the impact of *Counting On* on the professional development of mathematics teachers and support for student learning outcomes.

Background

Counting On is a numeracy program that targets Year 7 mathematics students who have not achieved Stage 3 outcomes in mathematics upon entry to high school. The program focuses on the professional development of teachers in addressing the student's learning needs and operates on a team approach involving, in each school, the Head Teacher, Mathematics (HTM), the Year 7 Classroom Teacher (CT), the Support Teacher Learning Difficulties (STLD) and the District Mathematics Consultant (DMC).

Key elements of the program are the training program, the assessment schedule, the initial assessment test (Term 1), follow up assessment test (Term 3), the videotaping of the assessment interviews, the team's analysis of student responses, the determined teaching and learning strategies, and the implementation of such strategies. The research base for the program is provided through the *Counting On* Numeracy Framework (Thomas, 1999). The evaluation which is the subject of this paper recognises and expands the evaluation of the pilot program which was undertaken in 1999 (Mulligan, 1999). As part of this evaluation, four case studies were implemented. This paper reports on two of these.

Methodology

Each case study school was visited three times. The purpose of the first visit was to establish contact with the school and, in particular, with the *Counting On* team in each school. Through individual and group interviews with teachers and students, which were audio recorded and transcribed, data were gathered on the sociocultural contexts of the school community, the needs of the students in numeracy, the team's perceptions of the *Counting On* training program, and the nature and adequacy of ongoing professional development for the teachers. Interviews were conversational in nature, led by researcher questions where necessary. Observation of *Counting On*

classroom activities was also undertaken. The second visit sought, through further interviews and classroom observations, to refine the researchers' tentative findings for each of the schools. The purpose of the third visit was to discuss the draft case study report for the school.

In Cusack High⁴, all the mathematics teachers involved with *Counting On*, the DMC and selected children involved in the program were interviewed. The STLD was not present at either interview. Each visit lasted for up to 4 hours to facilitate interviews and to accommodate the contextual needs of teachers' timetables and availability.

In Ridley Central, students were interviewed in groups while the mathematics teachers, the STLD and the DMC were interviewed individually, and as a group. On each visit, the researcher observed a *Counting On* class and was able to work with the students. As well, he was given access to a sample of the videotapes made during the first student assessment.

Data from the *Counting On* student assessment interviews in both schools has also been used in the development of the case studies.

Description of the case study schools

Cusack High

The school is a Year 7 to 10, co-educational, high school in a disadvantaged area of western Sydney with a student population of about 580 and a mathematics staff of six [2 female; 4 male]. The school is endeavouring to meet the needs of a cross cultural lower socio-economic community with a significant number of transient students across any given year. The school implemented *Counting On* in 1999 as part of the initial pilot. During 2000, fourteen students were involved in the *Counting On* program. There are eight graded classes in Year 7. Two children from each class except the top mathematics class were included in the *Counting On* program. The school did not target the weakest mathematics students rather 'our bottom middle and even some of our middle top kids.' The teachers selected students based on the 'judgement of which kids we would still have at the end of the year and who were more likely to talk to us during the interview' [Maths Teacher]. Across a fortnight, Year 7 had 6 maths lessons with *Counting On* being a focus for half the Thursday afternoon lesson.

Ridley Central

This school is a K–12 central school situated in a struggling rural town in north western New South Wales. It has a student population of approximately 300, with about 120 in the high school. In this part of the school, there are three mathematics teachers but only one (the Head Teacher) who does not take other subjects as part of his teaching load. The school is not classified as a disadvantaged school. There are

⁴ Cusack High and Ridley Central are pseudonyms.

three mathematics classes across the Years 7 and 8 — a straight Year 7 class, a straight Year 8 class and a class from both years consisting of students who were having trouble with their mathematics knowledge and skills.

From this class I intended to test them all and put them all onto *Counting On*. They are all learning it in class anyhow. The students who were tested were those whose parents agreed could be tested from that class. Anyone who was in that class would have been part of the program if their parents allowed them to. [HTM]

There were four students in Year 7 and six in Year 8 who commenced the program.

Findings on professional development

Cusack High

The HTM believed all mathematics teachers should have access to information about *Counting On*. He thought it important that all the mathematics teachers developed the teaching skills and strategies evident in *Counting On* program for the benefit of all students across Years 7–10. In his view, all the teachers do not have the time to develop their personal knowledge and competencies of the *Counting On* teaching strategies.

The STLD had helped prepare materials and worked in the classrooms as another pair of eyes and ears. Though ‘realistically, she says that her numeracy skills are quite weak. As mathematicians we do these questions really fast and often don’t think. She would say, “That’s interesting. I don’t add that way. I do it like this or whatever” [CT]. The mathematics teachers believed that, because they have done it so often and know it so well, they teach quite rapidly. ‘We forget that it isn’t so easy and to have someone there to kick us in the pants and say, “Hang on, not everyone thinks the way you do!” is a very good thing’ [CT].

The assessment tasks helped the teachers observe in more detail the strategies being used by the students. This was useful for ‘as a mathematician we think the things are easy but 90% of the population don’t think that way’ [CT].

Counting On forced the teachers to ‘learn how kids learn. We just assumed that they come to Year 7 and they can add up and they can subtract and if they can’t then they are weak at maths and that’s the end of the story. It taught us how kids actually do learn to add up. The awareness raising has been good’ [CT]. Another mathematics teacher commented

I guess in a lot of ways it’s made me aware of how to introduce these strategies and rather than just saying, ‘Oh well, just learn your times tables’. I can see that there are smaller steps in the counting. I have always been big in saying, ‘We can do it this way and that way,’ but I’ve never really thought of just simply counting on. I’ve always been someone that breaks the numbers down into pairs I suppose. I haven’t really thought of going back further. I wouldn’t say it’s affected my teaching in terms of dramatically changing it but it has just given me more idea to implement with more students.

Counting On helped the teachers ‘look at how children think in different ways... consider individual differences... notice how many children use their fingers to count’ [CT]. The teachers believed they were observing the students more.

I think it is a lot more focused. In the past I’ve tried to understand what kids are doing but because now I understand the small steps I know what to look for so I can see it a lot clearer. Most of the kids I could see when they were counting that they were using the ruler but the idea of them counting on or counting from scratch and pairing is giving me more focus to see exactly what they are doing. [CT]

Ridley Central

The *Counting On* team said that they had learned a great deal by being part of the program. Not only have they learned about some new activities but they have also had to justify — to themselves, individually, and in the team as a whole — the ways in which they teach the students. This reflection — based on the program and its assessment procedures — has helped them develop better procedures for the *Counting On* class. The HTM summarised the value of the professional component by saying that he thought ‘the teacher training part of it [*Counting On*] was probably as valuable a part of it as any’.

It was considered important for the STLD to be involved

because whichever students he works with in the school [he will know what they have done]. I think it is very good professional development for the STLD and even though he might not be directly involved in the classroom with [the mathematics teachers], I think he can use that information in his role anyway. [DMC]

The STLD identified two key benefits of *Counting On*.

The first will be seen when the kids that have been involved get through Year 10 with improved results. The other benefit is that there are a lot of secondary maths teachers who have been made to rethink the ways in which they teach certain streams of mathematics — they have been given new strategies. They have been given another insight into the way kids learn — not the way they teach kids but the way kids learn. ... The second benefit will probably be the more lasting one.

Analysing the videotapes and ascertaining the levels at which the students were performing was seen by the *Counting On* team as one of the highlights of the program. For example, the HTM could see many benefits in the assessment analysis process.

I think the strength of the assessment is that you can actually sit down and watch what the kid is doing. Often, when we are testing the class, there are so many students around that you really tend to focus on what they have answered rather than the thinking that they went about it. The strength is that we continually asked, ‘How did you get that,’ or ‘Talk to me as you’re doing it,’ so you are focusing on the strategies that they are doing which you don’t have the time in class to do. Having done the assessment I think now in class I can pick some of

that up a little bit more quickly. When they are answering in class I can say, 'Why did you get that?' and 'How did you get that?'

Teachers have indicated that the professional development opportunities afforded them through *Counting On* training days, activities development, assessment and analysis, whilst time-consuming and sometimes difficult to place into an already overloaded schedule, have been the highlights of the program for them. It was felt by the teachers that these opportunities should continue as the program continues, thus further expanding their opportunities to learn.

Findings on student learning outcomes

Quantitative data

Students were assessed in a one-to-one interview using the *Counting On* Assessment Schedule, consisting of 19 questions. The *Counting On* team then analysed the assessment using the interviewer's notes and the videotape record of the interview. In this paper, we have chosen three particular questions to illustrate the progress which the students in the two schools made between the two applications of the assessment schedule. Obviously, this is a preliminary analysis which will be expanded in later papers.

Correct				Strategies used							
				Circles 2 squares				Circles 20 squares			
Cusack		Ridley		Cusack		Ridley		Cusack		Ridley	
T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2
40	88	20	67	36	13	86	33	36	88	14	67

Table 1: Percentages of school cohort with question: 'Show the numeral 24. Point to 2. *Can you circle the number of squares that this part of the number stands for?*' (T1 — Test 1, T2 — Test 2)

Correct				Strategies used											
				Knows 10 tens in 100 but makes mistake				Uses 10 tens in 100 to answer question				Knows answer			
Cusack		Ridley		Cusack		Ridley		Cusack		Ridley		Cusack		Ridley	
T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2
22	29	14	43	43	60	67	50	29	20	33	50	14	20	0	0

Table 2: Percentages of school cohort with question: '*If you had 621 counters, how many groups of 10 could you make?*' (T1 — Test 1, T2 — Test 2)

Correct		Strategies used											
		Count by ones		Count perimeter		Counts but loses track		Counts by sevens		Counts by fives		Multiplies	
T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2
69	75	0	22	7	0	0	0	0	0	36	11	57	67

Table 3: Percentages of Cusack cohort with question: 'Here are 7 rows of 5 dots. How many dots are there altogether?' (T1 — Test 1, T2 — Test 2)

Correct		Strategies used											
		Count by ones		Count perimeter		Counts but loses track		Counts by sevens		Counts by fives		Multiplies	
T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2
100	100	0	0	0	0	0	0	0	0	80	78	20	22

Table 4: Percentages of Ridley cohort with question: 'Here are 7 rows of 5 dots. How many dots are there altogether?' (T1 — Test 1, T2 — Test 2)

Qualitative data

There were numerous comments during the school visits which suggested that both the students and the *Counting On* team felt that the students had made progress in their mathematics development. Some illustrative examples are provided here.

Cusack High

One teacher believed *Counting On* could help the students with their strategies.

In the Year 7 W class I have a girl who is having enormous trouble with number patterns — just finding the rule to something like 4, 7, 10, 13. She is finding it very hard to follow the pattern and that is just adding on 3s. She isn't really even counting on using her fingers. So she is really way back there and I can see that by making her more number aware and giving her some of the more basic strategies that she will benefit.

Interestingly, the same teacher saw *Counting On* benefiting the more able students as well as those having some difficulty.

For children, it is going to be a great benefit. They are the sort of students that slow the learning of the class down because you've got to spend so much more time with them. So I can see that they will benefit the better kids because if they learn those strategies then you will have more time.

The students thought they were learning 'how to solve the problems in your own way'. Some of them said that *Counting On* had 'made the work easier and helped a

lot' because 'I understand the questions'. The best part was 'that everyone helps you do the questions'.

Ridley Central

The HTM used a specific question from the Assessment Schedule to illustrate what he thought was a general improvement in the ways the students recognised tens and units.

I saw a big difference in the question where there were 24 squares and the number 24 written and you say to them, 'Circle the number of squares that the 2 represents'. Before there were one or two who circled 20 but this time all bar one or two circled 20. I think that comes back to ... the work on the blank number line... It is reinforcing to them all the time that there are tens and units.

In spite of such progress, the *Counting On* team had noticed that the students did not necessarily choose the strategies they had been taught as their first option when they were under the stress of the assessment interview. 'One thing I've noticed with these kids is that they may not still be using the most efficient strategies [DMC].

Both teachers and students commented on a development of confidence in the students and an increase in their willingness to 'have a go' at problems which they might have avoided altogether in the past. This is an outcome which cannot be measured by the Assessment Schedule.

Discussion

The selection of quantitative results show that the students in both schools have become more successful at getting the correct answer in all three of the questions described. Moreover, there had been some development in terms of the strategies used by the students. These results are most marked in Table 1 — the same question as has been mentioned by the HTM from Ridley. The increases are more marked for the questions dealing with place value than for multiplication, even though there is some evidence of advances in strategies used in the multiplication question. Obviously, the numbers of students are too small to allow much statistical analysis, which will be the subject of later papers dealing with the whole cohort of 671 students in 40 schools. However, the combination of the comments from teachers and students and the developments which can be seen in Tables 1–4 would suggest that *Counting On* has had some discernible effect on the learning outcomes of the students. This is particularly significant when one realises that the short time between the two assessment interviews, which, in Cusack High, was 23 weeks and, in Ridley Central, 20 weeks. In both schools, the teachers felt that the program should have started earlier, allowing more time for the development of the students' strategies.

Conclusion

In this paper, a brief description of the evaluation process for the *Counting On* program in NSW secondary schools has been provided. Preliminary data from both the quantitative and qualitative aspects of the evaluation show that the program has made a difference to the knowledge of the mathematics teachers in the two schools. Not only has their mathematical knowledge been developed, but so has their understanding of alternative approaches to assessment. As one teacher said:

I think it was really worthwhile. It was time consuming but to be able to sit down and look at it again — you just pick up so much the second and third time. I think it was excellent that we had the opportunity.

As well, the program seems to have given the students involved an opportunity to develop both their skills and strategies for solving certain mathematical problems and their confidence in doing so.

Teachers associated with *Counting On* believed that it had given them the opportunity to focus on ‘learning how students learn’. In turn, this emphasised the identification of individual student differences, particularly in terms of the strategies used, and the role of language in mathematics learning. Given that secondary mathematics is often characterised by whole class textbook teaching, this is a real advance. There were some disadvantages of the program highlighted in this evaluation but they were almost all in terms of the amount of time necessary to undertake the assessment interviews and analysis and the actual timing of the implementation of the program during 2000. The first of these is a feature of the program and probably cannot be alleviated to any great extent. The second can be easily overcome.

While there are some refinements to the program which have been suggested by participants, there are no overall changes that either the teaching teams or the students from the two schools would want to make. *Counting On* seems to have met a real need for a certain group of students and their teachers. It is accepted by all that the program has led to the students using different thinking strategies in their mathematical problem solving and being more successful in this problem solving. At least in Cusack High and Ridley Central, the program will be implemented again in 2001.

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About the presenters

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Research Project and was commissioned by AAMT to write a recent source paper in early childhood numeracy. He is the proud father of two year old Will who is redefining Bob's understanding of early childhood development, particularly in the numeracy field.

Peter Howard is Senior Lecturer within the School of Education, NSW, at the Australian Catholic University. He has a particular interest in the numeracy and 'literacy in mathematics' issues faced by students learning mathematics from low socio-economic communities. His other mathematics education research interests include collaborative work with mathematics educators within Australia and several South-East Asian countries into the nature of people's beliefs about mathematics, mathematics learning and mathematics teaching.

Assessing Numeracy in the Middle Years — The Shape of Things to Come

Dianne Siemon and Max Stephens

This session will report on the Middle Years Numeracy Research Project: 5–9 conducted in Victoria from November 1999 to November 2000. In particular, it will focus on the use of relatively open-ended, ‘rich assessment’ tasks and scoring rubrics that value mathematical content knowledge as well as strategic and contextual knowledge to evaluate the numeracy performance of students in Years 5 to 9. A case will be made that this form of assessment and performance-based assessment more generally represent the shape of things to come in relation to the middle years of schooling and school mathematics education.

Numeracy in the middle years

One of the major challenges confronting any attempt to improve numeracy outcomes concerns the notion of numeracy itself. In 1997, the National Numeracy Benchmarks Taskforce defined numeracy as the

effective use of mathematics to meet the general demands of life at home, in paid work, and for participation in community and civic life.

State and Territory curriculum documents refer to numeracy in a similar vein. In some cases, linking numeracy with a capacity for critical thinking and/or effective communication.

Numeracy involves abilities which include interpreting, applying and communicating mathematical information in commonly encountered situations to enable full, critical and effective participation in a wide range of life roles (Queensland Department of Education, 1994, cited in AAMT/DEETYA, 1997).

NUMERACY is an ability to cope mathematically with the demands of everyday life. Numerate and literate persons in mathematics are those who can appropriate mathematics as a tool to guide their reasoning, help them to solve problems in their everyday lives, communicate and justify their ideas, as well as to understand the ideas of others (ACT Curriculum Frameworks, 1996, cited in AAMT/DEETYA, 1997).

In *Numeracy = Everybody's Business*, the Report of Numeracy Education Strategy Development Conference, jointly published by AAMT/DEETYA in May 1997, numeracy is seen as

using some mathematics to achieve some purpose in a particular context ... To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work and for participation in community and civic life. In school education, numeracy is a fundamental component of learning, performance, discourse and critique across all areas of the curriculum. It involves the

disposition to use, in context, a combination of: underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic); mathematical thinking and strategies; general thinking skills; and a grounded appreciation of context (AAMT, 1997, p.15).

More recently, in *Numeracy A Priority For All: Challenges For Australian Schools* (DETYA, May 2000), numeracy is linked to success in school and to access to further study and training beyond school:

Numeracy like literacy provides key enabling skills for individuals to participate successfully in schooling. Furthermore, numeracy equips students for life beyond school in providing access to further study or training, to personal pursuits, and to participation in the world of work and in the wider community (DETYA, 2000).

The OECD's view of mathematical literacy is reported in the same document.

Mathematical literacy is the individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements, and to engage in mathematics, in ways that meet the needs of that individual's current and future life, as a constructive, concerned and reflective citizen. (OECD, Paris, 1999)

What each of these views encapsulates to varying degrees are the three foci identified by Willis (1998), that is, the underpinning nature of core mathematical understandings and skills (mathematical knowledge), the capacity to critically apply one's mathematical knowledge and skills in a particular context for some purpose (contextual knowledge), and the actual processes and strategies needed to connect and communicate one's mathematical knowledge to every-day problems and events (strategic knowledge).

This suggests that the development of numeracy will necessarily involve a consideration of each of these aspects in different ways and proportions at different ages and stages of schooling. In the early years where the focus is primarily on the development of the key mathematical ideas, skills and strategies. Numeracy becomes arguably more problematic in the middle and upper years of schooling where prior knowledge and experience, issues of identity, and a range of complex social, emotional and physical factors impact student's capacity to learn. For more general background information see the work of the *Middle Years Research and Development* project at www.sofweb.vic.gov.au/mys.

The particular challenges confronting the teaching and learning of numeracy in the middle years of schooling include the following:

- the enormous range in student ability and motivation, and the significant number of students whose experience of failure or sense of disconnectedness make them reluctant learners;
- the perceived demands of 'the mathematics curriculum' — too much, too soon, for too many, inhibiting attempts to cater for the learning needs of all;

- limited time, resources and availability of qualified mathematics teachers particularly in Years 7 to 9 and/or additional, appropriately trained staff to support strategic intervention;
- the relatively sterile, transitory, learning environments of most junior secondary classes which do not facilitate the display of artefacts that celebrate and record prior learning;
- procedural, 'surface' based approaches to learning mathematics, where there is little inclination to search for meaning and the primary focus is on 'getting the answer';
- little or no culture of communication which values explanations, justification and the elaboration of student reasoning and strategies.

Clearly, attempts to improve numeracy in the middle years will need to consider not only the contribution that school mathematics might make (that is, essential underpinnings as well as new knowledge, skills and strategies), but also how to impact entrenched classroom cultures, scaffold discourse elements, and engage learners more effectively. The Victorian *Middle Years Numeracy Research Project* is attempting to do this using an action research methodology with 20 trial schools. While it is too early to comment on the effectiveness of the approaches and strategies being trialed, it is possible to share some of the outcomes and observations derived from the initial data collection.

The Middle Years Numeracy Research Project (MYNRP)

The *Middle Years Numeracy Research Project* is one of a number of current research projects on literacy, numeracy and/or the middle years of schooling commissioned by the Victorian Department of Education Employment and Training (DEET), in partnership with the Catholic Education Commission of Victoria (CECV) and the Association of Independent Schools of Victoria (AISV).

The aims of the *Middle Years Numeracy Research Project* are:

- to provide advice to DEET, CECV and AISV which will lead to the development of a coordinated and strategic plan for numeracy improvement;
- trial and evaluate the proposed approaches in selected Victorian schools; and
- identify and document what works and does not work in numeracy teaching particularly in relation to those students who fall behind.

In addressing these aims, a major focus of the MYNRP has been to consider how valid, reliable forms of assessment can be used by teachers to inform their work and to provide a middle road between educational systems' desire for quality data, the implementation of Standards-based frameworks (e.g., CSFII, 2000) and the need to actively engage teachers as professionals in assessment approaches that support the reform agenda (see Crawford & Adler, 1996; Rhodes, Wiliam, Brown, Denvir & Askew, 1998).

Unlike many large scale testing programs where teachers have little or no role in the development and scoring of assessment items, the MYNRP has worked directly with teachers and schools with a view to changing teachers' understanding of students' thinking and of their own instructional practices. What sets this work apart from other large-scale systemic projects is the partnership between researchers and classroom teachers in the development of tasks and scoring rubrics and in the administration and marking of the assessment tasks. Another feature that sets this work apart is the project's clear commitment to realistic mathematics aligned to a standards frameworks and National Numeracy Benchmarks.

The project is essentially an ascertaining study involving the collection of quantitative and qualitative data and the implementation, trial and evaluation of a *Draft Numeracy Strategy*. A more detailed explanation of the research methodology can be found in the 2000 MERGA Conference proceedings (Siemon & Griffin, 2000).

Base-line data on the numeracy performance of a structured sample of Grade 5 to 9 students from 27 primary and 20 secondary schools in Victoria was collected in November, 1999. This involved a 5–6 item written test of approximately 45 minutes and an extended classroom task (also of 45–50 minutes duration). The extended classroom task, known as *Street Party*, was sourced from the INISSS Project in Tasmania (see Callingham, 1999). This task caters for a range of abilities and is aimed at assessing higher order cognitive knowledge and skills related to pattern recognition and generalisation. The short assessment tasks were largely derived from *Effective Assessment in Mathematics Levels 4 to 6* (Beesey et al., 1998). Both sets of tasks met the following criteria:

- they assessed numeracy performance of students in Years 5 to 9 (that is, mathematical, contextual and strategic knowledge (see Willis, 1998);
- they were broadly representative of the three aspects of numeracy, i.e., number sense, measurement and data sense and space sense 9 (the *National Numeracy Benchmarks for Years 5 and 7* were used a guideline);
- opportunities were provided for students to demonstrate what numeracy-related mathematics they did know or could do (referenced to *Levels 3 to 6, Victorian Curriculum & Standards Framework*);
- content as well as process outcomes were assessed, that is, conceptual as well as procedural knowledge and strategy usage;
- they modelled best practice (see Clarke et al., 1996) and facilitated performance assessment, that is, the use of scoring rubrics which evaluated student's performance including the capacity to choose, use and apply relevant knowledge, skills and strategies in context;
- they were relatively straightforward and cost-efficient to administer; and
- they could be locally assessed with some confidence and globally assessed using computer-readable score sheets.

All tasks were assessed by the teachers using previously trialed scoring rubrics and pre-printed scannable score sheets. The overall assessment procedure was referred to as the SNP or *Student Numeracy Performance* package.

Parallel versions of the short assessment tasks were prepared for Years 5/6 and Years 7 to 9. Versions 4.1 and 4.2 were designed to reflect Level 4 of the Victorian Curriculum and Standards Framework II, which most students are expected to achieve by the end of Year 6. Versions 5.1 and 5.2 were similarly designed to reflect Levels 5 to 6 of the Victorian CSFII version, which is appropriate for student sin Years 7 to 9. However, to support item analysis and numeracy performance across Years 5 to 9, common items appeared on both parallel and vertical versions of instrument. Five examples of short assessment tasks are given below together with the Scoring Rubrics for use by teachers.

Examples of Assessment Tasks

FILLING THE BUSES (Versions 4.1 and 4.2)

A school is planning a trip to the swimming pool for the school sports.

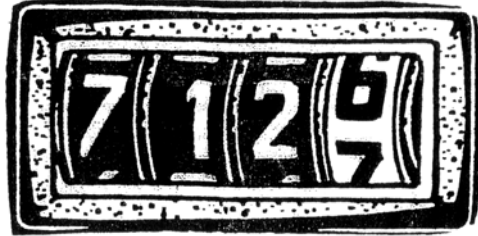
There are 489 students and 24 teachers at the school. Each bus can hold 45 passengers.

- (a) How many buses will be needed to carry all the students and teachers to the pool?
- (b) The teachers made a plan for the students and teachers to travel to the pool by bus. What do you think their plan is? Show how many students and teachers you think will be on each bus.

a.	No response	0
	Incorrect (e.g. 11 or 11.4 buses), recording shows not all information taken into account	1
	Correct (12 buses)	2
b.	Incorrect or no response	0
	Plan shows correct number of buses but little/no consideration given to the likely distribution of students and staff	1
	Plan is more systematically, thoughtfully presented, recognises context, i.e. need for some teachers on each bus	2

TRIP METER (all Versions)

The trip meter shows how far Susan's car has gone since it was last filled with petrol on a trip around New South Wales. The trip metre shows kilometres and tenths of kilometres.

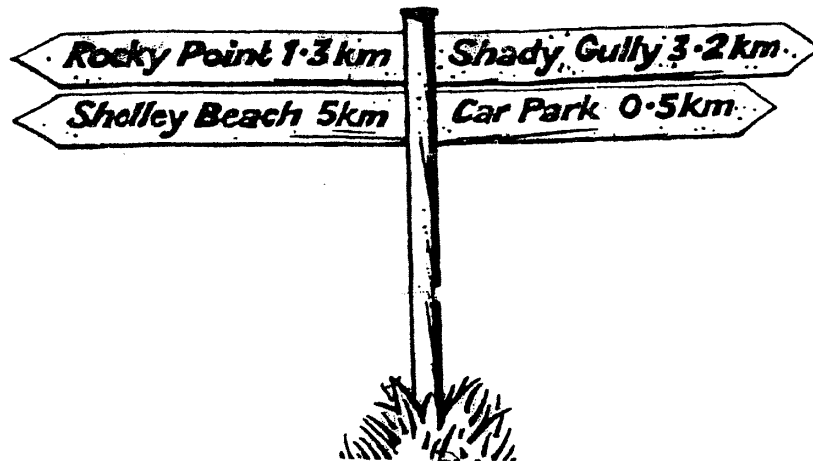


- (a) How far has the car travelled to the nearest kilometre since it was last filled with petrol?
- (b) From previous experience of long trips, Susan knows that the car uses about 10 litres of fuel for each 100 km (sometimes written 10 L/100 km). Approximately how many litres would have been used on this trip since the car was last filled with petrol?

a.	No response or incorrect response, e.g. reads as 7126 or 7127	0
	Partially correct, not to nearest kilometre (712.6 or 712.7 km)	1
	Correct (713 km)	2
b.	No response	0
	Irrelevant response, e.g. correctly uses whole number reading from (a) to get something like 713 L	1
	Correct (71 L)	2

HOW FAR TO WALK? (Revised item, Version 4.2)

The National Parks' Service has recently opened a new walking track from Shelley Beach to Shady Gully. Walking along this track from the car park you come across this sign:



- (a) Draw a rough map of the track from the signpost to Shelley Beach below. Be sure to show the signpost, Rocky Point and Shelley Beach.
- (b) You overhear some people talking about possible walks. One person says, 'I can see from the sign that the distance is 3.7 kilometres.' What might she have been talking about? Explain your reasoning.
- a. No map provided, map incomplete 0
 Map shows signpost, Rocky Point and Shelley Beach but not in proportion 1
 Map shows 3 locations in roughly correct relationship, that is, Rocky Point is about or slightly more than 1 quarter of the distance from signpost to Shelley Beach 2
- b. No response or incorrect, e.g. adds 3.2 and 0.5 km 0
 Partially correct (recognises difference, 5 km and 1.3 km), little or no explanation 1
 Correct (recognises and states as distance between Rocky Point and Shelley Beach), full explanation 2
- c. No response or incorrect, e.g. uses difference between Shelley Beach and Shady Gully 0
 Partially correct (adds 1.3 and 0.5 km), but little or no interpretation or explanation 1
 Correct (recognises and states as distance between Rocky Point and Car Park) 2

MEDICINE DOSES (Version 5.2)

Occasionally medical staff need to calculate the child dose of a particular medicine, using the stated dose for adults. The rule is as follows:

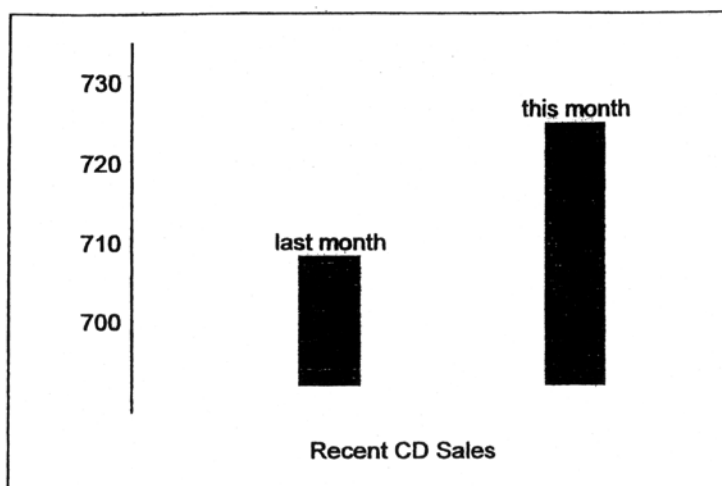
$$\text{child dose} = \text{adult dose} \times \frac{\text{child@ age}}{\text{child@ age} + 12}$$

- (a) If the adult dose for a particular medication is 15 mL, what would be the appropriate dose for a 6 year-old child?
- (b) What fraction of the adult dose is the 6 year-old child's dose?
- (c) A nurse uses the formula to work out the dose for an 8 year-old boy. She correctly calculates it as 8 mL. What was the adult dose in this case?
- a. No response or incorrect 0
 Information from formula used but incorrect or incomplete calculation 1
 Correct (5 mL), appropriate use of formula or recognition of proportion 2
- b. Incorrect or no response 0
 Fraction component identified but incomplete, e.g. recorded as 6 divided by 6+12 1

	Fraction correct (6/18) but not interpreted appropriate to context	2
	Fraction given as 1/3	3
c.	No response or incorrect	0
	Information from formula used but incorrect or incomplete calculation	1
	Correct (20 mL), appropriate use of formula	2

CD SALES (Versions 5.1 and 5.2)

The manager of a music shop showed this graph and said, 'There's been a big increase in the number of CD sales this month.'



Do you consider the manager's statement to be a reasonable interpretation of the graph? Explain your reasoning.

	No response or 'yes' or 'no' without an explanation	0
	Reasoning based on numbers alone, no recognition that 'big' is relative	1
	Reasoning shows some recognition that 'big' is relative to total sales, but unsupported conclusion, little or no explanation, e.g., 'It depends...'	2
	Reasoning concludes that increase is not 'big' relative to total sales, some attempt to relate this to notion of proportion, e.g., '15 out of 725 is not very big'	3
	Correct conclusion (not 'big'), %, fractions, ratio used correctly to support detailed explanation	4

The scoring rubrics

The development of Scoring Rubrics has been a significant aspect of the MYNRP. The prototypes of these tasks in *Effective Assessment in Mathematics Levels 4–6* (Beesey et al., 1998) included only a detailed description of what might be expected from an accomplished performance. Scoring Rubrics for partial credit were not provided. For the MYNRP, these rubrics were developed with several key purposes in mind:

- to provide teachers with a readily accessible and consistent means of assessing student numeracy performance;
- to draw the attention of teachers to crucial aspects of numeracy performance, in particular to emphasise and value the use of mathematical knowledge and skills relative to context;
- to provide the Research Team with a basis for consistent task analysis and of comparing performances across year levels both within and between schools.

For example, the Scoring Rubric for the 'Filling the Buses' indicated to teachers that merely providing the result of a computation was insufficient to score a 2. Students needed to recognise that an additional bus would be needed to accommodate remaining pupils. On another task where students needed to coordinate the information from bus and train timetables, students were expected to make allowances for the possibility of trains being slightly late and the time required to walk from the bus stop to the train platform. In the Scoring Rubric for 'Medicine Doses', students were expected to see that a more appropriate and practical way to arrive at the child's dose is to use the fraction $1/3$ rather than the corresponding fraction $6/18$. The Scoring Rubric to the task 'CD Sales' recognises that students are likely to develop increasingly sophisticated explanations as they extend their understanding to include percentage increase and ratio. The Scoring Rubric for this task allows students at different stages of schooling to achieve the maximum score using knowledge appropriate to their stage of schooling.

A large number of student work samples were collected and a random sampling procedure was adopted to explore the consistency of teachers' judgements both within and between schools. In addition, the MYNRP Contact Person was asked to establish moderation procedures within the school. Rasch analysis (Adams & Khoo, 1993) was used to test the suitability of the items and to identify key developmental levels of numeracy that indicate readiness to learn as well as providing a snapshot of achievement (e.g., Griffin, 1998). This analysis based on teachers' use of the Scoring Rubrics reflected the intended order of item difficulty within tasks and across CSFII Levels. An *Emergent Numeracy Profile* was developed from this analysis which was used to frame qualitative and quantitative feedback to schools on student performance. Item fit analysis was also used to modify tasks which had not performed as planned.

Of all the short assessment tasks used, only one task, the original version of *How Far to Walk*, lay outside the boundaries set by the Rasch item fit analysis suggesting that all the others were measuring a similar construct. The analysis confirmed that the use of teachers as assessors is a valid measurement procedure, and also that the degree of difficulty of the tasks chosen appears to be appropriate for the cohort tested (see Siemon & Griffin, 2000).

Preliminary conclusions from the MYNRP

'Hotspots' identified by the initial data collection, indicate that a significant number of students in Years 5 to 9 have difficulty with some or all of the following:

- explaining and justifying their mathematical thinking;
- reading, renaming, ordering, interpreting and applying common fractions, particularly those greater than 1;
- reading, renaming, ordering, interpreting and applying decimal fractions;
- recognising the applicability of ratio and proportion and justifying this mathematically in terms of fractions, percentage or written ratios;
- generalising a simple pattern and applying the generalisation to solve a related problem;
- working with formula and solving multiple steps problems;
- writing mathematically correct statements using recognised symbols and conventions;
- connecting the results of calculations to the realities of the situation, interpreting results in context, and checking the meaningfulness of conclusions;
- maintaining their levels of performance over the transition years.

One of the most promising outcomes of the initial data collection has been the development of an *Emergent Numeracy Profile* with rich descriptions of distinct developmental levels of numeracy performance based on the content and process analysis of the items included in the Phase 1 data collection (see below). This has important implications for the design of structured, numeracy-specific teaching and learning materials which not only support students to acquire the necessary content knowledge and skills but also scaffold a hierarchy of skills, strategies and dispositions concerned with mathematical thinking and problem solving (Siemon, 1993). Callingham (1999) has reported a similar developmental pattern for the *Street Party* task which she has described using the SOLO taxonomy.

Emergent MYNRP Numeracy Profile

- H Well established in the use of fractions/ratio. Able to generalise and apply number relationships to solve problems. Monitors cognitive actions and goals (i.e., almost always evaluates what they are doing for meaning and relevance to problem solution).
- G Established in using and interpreting data and/or information appropriate to context, fraction representations, and in describing patterns and relationships. Able to explain solutions to problems.
- F Consolidating use of data and information appropriate to context. Established in recognising 2D representations of simple 3D space. Beginning to monitor cognitive goals as well as actions (i.e., evaluates what they are doing for sense and relevance).
- E Consolidating fraction and % knowledge. Monitors cognitive actions (for 1–2 step problems). Little/no monitoring of cognitive goals (i.e., checks procedures but not their meaningfulness and/or appropriateness to problem context and/or conditions).

- D Beginning to understand and represent simple fraction situations. Generally solves one-step problems involving 3-digit whole numbers, ones and tenths. Describes simple patterns.
- C Able to use a number pattern to solve a problem. Monitors cognitive actions and/or goals some of the time (e.g., recognises relevant information but unable to use it effectively).
- B Recognises a number pattern and represents it in one way. Makes judgements about data more on the basis of perception than analysis. Little evidence of cognitive monitoring, e.g., estimates or calculates without regard for meaning or applicability.
- A Uses make-all, count-all strategies to solve a simple number pattern problem

While the *Emergent Numeracy Profile* will be informed by further trialing of the SNP, it will be used in the Trial phase as a framework to guide the design and implementation of school-based teaching materials and assessment tasks. During the Trial phase it is also planned to collect data to help frame advice concerning the design elements under consideration. That is, structured mainstream classroom programs, additional assistance, the role of parents, mentors and peer support, and the role of professional development in improving numeracy outcomes.

Feedback on the implementation of the assessment tasks in Phase 1 schools indicated that although the assessment took place at a very difficult time of the year, it was generally viewed as a worthwhile exercise. Teacher journal entries from the Trial Phase suggest that teachers are more likely to accept the outcomes of assessment if they have been involved in the assessment themselves. For instance, there appears to be a greater acceptance of the importance of students' explaining and justifying their mathematical thinking and/or conclusions, even though this message has been part of the reform agenda for some time (e.g., see Victorian Curriculum Standards Frameworks, 1995, 2000). This in turn appears to have led to a greater willingness to use the data to inform future teaching (evident in Trial School Action Plans). In some cases, the assessment tasks actually prompted discussions on task-specific solution strategies (e.g., 'rotators' or 'left/righters' approaches to the *Bird's Eye View* task).

To date, the work of the MYNRP suggests that it is possible to measure a complex construct such as numeracy using rich assessment tasks incorporating performance measures of content knowledge and process (general thinking skills and strategies) and teachers as assessors. While it appears that the *Emergent Numeracy Profile* represents an important first step in helping teachers plan more effective instruction, it must be stressed that the profile represents work in progress that needs to be elaborated by further data collection and analysis. The research team would welcome any comments and/or feedback on the work so far.

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About the presenters

Di Siemon is an Associate Professor of Mathematics Education at RMIT University. Her research interests include children's number ideas and strategies, mathematical problem solving and the role of classroom cultures and communication in the teaching and learning of mathematics. As Immediate Past President of AAMT, Di has a keen interest in numeracy education and the development of professional standards for excellence in the teaching of mathematics. She is the Director of the *Middle Years Numeracy Research Project: 5–9*, a fifteen month project commissioned by the three employer groups in Victoria. Di enjoys working with teachers and school communities to make mathematics more meaningful and accessible.

Max Stephens is collaborating with Di in the *Middle Years Numeracy Research Project: 5–9*, especially in the design of assessment tasks and rubrics. His current work involves teaching at the Australian Catholic University and continuing project work at the Victorian Board of Studies where he was Manager of the Mathematics Key Learning Area until 1998. Max is President of the Mathematical Association of Victoria, and a former President and Life-Member of AAMT.

Numeracy for National Development

Beth Southwell

Claims are made that the quality of people's numeracy contributes to the nation's development, so it is appropriate to reflect on the role of numeracy for citizens of the twenty-first century. Accordingly, some of the growth in some overseas countries will be explored as will some aspects of citizenship and the contribution that mathematics makes to that. Other points for consideration will include the tensions raised by the Delors (1996) UNESCO report and the practical implications for numeracy in Australian schools.

The word 'numeracy' is often heard both in educational settings and also in the media. We use the word freely but there does not seem to be complete agreement on its meaning. In this particular paper, the simple explanation that is being used is as follows:

Numeracy is the ability to understand and apply mathematical ideas in solving problems encountered in everyday life.

The ideas that are emphasised in this brief description are that:

1. it is related to mathematics and therefore covers a range of different branches of mathematics, not just arithmetic,
2. it involves understanding and application,
3. it is related to every day life and is therefore relative to one's life style and occupation.

How, then, does this relate to national development? Ultimately, national development requires every citizen to be able to make a worthwhile contribution to the nation in an environment of harmony and justice. What does being numerate mean in this context? How can being numerate help this development?

What students do outside of school is the critical aspect for numeracy because in school, they are being equipped to make responsible decisions related to their own and others' lives. As numeracy has to do with everyday life, it contributes to the development of responsible citizens. Cotton (1999) reports a project he has undertaken to 'explore the ways in which mathematics in school can affect the ways in which individuals lead their lives outside school, (p.6)' His idea of social justice for this project was that of a society in which individuals feel they are in control of their own lives and can help improve things for others. He thinks 'mathematics is important because it offers both access to positions of power, and also a way of critically viewing the pronouncements of those who are involved in making decisions over which we have no control. This critical view allows us to challenge policy decisions we feel are unjust (p. 6)'. Cotton's scheme of work for his year 7 class

involves the use of themes for each half term and, because they have been carefully designed, through them, the students cover the areas set down in the National Curriculum. Numeracy skills are therefore of great significance in citizenship and therefore in national development.

Interestingly, a comparison of adult (16–60 years) numeracy skills from seven countries conducted in 1996, found that those with the poorest numeracy skills were from working class homes and that women performed worse than men. A further result was that those with the poorest numeracy skills came from the 16–34 year old group (Wells, 1999).

Not everyone sees the contribution of education to national growth without also recognising there are some concerns. Perraton (1998) does recognise that the development of skills and education are the keys to prosperity but has reservations as to whether further levels of education will affect growth when there is universal education and, for this reason, advocates that more research is needed into a number of different aspects of the national political economy. Robinson (1998) makes a distinction between raising average levels of literacy and numeracy and reducing the number of children with the lowest levels of achievement. He claims that the former makes no difference to national development but that the latter could have ‘an important impact on their labour market prospects (p. 148)’ One might ask, however, is it literacy or numeracy or the two together that contribute to this result? How does this tie in with Wells’(1999) findings?

Internationally, in 1989, the Total Literacy Project (TLP) was begun in the district of Kerala and proved very successful. Kerala became India’s first totally literate state within one year. In writing about this project, Saini (2000) reports that the project emphasised literacy and numeracy and attempted to raise people’s self-respect, motivation and sense of responsibility. Saini claims that the project demonstrated that literacy, and presumably numeracy, encouraged people to become more responsible citizens and more competent parents.

The acquisition of literacy and numeracy is seen as contributing to national development by the Ghanaian minister of education. He is reported to have stated that ‘education in the new millennium is not about mere literacy and numeracy, but rather about equipping a nation’s human capital for rapid development’(COMTEX, 2000). Similar thoughts have been expressed in Kenya where 35 per cent of the population are considered illiterate. The claim is made that ‘there are millions of people who would be leading better lives today if they acquired numeracy and literacy skills (COMTEX, 2000).’ Presumably, they would be able then to contribute to national development.

Brand (2000) looks at literacy and numeracy in a different way. Instead of looking only at the benefit of literacy and numeracy for national development, he makes the statement that a sound educational base includes ‘the acquisition of “foundational” skills, such as literacy and numeracy — skills which should be “accessible and mandatory” for all, and should be regarded as social rights’(p.43). The welfare of each individual is a critical factor in developing literacy and numeracy skills, and as

Cotton (1999) advocates, social justice must be a necessary accompaniment to national development.

At the UNESCO-ACEID conference in 1997, Heyn presented data that substantiate the claim that literacy and numeracy acquisition is a critical factor in the reduction of poverty in both developed and developing countries (Heyn, Lythgoe & Myers, 1997). Heyn's data is supported by figures published by Day (1999) who states very strongly that basic education is the key for the eradication of poverty. He claims that 125 million children will never have the chance to attend school and that one in every four adults in developing countries are illiterate. Ensor (1999) considers that the real tragedy is not only that communities are the victims of drought, floods, earthquakes or civil conflict, but they are also victims of the poverty, ill-health and powerlessness that are the result of illiteracy and innumeracy. This claim is echoed in various documents that have appeared in the past few years. Hughes (1997, p. 7-7) is one example of an educator who sees the first priority in thinking about the future as basic education for all.

Similar data were given in a news release (COMTEX,2000), in which the following statement was made in relation to basic education:

Of the estimated 120 million children not enrolled in school, an estimated 60 to 70 percent are girls. Forty percent of African children are out of school (42 million total), as are 26% of South and West Asian children (46 million). At least four years of quality education are necessary for sustainable literacy and numeracy skills, but about 150 million children drop out before completing fourth grade. A 1995 UNICEF/UNESCO study found that about one-third of students don't have classrooms with blackboards, and a similar number lack desks, chairs and access to safe water.

It is in this context that Delors' (1996) report of the UNESCO Commission on Education for the Twenty-first Century is relevant.

UNESCO Report

In his report of the UNESCO International Commission on Education for the Twenty First Century, Delors (1996) has structured an approach to education that is based on four pillars. These are *learning to know*, *learning to do*, *learning to be* and *learning to live together*. The last of these is seen as the ultimate desired outcome which follows from the other three and greater emphasis has been placed on it. He states, quite unequivocally, that if we see the world as a unity,

everything falls into place, whether it be the requirements of science and technology, knowledge of self and of the environment, or the development of skills enabling each person to function effectively in a family, as a citizen or as a productive member of society' (p. 19).

...nothing can replace the formal education system, where each individual is introduced to the many forms of knowledge.

Delors based his recommendations and hopes on an examination of the tensions that he sees as operating in the world as we move almost inexorably into the 'global village' and as educational policy is being rethought, redeveloped in some countries

and relegated to lower levels of interest and influence in others. The tensions he draws to our attention are:

- the tension between the global and the local,
- the tension between the universal and the individual,
- the tension between tradition and modernity,
- the tension between long-term and short-term considerations,
- the tension between the need for competition and the concern for equality of opportunity,
- the tension between the expansion of knowledge and the capacity to assimilate it,
- the tension between the spiritual and the material.

These are all active tensions, even in a country like Australia, and many apply to the area of mathematics, despite the best efforts of many teachers and students. We are not always conscious of these tensions but this is probably a good time to consider them.

The tension between the global and the local

This tension is seen in several ways in mathematics education. How often do we use local knowledge and contexts in teaching mathematics and developing numeracy concepts? How often do we choose exercises and problems, and even textbooks, from local sources? The best problems, investigations, projects are those that arise out of the experiences of the students in the classroom (Cockcroft, 1982, p. 74). We tend to rely on what other people have done rather than develop our own ideas and it is not always because of lack of time.

The tension between the universal and the individual

There has always been a plea to teach the individual rather than just consider the class as a whole all the time. At various times in the history of education there have been strong pleas for individual instruction. Then the teacher finds there are 30 students in the class so the struggler and the gifted student get lost in the attempt to reach the majority of students in the middle. What do we do with the student who needs extra help or the one who knows more than we do?

The tension between tradition and modernity

This tension is particularly obvious in the neglect of Aboriginal and other cultures in the mathematics classroom. We forge ahead with computers and the internet and forget that some of the richest technology is available to us through connecting with some of the traditional ways of doing things.

The tension between long-term and short-term considerations

There is a view abroad that what we do in schools is to prepare children for the future. In other words, we are looking at a long-term result. That is, of course, true, but it is only half the whole truth. We are also enabling students to live today, tomorrow — the next day. It is today that these students of ours will be forming their view of the world and unless it makes sense to them today, it probably won't in ten years time either.

The tension between the need for competition and the concern for equality of opportunity

Competition, as we saw last year in the Olympic Games, is motivating. It encourages the individual to strive for a goal, to do one's best. It can also develop into conflict, as we have seen in many parts of the world. Delors (1996) says that although the death toll in the last world war was 50 million, not many people realise that there have been about 20 million deaths in about 150 wars since the end of that world war. In numeracy and mathematics, how do we treat competition and co-operation? Do we ensure that all students have the opportunity to have access to the best teaching and the opportunity to succeed? This is particularly relevant in the current climate in relation to gender, but also to Aboriginality, students from non-English speaking backgrounds and other exceptional students.

The tension between the expansion of knowledge and the capacity to assimilate it

We know the feeling of being overwhelmed by the rate at which new knowledge is being discovered and developed. We are also very conscious of our inability to assimilate all the knowledge that comes to us. What do we do about it? There are some essentials that we need to learn, and numeracy concepts are some of those, but there are other things that are not so essential. We need to be able to distinguish between the knowledge that is necessary and that which is not, remembering that some knowledge is not essential in itself but it is through that knowledge that we learn the processes that are necessary to enable us to access the relevant knowledge when we do need it. In other words, we need to learn how to learn.

The tension between the spiritual and the material.

What are the values we are passing on to our students? Are they the values that respect the human being and human service or are they values that emphasise self-aggrandisement or the acquisition of possessions for their own sake? Many would say that mathematics is a subject that is valueless. One only has to study the various philosophies of mathematics that have emerged over the centuries to realise that at times these philosophies have placed values on mathematics. In one sense the current emphasis on constructivism is respecting the children in that they are able to construct knowledge in their own way. So why this current emphasis on numeracy? Is it to enable students to use their mathematical knowledge to build their own

wealth? Is it to enable students through their knowledge to develop their own self-esteem and consequently to contribute to the greater good? To their country? Degenhardt (1999) looks at mathematics in a slightly different way and claims that ‘there is spiritual and moral worth in studying mathematics because in doing this we rise above our physical nature and transcend the limitations of our particular culture’ (p. 12).

The four pillars of learning are more evident in developing countries than in the recent Australian context though that is debatable. *Learning to know* has involved many countries, with the help mainly of external funding sources, in developing education systems that have aimed at providing basic education, at least, to all children and, in many cases, to adults as well. In places like Indo-China, where there were conflicts almost continually from the end of the second world war till quite recently, education systems set up by colonial powers were completely destroyed and needed rebuilding from nothing. In the Lao Peoples’ Democratic Republic, for example, all textbooks were burnt and many teachers were sent to re-education camps so the initial post-war developments were limited to the help that could be given by Vietnam and many teachers at the primary level had no training at all. From this serious situation in 1975, the government has been able to develop school syllabuses, prepare resource material for several levels, establish a teacher training scheme with new courses, implement professional development programs and initiate basic education programs for all. Other countries, such as Cambodia and many of the Pacific nations have also been through educational renewal programs.

Learning to do has emphasised the need to develop skills for doing a job of work and to gain a competence that would enable people to deal with a range of situations. Numeracy, as we define it, is critical in this regard even when it is not perceived to be so. Again, with the developing technology throughout the world, this has often meant people need to retrain. No matter what the area might be, numeracy skills contribute by enabling people to go from one area to another with relative ease.

Learning to be is one that is very relevant for numeracy for national development. It relates directly to the need for everyone to exercise greater independence and judgment with a stronger sense of personal responsibility for working towards common goals. The better we feel about ourselves, the greater our sense of independence and desire to contribute to the common good. We feel better about ourselves if we have control of knowledge and its application to everyday life and this is the expectation for numeracy: that students will gain numeracy skills and concepts and develop their self-confidence in being able to successfully apply those concepts and skills to their everyday experiences.

Learning to live together is the greatest challenge. Again, through discussing how numeracy concepts and skills apply to real life situations, we can help students understand each other and hopefully create a new spirit of co-operation and interdependence in the classroom. The kind of things we get students to do will assist in this regard. For example, the use of cooperative problem solving, group investigations, projects and problem-based teaching encourage students to work together.

As long ago as 1969, Christianson (cited by Christiansen, and Walther, 1986) advocated activity or projects as a method of teaching mathematics. He described his conception of inductive approaches to teaching mathematics under four headings; experimentation, observation, the formulation of an hypothesis and then testing of the hypothesis by further experimentation (p. 281). This process, which we attribute more recently to the process required for mathematical investigations incorporates the skills and understandings necessary for informed decision-making which in turn is what is required for good citizenship. The role of tasks in this approach is considered on two levels. There is first of all the level of the students' activity initiated by means of tasks and then there is the level related to the need for specific tasks to motivate for specific types of activities such as exploratory activity or problem solving activity. Christiansen and Walther see these activities as political concepts (Mellin-Olsen, 1987, p. 37) contributing to the welfare of the learner.

What are the mathematical benefits of such an approach?

What other benefits are there for the individual? For the school? For the nation? Christiansen and Walther (Stellin-Olsen, 1987, p. 198) claim that the kind of cognitive and affective activity that takes place in a project such as the ones they describe will contribute not only to the individual's mathematical and numeracy achievement but also to their development as a good citizen of their country. These include questioning, checking, analysing, synthesising and decision-making. Cotton (1999) lists fifteen strategies he uses in his social justice curriculum. Some of these are particularly interesting. Cotton (p. 7) has recommended the use of learning journals, pupil 'focus' groups and the use of colleagues to observe lessons and provide feedback.

A final word from Delors (1996, p. 21): 'There is no substitute for the teacher-pupil relationship, which is underpinned by authority and developed through dialogue'. What is our authority? Some might say our knowledge and that is true but the greater authority is our concern for each individual student in our care. How do we develop dialogue? There are many ways and one that might be suggested is setting our teaching in contexts of interest to our students and using local knowledge in this regard. Another is getting students to keep journals which we read at workable intervals. The most obvious and in a sense, the easiest, is to develop our powers of observation to pick up clues that indicate when particular forms of dialogue might be necessary. Another still is to use content and teaching strategies that enable students to dialogue, not only with the teacher, but also with each other. Some of these techniques, hopefully, will help us develop numeracy concepts and skills that will contribute to the national development of Australia.

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About the presenter

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Calculators: Shaping the Way Children Think

Len Sparrow and Paul Swan

The use of the calculator in primary mathematics is much maligned. The reality is that most teachers do not, in fact, use calculators with their classes because they do not have sufficient knowledge of sensible and suitable activities to engage children. This paper will provide examples of activities with the new TI-15 calculator to shape the way the way children think about and learn mathematics.

Introduction

For many years people have suggested that calculator use in primary schools will diminish children's need and ability to think. Many of the examples of calculator use offered in arguments and examples do in fact portray a lack of thinking by children and adults. Numbers, in these cases, are keyed into the machine without any thought as to what is happening. Proponents of this argument develop it to suggest that if society wishes to produce a generation of thinkers about numbers and mathematical situations then calculators should not be used in primary schools.

A counter argument, that calculators are needed to encourage and support children thinking about numbers, calculations and numerical situations, can also be made. This paper will take, illustrate and expand this argument.

Calculators shaping thinking

By the age of ten most children have learned to a greater or lesser extent, how to apply standard methods for calculating. For many teachers the adoption of calculators into the primary classroom at this point is less contentious as they feel that most children have grasped the 'basics' of standard algorithmic procedures and tables facts. There are, however, two major constraints to easy integration of calculations into the classroom. They are firstly that many teachers do not know how to use a calculator personally beyond a very basic level, and secondly do not know how to use one as a teaching aid to help children learn and understand mathematics.

The move from a relatively simple four-function calculator of the primary years to a sophisticated, multi-functional graphics calculator is large. A useful bridge between the two is a 'more function' calculator, for example, the more function TI-15. The idea of children being introduced to calculators with ever increasing degrees of sophistication and function is one way to develop children's ability in mathematics as well as their skill in using the machine appropriately (Kissane, 1997). An example of this development of 'calculator right of passage' using the Texas Instruments range is illustrative — for younger children (K–4) to use a TI-108, for children in years 5 to 7 to use the more function TI-15, and for secondary children to move to the

multi-function TI-83. Not only do children grow with the calculator but this is also a growth for the teachers.

Thinking with technology

The main aim of this paper is to present an argument for and examples of children thinking with technology. In this case technology is limited to the calculator – only a small part of the big picture. Specifically we want children to think about numbers and mathematics in general with the use of technology and also to be able to think about problem situations with the tools of technology that are available. This is what happens in the workforce and what is needed by modern society.

The calculator, with appropriate activities and tasks, can help children think about and develop a range of strategies for calculating mentally, mechanically and with technology. A calculator can open up avenues for exploration and understanding for children. We propose in this article to use the calculator to help children think about the methods for calculating that many will already have learned for working with pencil and paper. This calculator will also take them into the murky world of fractions and decimals.

The TI-15

We have selected to use the TI-15 calculator as it is relatively new to the Australian market, not too expensive and has a number of functions extra to the normal primary school calculator. It is a useful example of the ‘more function’ calculator that is appropriate for older primary children.



The TI-15 is different from previous primary calculators, not only in appearance but also in functions. Firstly, it is a larger size than a typical primary calculator is. It fits comfortably into an adult hand and should not present problems of manageability

with older primary children. There is a modern, 'teccy' appearance to the TI-15. The blue, see-through casing allows a view of the calculator innards. There are also lots of extra keys compared to usual primary school calculator models. The keys follow a colour-coded pattern of yellow number and simple function and on/off buttons. More advanced functions are dark blue with red signifying keys related to place value. The next section will outline some activities that will be most appropriate for older primary children. All activities are underpinned with the notion of children investigating, explaining and thinking with technology.

Lost instructions

One of the interesting things we have found that children and adults wish to do when presented with the calculator is to play with it and find out what it can do.

Activity

Present the children with a calculator each. Tell them that you have lost the instruction book and that it is their job to find out how the calculator works. Children then explore what the keys appear to do and the functions they perform.

After a short while ask each person to explain to a partner what has been discovered. A final stage could be for the partners to write a part of the instruction manual outlining what they have found so that other class members (and the teacher) can use the calculator effectively.

Road testing the calculator

An important thing for children to realise and understand is a calculator's limitations. With this information children are more likely to use the calculator sensibly and obtain appropriate answers to their problems.

Activity

Ask children to see what rules the calculator uses when calculating. Try a series of calculations to see what answers are obtained:

$$2 + 3 \times 6 \quad 2 \times 3 + 6 \quad 20 - 10 \div 2 \quad 26 + 5 - 7 \times 3 \quad 2 + 8 - 3 \times 4 \div 2$$

This should establish if the calculator uses rule of order in its functions.

Ask children what size of numbers the calculator will use. Is it limited to eight digits like most calculators? What happens when something larger is tried for example:

$$100\,000\,027 \times 6\,952?$$

How could they use the knowledge of the answer to calculate other large numbers?

Leftovers

One of the useful functions of the TI-15 is that it will complete division calculations in two formats — with decimal places and as remainders (or leftovers). Many

children, who are familiar with remainders from their earlier experience of sharing situations, often misinterpret the calculator display of 6.4 as 6 remainder 4. One of the important skills children need to acquire is that of interpreting the display correctly.

Activity

Ask children to investigate other sums that give an answer of 6 with 4 left over using the 'int÷' key. Ask them to do the same calculations but use the normal division key and then to comment on the answers.

Fractions to decimals

The TI-15 calculator has functions that will change decimals to fractions and vice versa and will simplify fractions either electronically or manually.

Activity

Ask children to investigate what happens when they change decimals such as 0.25 into a fraction.

$$0.25 = \frac{25}{100}$$

Ask them to simplify the fraction in two ways:

- by the automatic function ($0.25 = \text{F-D simp} = \text{simp} =$)
- manually ($0.25 = \text{F-D simp } 5 = \text{simp } 5 =$)

Children keep a record of what they did and what resulted. Discussions can follow to identify what has happened and if there are any quicker ways to simplify the fraction.

Converting

The opportunity to discuss terminating and recurring decimals may be taken as a result of completing the following activity. The activity also involves looking for patterns.

Activity

Convert the first four fractions and then predict the rest based on the patterns you observe.

$$\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}$$

$$\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{7}{7}$$

Power patterns

A typical problem found in many books asks ‘What is the units digit of 7^{23} ?’

In order to solve this problem and others like it children need to observe patterns. The power key ‘ y^x ’ and the use of table will help solve the question. Further interesting patterns may be found by examining the unit’s digit of various powers.

Powers

# ¹	# ²	# ³	# ⁴	# ⁵	# ⁶	...
1	1	1	1	1	1	
2	4	8	16	32	64	
3						
4						
5						
6						
7						
8						
9						
10						

Last Digits

Number	# ¹	# ²	# ³	# ⁴	# ⁵	# ⁶
1	1	1	1	1	1	1
2	2	4	8	6	2	4
3						
4						
5						
6						
7						
8						
9						
10						

For a detailed discussion of this activity see Lappan et al. (1982).

Ask the children to investigate what happens when numbers are raised to series of powers. Children discuss and record patterns, connections and relationships they notice. These connections can be reported and explained to the class.

Sobel and Maletsky (1988) described an extension to this activity where students are encouraged to examine the first ten powers of a number. The reciprocal ($1/x$) of this number is found and the reciprocals added to memory ($m+$). Students should be encouraged to examine the values that appear in the memory. The display will approach a specific number. For example consider the following table of powers of four.

		Reciprocal	Sum of Reciprocals
4^1	4	0.25	0.25
4^2	16	0.0625	0.3125
4^3	64	0.015625	0.328125
4^4	256	0.0039063	0.3320313
4^5	1024		
4^6	4096		
4^7			
4^8			
4^9			
4^{10}	1 048 576		

If the procedure is repeated for powers of three and five then further interesting patterns may be found. Prior to the advent of calculators, children would have been precluded from participating in this type of pattern searching behaviour. The advent of a more function calculator in the primary school has meant those activities with powers, fractions and reciprocals can now be added to the problem solving curriculum.

Fascinating Fibonacci

Recent mathematics curriculum documents in many parts of the world have, as part of their aims, a section on appreciating mathematics with its social, cultural and historical contrasts.

Activity

Ask the children to look at the Fibonacci series

1, 1, 2, 3, 5, 8, 13 ...

Discuss how the series is formed and therefore how it will continue. Ask children to use the calculator to divide a number in the series by the number before it and compare the answers; for example $\frac{5}{3}$ or $\frac{8}{5}$.

The answers will begin to converge on the golden ratio of 1:1.618. Further discussion and investigation could be established to find the historical and cultural significance of Fibonacci and also the Golden Ratio.

What's happening?

An important feature of working with calculators is that of asking children to think about what has happened on the calculator. This is an effort to take children beyond just accepting what the calculator does and says without any thought. It is a useful generic technique for developing activities that require children to learn and understand a new idea. The activity and the subsequent reflection are powerful ways to help children build connections and knowledge. Some examples based on this technique are offered below.

Activity

Multiplying and dividing by fractions. Ask the children to program their calculators with the constant option (Op1) and either multiplying or dividing by a fraction.

For example: Op1 \times 1/2 Op1 14 Op1

Pressing the yellow semi circular key will display the sum performed by the calculator, which can provide a stimulus for discussion. Many children will have the idea that multiplying always makes something larger. The new theory can be tested by substituting the 14 in the key press sequence above with another number, for example 1/4.

A similar sequence can be established for the Option key by substituting \times 1/2 with \div 1/2 and discussing what happens. The calculator in these cases is being used to put aspects of children's thinking into conflict and by discussing and reflecting on this children can build new connections in their understanding of concepts such as multiply and divide.

Conclusion

Many of the sensible activities involving calculators as teaching and learning aids can be used with more function calculators such as the TI-15. Used sensibly as part of an enquiry approach to mathematics learning, calculators can help to shape the way children think about and understand mathematics. The TI-15 can add an appropriate challenge to these activities for older primary children. All these should engage children in thinking with technology about numbers and mathematics. At the same time, children are practicing and developing their experience and ability to *think with technology* to solve problems.

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How Might Computer Algebra Change Senior Mathematics: The Case of Trigonometry

Kaye Stacey and Lynda Ball

Computer algebra systems (CAS) have been available on computers for many years. However, they will soon be available on affordable hand-held machines, and at that stage, are poised to make an impact on school mathematics curriculum and assessment. This paper will demonstrate how the teaching of trigonometry and circular functions may change. Good basic algebra will remain essential, the order of topics may change, and some topics will no longer be justifiable whilst others will be tackled in new ways. The interplay between graphs and symbols can be strengthened.

Purpose

Computer algebra systems have been available for use on computers now for many years and it has long been common for tertiary mathematics and engineering courses to make some use of them. They have not, however, made a large impact on common practice and their impact in schools is minimal. This may change in the near future, when the capability of graphics calculators is extended to include a computer algebra system (CAS) at a price within reach of most Australian senior mathematics students. When a technology is regularly available at the time and place where mathematics is done, it can make a significant difference to practice.

This paper reports preliminary thinking on how one of the topics in Australian senior mathematics courses (trigonometry) may be impacted by readily available CAS. It has been prepared as part of the preliminary thinking for conducting an experimental Year 12 mathematics subject using CAS in Victoria. Further details of this, the CAS-CAT project, are available from <http://www.edfac.unimelb.edu.au/DSME/CAS-CAT>. This paper aims to provide a general discussion of how various topics related to trigonometry and circular functions might be taught with CAS. It is not confined to one current syllabus, but looks at a range of topics that may be included at senior secondary level. Sample questions were mainly derived by examining textbooks such as Evans et al. (1999) and Fitzpatrick et al. (1992). The paper outlines some of the new approaches that should be considered for adoption in a CAS environment, proposes topics to receive different, more, earlier or less emphasis and points out some of the differences between systems that will need to be considered for assessment. A range of examples is considered to illustrate particular points.

In preparation for this paper, we have used several different systems: three hand-held calculators (TI-89 manufactured by Texas Instruments, the Casio FX-2.0, the Hewlett-Packard HP-49G) and two computer packages *Derive* and *Mathematica*. As might be expected with emerging and complicated software such as a CAS, there are

currently substantial differences between the modes of use and the form of the answers obtained (which are important for setting examinations) although there is a broad set of common features.

Trigonometry and circular functions are excellent topics to remain in a CAS-active curriculum because of the very accessible and important fields of applications; finding lengths and angles in two and three dimensions and the modelling of periodic phenomena. There is also important theoretical work because the properties of the family of trigonometric functions (sine, cosine and tangent) are accessible by elementary techniques including calculus but they contrast with properties of polynomial and exponential/logarithmic functions. For this paper, senior secondary work with trigonometric functions is divided into three main areas treated in turn: firstly solving triangles, secondly working with identities, and thirdly the function properties.

Solving triangles

Numerical or exact values

The advent of the scientific calculator revolutionised the arithmetic involved in solving triangles and made the use of tables of values of the functions obsolete. Computer algebra extends this revolution in two ways, by using exact values and by performing some of the equation solving arising in multi-constraint problems. Finding an 'opposite' side in a triangle given an angle of 60 degrees and a hypotenuse 12 units long is straightforward if the calculator is set in numerical mode and to use degrees: $12 * \sin 60 = 10.39$. However, in exact mode, the CAS gives $6\sqrt{3}$. In an examination system with CAS, no longer would a question such as this test any special knowledge of the values of the sine function. The role of work with exact values needs to be carefully reconsidered: it could be a major feature, supported by technology, but there must be a clearly specified purpose.

The computer algebra systems can also provide exact values beyond the commonly memorised repertoire. For example, entering $\cos(75^\circ)$ on the FX-2.0 with exact mode set will give the $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ in one step. *Mathematica* gives $\frac{-1+\sqrt{3}}{2\sqrt{2}}$. This variety illustrates that students will frequently need basic algebra skills, if they are to match output to given forms (e.g. for checking answers in a textbook, working with others using a different system, getting a specified answer on a test etc.).

As with a scientific calculator, students need to be aware of degrees and radians as two parallel systems for measuring angles. However, radian measure is more important on a CAS than for a scientific calculator because it is used in more advanced features such as the calculus and even in some equation solving.

The equation solving facility can also be used with advantage. Kendal and Stacey (1996) studied the trigonometry problem solving of all the Year 10 students at a large Victorian high school. They found that the most frequent cause of errors in one-step problems to find an unknown side in a right angle triangle was not the trigonometry

but the basic algebra, especially when the unknown side is in the denominator of the trigonometric ratio. CAS can, however, solve an equation such as $100/x = \tan(15^\circ)$ immediately. For example, the TI-89 in exact mode gives the exact answer $\frac{100}{\sqrt{3+2}}$. A numerical answer is also immediate.

A word of caution is due here. Kendal and Stacey also found that students who encountered equations such as $30 = \frac{0.0003}{x}$ in the course of learning trigonometry made significant improvements on solving algebraic equations like this. In schools such as the one where Kendal and Stacey collected their data, trigonometry is a vehicle for improving basic algebra, although progressing in the new topic is also made more difficult by the students' lack of skill. In considering changes to in one topic such as trigonometry, the indirect roles in the curriculum need to be considered, as well as the direct role.

Introducing new methods into problem solving

Solving triangles in multiple-step problems can often be done using the facility to solve linear equations. In these questions, students will be able to think through the whole process to solve the problem, rather than just what operation is required at each stage to give the next line in the solution. Consider this problem:

A flagpole of height h metres is on the top of a tower of height H metres. From a point horizontally 85 metres away from the base, the angle of elevation is 40° to the top of tower and 43° to the top of the flagpole. How tall is the flagpole?

Writing down the information from the triangles:

$$H = 85 \tan 40$$

$$H + h = 85 \tan 43$$

These equations can be solved as

$$h = 85 \tan 43 - h = 85 \tan 40$$

In this problem, student understanding is evident through the setting up of the problem. The mathematical formulation of the problem is essential in order to be able to use the CAS to find the numerical solution.

If these equations were a little harder, we could see them as two linear equations in two unknowns and use the matrix solving facilities — this is a general method that could be used in a wide range of trigonometric situations. For example, consider the problem in Figure 1.

$$H + h = (35 + D) \tan 48$$

$$H + h - \tan 48 D = 35 \tan 48$$

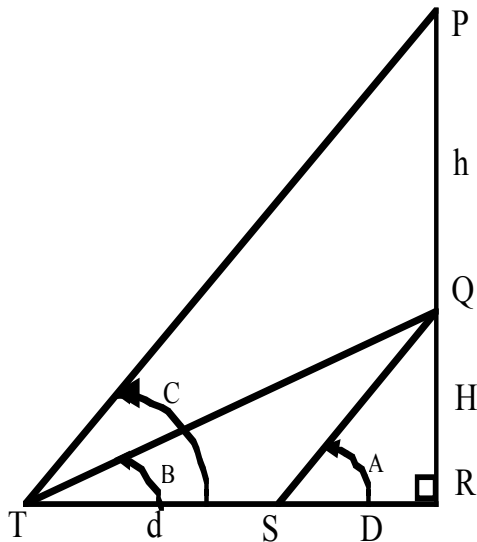
$$H = D \tan 43 \quad \text{rearranged as} \quad H - D \tan 43 = D$$

$$H = (35 + D) \tan 40$$

$$H - \tan 40 D = 35 \tan 40$$

After rearranging these as three linear equations in three unknowns (H , h and D) as shown, the matrix facility can be used to solve them automatically. This emphasises an orientation to writing down the whole problem and then using an automated

procedure for the solution, rather than working step-by-step through a solution. It also shows new links between previously isolated topics. A discussion of this problem in the case where A , B , C and D are unknown, and the symbolic algebra facility is used, is given by Stacey (1999).



A flagpole (PQ in the diagram) is placed on top of a castle wall (QR), which is surrounded by a moat (RS). From point S , the angle of elevation of the top of the wall is $A = 43^\circ$. From point T , the angle of elevation of the top of the castle wall is $B = 40^\circ$. From point T , the angle of elevation of the top of the flagpole is $C = 48^\circ$. The distance ST is 35 metres. Find the height of the flagpole.

Figure 1. An illustration that matrix equation solving can be used in trigonometry problems.

Opportunities for more 'algebraic' approaches

There are other opportunities for more 'algebraic' approaches when using CAS e.g. less step by step evaluation, more focus on writing down the constraints for a problem, more opportunities to work with unspecified values. Students will be able to show how a complete expression can be made, rather than just working everything out step-by-step and evaluating at each stage. The reasons why such an approach may be considered more algebraic comes from a consideration of the differences between arithmetic and algebraic thinking, as proposed, for example, by Stacey and MacGregor (2000).

Example:

Two ports, A and B are such that B is due West of A . A is due North of a ship, S . The ship is on a course $328^\circ T$ and reaches B after travelling for 3 hrs at 25 km/h. Calculate the distance between the two ports, and the time it would have taken the ship to reach A from S . (Fitzpatrick, Galbraith and Henry, 1992, p. 237).

Outline of a solution using CAS:

Define $AB = 75.\sin(360^\circ - 328^\circ)$

Using Pythagoras' theorem, the distance from A to S is $D = \sqrt{75^2 - AB^2}$

Define the time from A to S to be equal to $D/25$

Then the whole expression can be evaluated at the end in one step to get 2.54 hours

We propose that the written record from a student using CAS to assist in solving this problem might look something like the outline above and welcome debate on this point. (A discussion paper on this issue is available on the CAS-CAT website, address above.)

Trigonometric identities

Beyond simple relationships, working with trigonometric identities on CAS calculators can be quite difficult and there are substantial variations between brands. This aspect of CAS may well become easier to use in the near future, but with the current capability, it often seems that substantial by-hand skills are required for a user to persuade the CAS to work with trigonometric identities.

Well known relationships between cos, sin and tan

Relations such as $\sin(90 - x) = \cos(x)$, $\sin(180 - x) = \sin x$, $\cos(\pi - x) = -\cos x$, $\tan(\pi - x) = -\tan x$ etc. are mostly easy to find on all CAS machines. There is little purpose for these identities in calculation now and so we propose that the emphasis should be on how these translate into properties of the functions, rather than for calculation. Students should be able to relate these identities to the various symmetries of the functions and link the algebraic with the graphical. These identities are particular instances of the compound angle formulae such as $\sin(A + B) = \sin A \cos B + \cos A \sin B$, which the systems use frequently. For this reason, students may need to learn about the existence of these relationships earlier than before to make sense of some output.

Identities can be surprisingly hard

Simplifying an expression such as

$$[(\cos t / \tan t - \sin t \tan t) \sin t \cos t] / (\cos t - \sin t)$$

demonstrates the variability between currently available computer algebra systems. *Mathematica* will simplify the expression in one step to $(\frac{1}{2})(2 + \sin(2t))$. On the hand-held calculators, the user needs to enter parts of the expression separately before combining them. The challenge for students is in deciding when it is appropriate to collect trigonometric terms versus expanding, and how much of the expression should be entered at each stage. This shows that although it may not be possible to simplify an expression in one step, through using inbuilt features of the calculator it may still be possible to simplify the expression. Students need to use their understanding of trigonometry to decide on how they can work towards a simplified expression, but problems like this may well be easier by hand.

Easy by hand, harder by CAS

Solving equations involving trigonometric functions can also be easier by hand than with the machine. For example, it is quite easy to solve the equation $5 \cos^2 x + 2 \sin^2 x = 2$ by hand because the substitution of $\cos^2 x + \sin^2 x = 1$ is immediately recognisable. The CAS machines, however, gave a variety of answers, sometimes after substantial guidance from the user. The HP-49G solved the equation numerically immediately giving one answer of 1.5707, but solving the equation exactly had to be guided step by step by the user. Using the solve command on the TI-89 gives the exact answer $x = ((2n - 1)\pi)/2$. The FX-2.0 gives an answer of $x = (2\pi)k - \pi/2$ and $x = (2\pi)k + \pi/2$. Even though the systems give general solutions for the problem, the forms of the answers are different and students must be able to interpret these different forms. Using *Mathematica* only two answers were obtained; $x = -\pi/2$ and $x = \pi/2$.

This highlights the differing capabilities of various brands of calculators (and CAS software packages) for given content areas. The trigonometric features on the various brands of CAS calculators seem quite different. Students will need to know a range of trigonometric relationships to make sensible decisions when using the trigonometry menus.

Properties of trigonometric functions

The properties of trigonometric functions provide an excellent contrast in the curriculum to the properties of polynomials and exponential/logarithmic functions. Students learn how to manipulate the functions to match given amplitudes and periods for modelling and to deal with multiple solutions in a regular pattern. When using CAS calculators, all the graphing capabilities of graphics calculators are retained, so that many of the adjustments required have already been widely discussed and are in the process of being implemented in some examination systems. Students can plot graphs readily, read off periods and amplitudes and maxima and minima etc., see families of multiple solutions graphically, and solve equations graphically.

Students will need to know more about parameters

To use an algebraic language such as CAS, students need to know of many different uses of algebraic letters, including as parameters not just as unknowns and variables.

For example, to find all values of x such that $\sin x = \frac{-\sqrt{3}}{2}$ in exact mode, the HP-49G gives $x = -\frac{(6n1-4)\pi}{3}$. To find solutions that satisfy specified constraints such as $0 \leq x \leq 2\pi$, students must deal with the parameters.

Using *Mathematica*, the only solution that is obtained to this equation is $-\pi/3$, although the program reported that some solutions might not have been found. Students would need to recognise that the expected form of the solution is a family

of solutions and use this understanding to generate the correct solution for the problem.

Finding all families of solutions is a delicate operation and students using CAS will need to be very careful not to rely on it mindlessly. The following investigation, illustrated in Figure 2, could be used to show some of the difficulty of dealing with functions which are not one to one and the consequences of defining inverses by limiting the domains.

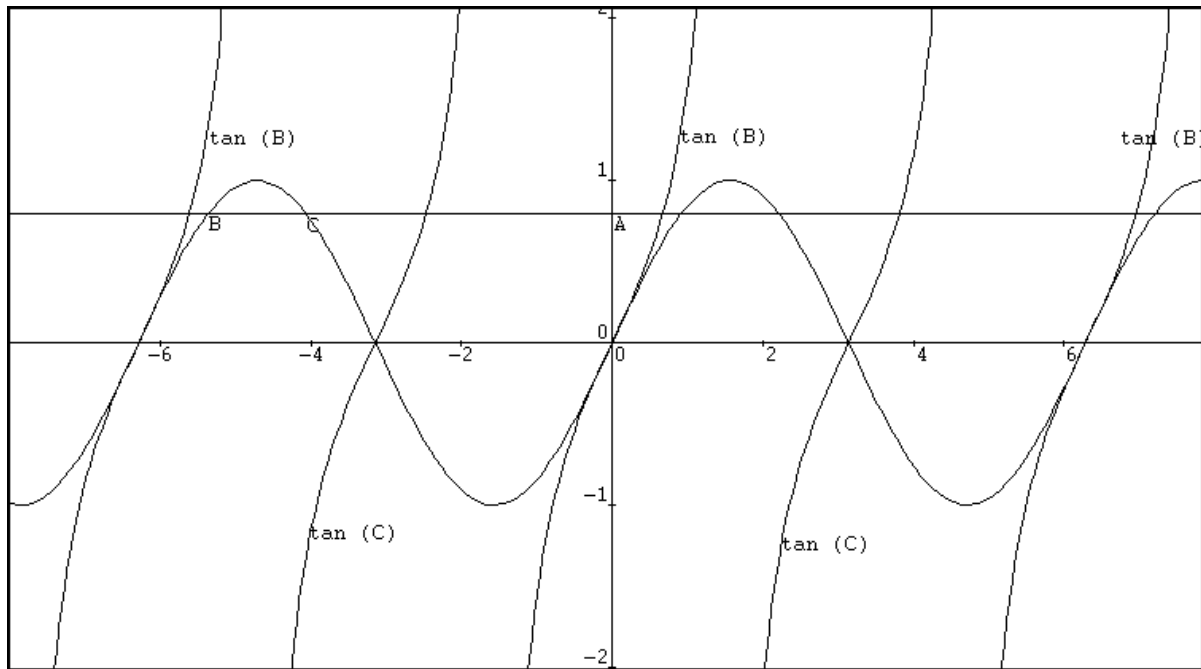


Figure 2. Investigation. If $\sin x = A$, what is the value of $\tan x$?

Figure 2 shows the graphs of $y = \sin x$ and $y = \tan x$. The horizontal line $y = A$ intersects the graph where $\sin x = A$ (an arbitrary value). Two such points, B and C, are labelled. The problem asks us to find the value of the tangent function at points like these. Moving up from point B gives $\tan(B)$, marked on the graph and moving down from point C gives $\tan(C)$. The graph shows that there are a series of points where the value will be $\tan(B)$, spaced at intervals of 2π and a series of points where the value will be $\tan(C)$, also spaced at intervals of 2π .

What does the algebra facility do? Quick calculation of $\tan(\arcsin A)$ gives one solution $A/\sqrt{1-A^2}$ immediately, for example using the HP 49G, but this is only one of the two solutions (in the case shown in Figure 2, it is $\tan(B)$) and there is no indication that there are families of solutions. The relationship between $\tan(B)$ and $\tan(C)$ can be found by considering the symmetries of the graphs.

Directions for doing trigonometry with CAS

The examples above have discussed some of the ways in which the trigonometry topics in senior secondary school may change when CAS is widely available. The strengths (and hence challenges to the mathematics curriculum as it is) of the current systems over the graphics calculators lies in their ability to deal with exact solutions and the equation solving (both numeric and exact). Some identities which are now memorised, such as $\sin(-x) = -\sin(x)$, are available at the push of a button, and compound angle formulas are readily used. On the other hand, the skill involved in persuading the machine to solve some trigonometric identities surpasses the skill required to do them by hand.

Several topics might be introduced earlier to students using CAS. The existence of radians must be known early, at least to set the MODE to degrees. Because the CAS needs to work in radians in unexpected places, radians cannot be ignored as easily as on a graphics or scientific calculator.) Compound angle formulas may be used by the CAS in unexpected places for simplification and so at least their existence needs to be known early. (For example, the HP 49G frequently replaces $\cos^2 x$ by $\frac{1}{2}(\cos 2x + 1)$ unexpectedly to me). However, since facilities like this are so strong, possibly students need to learn little other than their existence. This is a specific instance of the question of what essential skills (or understandings) students will need to be able to use the trigonometric functions effectively. For example, will students need to know the different forms of $\cos(2x)$ etc. so that they can make sensible decisions when using the trig menus? Students will also need to be able to work with parameters. Finally, it is time to bid goodbye to the cosec, cot and sec functions. The CAS manages without them, and we think we can too.

In this topic as with others, CAS provides opportunities for what can be seen as a more 'mathematical' approach to solving problems; one where there is less step by step evaluation, but more focus on writing down the constraints for a problem, and more opportunities to work with unspecified values. This may help students to display in their written work their overall plan and the reasons for it, rather than principally recording the details of the calculations.

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Mathematics is Stuffed

Paul Swan

The majority of students leave school with a dislike for mathematics. Why is this the case? Is it perhaps the nature of the subject, poor teaching, maybe conditioning from home or possibly a change in the way children learn? These issues are examined in the light of current research and experience. In particular the role of the teacher is examined as a means of improving affective outcomes in mathematics.

Introduction

The title of this paper was deliberately chosen to evoke a response. While it is disturbing to hear children make comments like 'I hate maths', 'maths is boring' or in the vernacular of today's teenager, 'maths is stuffed', it is even more disturbing to hear adults use similar expressions such as 'I was never any good at maths', 'I never understood maths at school' or 'I never liked maths'. What is most disturbing, however, is that many first-year teacher education students express similar sentiments. Many express a desire to teach junior primary classes because the mathematics is easier. Unfortunately the early years at school are the most formative and therefore children need teachers who are enthusiastic about the subject.

It is with fear and trepidation that many first-year teacher education students enter their first mathematics education unit. What has caused these intelligent adults to feel this way? Westwood (2000) cited Wain (1994) on this issue of disenchantment with mathematics.

He points out that many intelligent people after an average of 1500 hours of instruction over eleven years of schooling, still regard mathematics as a meaningless activity for which they have no aptitude. He concludes that 'it is difficult to imagine how a subject could have achieved for itself such an appalling image as it now has in the popular mind ... to think that all our effort has led to a situation of fear and loathing is depressing' (p. 31).

This phenomenon is not a local one, peculiar to Australia. The words panic, anxiety and phobia have all been linked to mathematics in the literature (Buxton, 1991). The National Statement on Mathematics for Australian Schools even acknowledges the feelings children have toward mathematics. 'There is considerable anecdotal and research evidence to suggest that many people dislike maths and may feel intimidated in situations which it is used' (p. 7).

What is of even more concern, however, is that it has become 'cool' to dislike mathematics. Gordon (1992) made the following comment in the context of his discussion of dyscalculia, a disability that is thought to affect some 4–6 percent of the population. 'A child may well not be referred for assessment because such a

disability is much more socially acceptable than an inability to read or write, and it is not thought to be a serious educational problem. Often people will say, with a degree of pride, 'I was never any good at maths' (p. 459).

Australia is currently in the grips of an outcomes based education revolution. At a time when the only universal mathematics outcome seems to be that people end up hating mathematics it would seem appropriate to examine the reasons behind this dislike of mathematics.

Why is this the case?

It is one thing to identify a problem and quite another to find the cause of the problem. No single factor can be blamed for producing generations of students who loathe mathematics. There are several factors that might be examined. In broad terms these include:

- the nature of the subject;
- the nature of the learner;
- the nature of the teacher.

The subject

Students often complain that the subject is boring, too difficult and divorced from reality. Teachers complain that they are rushed and that there is too much to teach, too little time and too many interruptions. While teachers have control over *how* they teach often they have little say over what is taught. This issue is too broad to be discussed in brief, suffice to say that the issue is broader than the confines of the classroom.

The nature of the learner

Much has been published about the changing nature of the learner. For example Healy (1990; 1998) suggested that television, video games and other aspects of modern culture have all affected children's ability to absorb and analyse information. Anecdotal evidence gathered from teachers and parents along with the increase in the incidence of conditions such as ADD and ADHD would give some credence to her argument. There is evidence to suggest that the home environment can affect the learner. For example it has been acknowledged that reading to a child in the formative years and surrounding them with books has an impact on reading in later years. Parental modelling of reading can also help set up a pattern of reading in the home. Similarly parental dislike for mathematics can affect the way children feel about mathematics.

The advent of new technologies that allow for scanning of the brain have also fuelled new ideas about how the brain works. Butterworth (1999) in his work entitled *The Mathematical Brain* examined the question of 'from where does the ability to use numbers come?' Butterworth argued that the ability to work with numbers is programmed into our genes. Likewise Devlin (2000) suggested that everyone has the

ability to do maths but most people do not use that ability. Both authors cite example of people with brain lesions, or stroke victims who have lost facility with number in support of their arguments. The implications of such a suggestion are far reaching and beyond the scope of this discussion. Of interest, however is Butterworth's question *Why does schooling sometimes leave us so muddled and discouraged that we close the door on our mathematical brain?*

The teacher

Teachers make a difference. Remember, however, that not all teachers are paid and that a child's first teachers, the ones that influence the formative years are usually the parents and in particular the mother.

In relating stories of school mathematics nearly all the positive stories given by adults relate to specific teachers who were able to inspire and make the subject live. Stories relating to bad experiences often include teaching practices such as tables drills and completed endless work-sheets or exercises out of a text.

The job of a teacher is not an easy one, particularly when a primary teacher has to juggle a command of several subject areas and cope with changes in the nature of the child and the family. Many of these changes are beyond the realm of the classroom teacher to affect. It is for this reason that a job description provided by Sobel and Maletsky (1988) appeals. They sum up the job of a teacher in three profound statements:

Teachers must know their stuff.

They must know the pupils they are stuffing.

And above all, they must know how to stuff them artistically.(p. 1)

The title of the discussion was really drawn from this job description as clearly I do not believe 'mathematics is stuffed', nor do I wish to imply a method of teaching based on stuffing or transmission of knowledge is appropriate in the twenty-first century.

Teachers must know their stuff

To illustrate the importance of teachers knowing their stuff, that is understanding the mathematics they are teaching consider the teaching episode depicted in the Roald Dahl book, *Willy Wonka and the Chocolate Factory*. To set the scene the children are eager to learn about percentages as they are motivated by the thought of winning a tour through the chocolate factory. The teacher then begins the lesson.

I've just decided to switch our Friday schedule to Monday which means that the test we take each Friday on what we learn during the week will now take place before we have learned it, but since today is Tuesday it doesn't matter in the slightest. Pencils ready...

Realising they are about to start a mathematics lesson the class lets out a collective groan. The teacher in question then launches into a big spiel about percentages using 1000 Wonka bars as the base. His explanations are somewhat confusing and what he

writes on the blackboard does not match what he is saying. The teacher really comes unstuck when Charlie explains that he only bought two Wonka bars. As this is too difficult for him to work out he changes Charlie's response to two-hundred. By the end of the lesson the students are totally confused. While this segment was designed to elicit a humorous response, the real issue is that many people can identify with that feeling of bewilderment that comes from mathematics classes. Too many children and adults their experience of mathematics is just like that. Unfortunately that is *not* funny.

The work of Liping Ma (1999) has been quoted to show that teachers' understanding of mathematics has an effect on the learner. The poor showing of American students in comparison to their Asian counterparts has been the catalyst for several cross cultural; studies comparing Asian and American Teachers (Ma, 1999; Stigler et al., 1996). In brief, Ma found that Chinese teachers tended to develop a more 'profound understanding of fundamental mathematics' than their American counterparts. This was despite the comparatively shorter amount of time Chinese teachers spent studying. Chinese students grew up with a better understanding of mathematics than their American counterparts. The Chinese teachers do not attend a University to learn to teach but tend to follow an apprenticeship model where knowledge is shared among colleagues. Care needs to be exercised that the comparison not be stretched too far with Australian teachers. Australian students perform much better on international comparisons than American students but they still lag behind their Asian counterparts. Remember, also, that in some cases Asian teachers only teach mathematics, Asian children are often more willing to learn and more time is devoted to school-work

What can be gained from comparative studies is that the classroom environment and the understanding of the teacher do play a key role in helping children to learn mathematics. The answer to improving teachers understanding of mathematics, however, may not lie in spending more time at University but rather in collegial support and discussion of mathematics and how to teach it.

They must know the pupils they are stuffing

A simple answer to the question of improving teachers' profound understanding of knowledge would be to employ specialist mathematics teachers in primary school. I believe that would be counterproductive in the light of the second aspect of a teachers' job — to know the students. One advantage of employing generalist rather than specialist mathematics teachers in primary schools is that teachers have the chance to develop relationships with their pupils and therefore are able to extract the best from them.

A positive classroom climate is something that takes time to build and nurture and can have a profound effect on the learner. Imagine a classroom where children are free to ask questions without fear of reprisals. Imagine children working collaboratively in groups, discussing various ways to solve a problem. Compare this to a classroom where the climate is hostile. Roald Dahl, provided the perfect example of a classroom designed to turn anyone off mathematics and school. Dahl, who one

suspects must have grown up with an aversion to mathematics teachers, described the stereotype of a maths class with children sitting in rows, working from text-books and in silence. Consider the description of a school mathematics lesson in *Danny The Champion of the World*.

To set the scene Danny reluctantly attends school on Thursday, knowing that on Friday he will be helping his father get ready for pheasant poaching. Danny is obviously excited at the prospect and the last thing he is interested in is doing some mathematics. His teacher, Captain Lancaster, who is described as having a fiery temper and a small clipped moustache begins the mathematics lesson by instructing the children to turn to page thirty-one of their books. Next the children are commanded to work out all their multiplication sums in their exercise books 'without a word'. One of Danny's friends asks him the question *What are eight nines?* to which Danny replies *seventy-two*. Unfortunately this evokes the wrath of Captain Lancaster who calls Danny a cheat and administers corporal punishment.

I would rather children learn less mathematics but had more fun learning it. I would also rather children learn less mathematics well, than a great deal of mathematics that they did not understand or could not apply. The expression *Its not what you say but how you say it* comes to mind when considering the various approaches that teachers adopt in teaching mathematics. Two teachers can teach the same topic and yet the children from one class can come out bored to tears while the students in the other cannot wait for the next lesson. What is the difference? It is in the way the mathematics is stuffed.

Stuffing artistically

Sobel and Maletsky (1988) classify most lessons according to the three Ds:

- Dull
- Deadly
- Destructive of all interest (p. 2).

Not all teachers can 'act'. Not all teachers can inspire. Not all lessons can captivate the imagination. In this section I would like to examine a little more how our Asian colleagues 'stuff their pupils' and look at some approaches that might be used to liven up a mathematics lesson, artistically.

How Asian teachers teach

We can also learn from our Asian colleagues who, on the whole tend to work toward the development of conceptual knowledge rather than simply procedural knowledge. In general terms Asian teachers tend to:

- spend a lot of time planning lessons;
- use predominantly whole class teaching methods;
- spend less time on revision and drill than was thought by Western commentators.

It should be pointed out that Asian teachers have less disruption especially in terms of student behaviour and administration demands. Shimazu (1995) described the typical four stages that most Japanese teachers followed.

- The problem is presented.
- Students try solving the problem either individually or in most cases with a partner or in a group.
- There is a whole class discussion about the various methods used to solve the problem.
- The teacher provides a summary and the students then apply this thinking to another problem.

The use of literature as a starting point

It should be acknowledged that many teachers of mathematics at primary school feel more comfortable teaching language rather than mathematics. Many children like stories, so why not combine mathematics and literature? Let me give some examples. The Harry Potter series of books have become a worldwide phenomenon. Most children have an interest in the adventures of Harry and his schoolmates. In J. K. Rowling's Harry Potter and the Philosopher's Stone, Hermione Granger and Harry have to solve a logic puzzle in order to stop the philosopher's stone falling into the evil hands of 'you know who' (Voldemort). The puzzle involves working out which of seven bottles, all of different sizes, contain poison, nettle wine or a potion that allow them to pass through fire. Three bottles contain poison, two wine and two magic potion. Hermione solves the riddle which she acknowledges 'isn't magic — its logic — a puzzle.' (p. 207)

Harry Potter is fine for the children who read, but what about those who only watch TV. Bruce Willis supplies the answer. In the third instalment of the *Die Hard* trilogy the two main characters are only given five minutes to solve the famous water decanting puzzle or be blown up. Talk about motivation! While it would not be advisable to show such a clip in class, or blow up children, the problem is reproduced below. For those who have not seen the movie, Bruce Willis, chest heaving in a designer singlet, sweat dripping from his 'follicly challenged' forehead solves the puzzle with, you guessed it, seconds to spare.

Two jugs (five and three gallons respectively) are found in an ornamental pond in a park along with a scale. The instructions state that our heroes must place exactly four gallons onto the scale.

There are several sources of literature that may be used to stimulate thinking in mathematics. There have also been several books written on how to make use of children's literature in mathematics (See Doig, 1989; Griffiths & Clyne, 1988). Let me share two recent titles, *Aquilla* and *Matt's Millions*, both by the same author, Andrew Norriss. *Aquilla*, which appeared on television revolve around two boys who find a spaceship. The two boys manage to leave the spaceship running while it is invisible. When they return to where they left the ship it has moved. The boys try all sorts of ideas to locate the ship until one realises that the answer lies in mathematics. He

applies his problem solving skills stating *What we need ... is someone who knows about maths.* (p. 67) Later in the staffroom the teachers are in disbelief as one of their number relates the story of how these two boys were in the form room doing maths. The conversation proceeds:

‘What sort of maths?’

‘They were trying to work out a problem. I don’t know where they got it from. Something about if A travels X centimetres in Y hours, how long is it before it gets to Z. Seriously! I couldn’t believe it!’

‘Was it homework or something?’ asked Miss Poulson.

‘No.’ Mr Duncan shook his head. ‘I haven’t set any this week. It had nothing to do with their class work.’

‘So why were they doing it?’

‘Fun,’ said Mr Duncan. ‘That’s what they told me. Fun.’ He shook his head in disbelief. ‘And when I offered to explain what they were doing wrong, their little faces lit up. They were interested. They were hanging on every word like they really wanted to know... and then when I told them the answer — I forgot what it was, four hours and something — they were so grateful. Extraordinary, isn’t it? I really hadn’t thought they were the type!’

Miss Taylor went back to her office wondering if the world had gone quietly mad. Baxter and Reynolds doing maths... (p. 68–69)

Amazing what a little motivation can do! The second story by the same author is about a boy who invents a computer game and cannot spend the royalties fast enough. The book highlights a great deal of mathematics including the exponential nature of compound interest.

Stuffing the basic facts — artistically

Consider a relatively boring topic such as the multiplication tables. There are several approaches that can be used to ‘stuff the tables’ into a child’s head. The most common methods would involve some form of drill and practice. Drill and practice activities can either be boring and promote maths anxiety or interesting.

An example of an activity once commonplace in many schools was *Beat the Tape*. Recently I found a set in use in a school. Let me share the introduction to Tape 27. There are fifty tapes in all I might add so at this point the children would be a little over the half-way point in their beat the tape course.

Well girls and boys if you have battled your way through to tape number 27 you’ve done very well indeed and you deserve a letter and you shall have one. On this tape we examine the basic processes in which you’ve had success on the earlier tapes so I expect you all to do very well. If you make any mistakes it means you need a little more practice on these tapes. Now I don’t think any of you should be satisfied until you have sixty correct [out of sixty]. On your worksheet write tape number 27 and your name.

Although the introduction is a little patronising by today’s standards the questions that follow and the pace are similar to what might be given in many classrooms

today. The taped questions feature a variety of words for addition, subtraction and so on, but one might imagine that we have moved on from here. In a way we have, now instead of 'Beat the Tape' we have a plethora of rap CDs that essentially do little more than set the tables to music. While there may be an argument from a multiple intelligences perspective to associate tables with music there seems to be little other argument for using this method other than the possible motivational appeal of 'singing the tables'.

There are several alternatives to the ten-a-day mental sessions that children have had to endure for the better part of a century. Often when reflecting on negative school experiences adults refer to tables drills of this nature and so called games that put them on the spot and humiliated them. Several alternatives for mental sessions are possible including the use of games and routines that involve children in discussion. Swan (1995) provides some other alternatives to standard drills including the use of patterns to make learning the tables less of a chore.

Mathemagic

Gardner (1956, p. xi) described mathemagic as combining 'the beauty of mathematical structure with the entertainment value of a trick.' For detailed discussion of mathemagic see Swan (1998). An example of mathemagic is the mind reading trick based on using a set of cards to determine a secret number. A volunteer is asked to choose a secret number and then is shown a series of cards and asked to state whether their secret number appears on the card that is shown. Based on the answers the mathemagician can determine the secret number. A set of cards for determining a secret number between one and thirty-one are shown below.

1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

Card 1

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

Card 2

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

Card 3

8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31

Card 4

16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

Card 5

Figure 1. Mind Reading Cards.

This piece of mathemagic works by adding the numbers on the top left hand corner of each selected card. For example if the second, third and fifth cards were chosen the secret number would be twenty-two ($2 + 4 + 16$). Of more interest is why the trick works and how it may be used to stimulate the development of some mathematics. The cards are based on powers of two. In order to create the set of cards numbers

need to be placed according to the powers of two that are added to make the number. For example the number twenty-two should appear on the 16, 4, and 2 cards, because these add to make twenty-two. The trick may be extended to include a sixth card which increases the range of numbers may be chosen from 1 to 61.

I do not advocate the use of tricks and puzzles simply for the sake of doing so but where clear links can be made to the development of some meaningful mathematics I believe teachers are more than justified in using them. Teachers looking for further mathemagical ideas may like to try Swan (1993, 1994, 1998).

Conclusion

A National Statement on Mathematics for Australian schools made the following comment.

We must increase the number of students who are enthused by and successful with mathematics and who wish to remain involved in it... Therefore, mathematics curricula should explicitly address the development in students of positive attitudes towards mathematics and towards their continued involvement in mathematics (p. 12).

Teachers especially at the primary level are in a unique position to be able to instil an interest in mathematics. Children of a primary school are still malleable, able to be moulded in such a way to appreciate the beauty of mathematics. If we fail to act then not only will the children be 'stuffed' in later life, so will the country.

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About the presenter

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Strategies for Going Mental

Paul Swan and Len Sparrow

Much is known about mental strategies and how children use them. Evidence suggests that discussion should play a key part in the development of mental strategies and yet many mental computation sessions are still characterised by the traditional ten or twenty quick question approach. This paper reviews what is known about mental strategies, examines why a certain level of inertia exists and suggests a way forward.

Introduction

Mental arithmetic is often a focus for debate in the media and across the dining room table. In most cases it refers to the learning and recall of basic number facts and multiplication tables. It is possible however, to use the term mental arithmetic to mean something similar but fundamentally different. Many people would now see mental arithmetic having two parts. The first part is concerned with the recall of facts and the second with the development and use of strategies for calculating mentally. This paper is concerned mainly with the second aspect of mental arithmetic, that of strategy building.

Within the idea of mental strategy building is the issue of whether or not to teach explicitly various strategies or to let them grow and develop as children face and solve problems concerned with mental calculation. This paper adopts the position and assumption that children develop a range of mental strategies by being exposed to rich situations requiring them to explain and describe their method of solution to their peers. In this way they hear and see other strategies to solve problems involving mental computation. This is, however, not an ad hoc, laissez-faire approach as the skilled teacher is aware of the possible variety of strategies and can draw and highlight them in the situation.

The strategies children use to calculate mentally have been researched to the point where we know:

- children invent their own strategies for calculating mentally (Kamii, 1994; Kamii, Lewis & Livingston, 1993);
- children often adopt one method in school and another out of school (Carragher, Carragher & Schliemann, 1985);
- methods vary from child to child and even the same child may choose to use different methods to solve similar problems at different times (Hope & Sherrill, 1987);
- mental strategies differ from written methods: for example, many mental strategies for addition, subtraction and multiplication start from the right,

whereas most mental methods start from the left (Askew, 1997; Hope & Sherrill, 1987);

- the teaching of written methods, particularly at an early age can stifle the development of mental strategies (Carraher & Schliemann, 1985; Kamii & Dominick, 1989);
- some mental strategies are more efficient than others: for example, counting on in ones from a smaller number rather than the larger of two numbers if adding (Hope & Sherrill, 1987);
- strategies have been identified and coded, although strategies are often referred to by different names and codes in the literature (McIntosh, deNardi & Swan, 1996).

Mismatch between what is known and what is taught

Yet despite the increased awareness of how children calculate mentally, many textbooks containing lists of basic fact questions continue to be produced, tables tapes abound and the ten quick mental a day is still practiced in many classrooms. Why is this the case? We would like to suggest there are several reasons for this apparent mismatch between what is known and what is taught related to mental arithmetic.

Tradition

Tradition is very powerful and difficult to change. The rapid-fire, tables drill has been a part of classroom practice since stimulus response theories became popular in the early part of last century. Rigour was valued and drill was viewed as a way of exercising the mind. Parents have come to expect it to be part of mathematics teaching. Principals and teachers value it. Children suffer it. Tradition, by its very nature, is often not questioned. In order to change tradition one must present powerful arguments.

Ease of assessment

Clearly ten quick mental recall questions a day represents a testing rather than a teaching situation. Assessment is clear-cut, results can be monitored, graphed and progress measured. How to measure the development of mental strategies is somewhat more cumbersome.

Defined teaching approach

The rapid-fire approach to mental computation is much more defined and easy to pass on. Simply choose a set of questions which may or may not be related, present them orally, have the children mark the answers and record the result. In ten to fifteen minutes the teacher is able to deliver a neat package.

Discipline

A common practice in schools is to timetable mental arithmetic just before or after a break. It is suggested that mental arithmetic is the ideal activity to settle children to work. The children are controlled, sitting in their seats and have to listen carefully for fear of missing the next question or worse still mixing up the sequence of answers.

Unclear direction

Change will not be effected unless a viable alternative is provided. The old adage 'If it ain't broke why fix it' is often at the core of arguments in favour of keeping the status quo. Viable alternatives that encompass many of the elements that teachers require, while also embracing sound educational principles, need to be offered. The purpose of this paper is to offer a clear alternative to the ten or twenty rapid-fire questions that dominate mental computation sessions in many classrooms around the country.

Developing mental strategies: a way forward

The development of mental strategies is a key element of many Australian and international curriculum documents. The following sample from Western Australia (EDWA, 1998) illustrates the use of the term (*Italics added*).

Students choose and use a repertoire of *mental*, paper and calculator *strategies* ...

N1.3 uses *counting and other strategies* to mentally solve ...

N2.3 ... add and subtract one and two-digit numbers drawing mostly *on mental strategies for one digit numbers*

N3.3 adds and subtracts whole numbers and amounts of money and multiplies and divides by one-digit whole numbers, drawing mostly *on mental strategies* for doubling, halving, adding to 100, and additions and subtractions readily derived from basic facts.

N4.3 calculates ... drawing mostly *on mental strategies* to add and subtract two-digit numbers and multiplications and divisions related to basic facts

Thompson (1999) described the phrase 'mental strategies' as:

The application of known or quickly calculated facts in combination with specific properties of the number system to find the solution of a calculation whose answer is not known. They also incorporate the idea that, given a collection of numbers to work with, children will select the strategy that is the most appropriate for the specific numbers involved (p. 2).

It could be argued that children making use of mental strategies are 'working mathematically' and thinking about numbers rather than remembering procedures.

Mental strategies: to teach or not to teach?

A constructivist approach to mental computation relies on the generation and sharing of mental strategies. This places the onus on the teacher to examine and

interpret the responses given by children. The teacher therefore needs to have knowledge of mental strategies in order to offer appropriate responses to the children. The response can take many different forms, for example the offering of a question or asking the children for clarification. Armed with this knowledge, teachers then need to make judgements as to how much advice and help should be offered. Allowing children to discuss and describe the strategies requires teachers to comprehend the nature of the strategy being described. They also need skills to be able to assist children to verbalise their thoughts and communicate them clearly so that the rest of the class can understand.

An alternative approach to developing mental computation strategies from the children's thoughts is to teach a specific strategy in a particular lesson. This teaching approach could be considered almost algorithmic in nature and teachers run the risk of streamlining the use of strategies to the point where flexibility is lost. Flexibility is really the key to the development of skilled mental calculators so it is important to keep this to the fore. It is much easier to teach a specific lesson about a specific mental computation strategy than to work with the possibility of a multitude of strategies. The teacher can focus on a single line of reasoning rather than have to cope with a variety of strategies all at once.

To help with the first approach, which is reliant on the use of student, generated strategies, we suggest the use of routines. The pedagogy of such an approach frees the teacher to focus on discussion rather than on transmitting information in the lesson structure. Teachers and students become familiar with the routine at the same time increasing the amount of time available to focus on strategies that come about due to discussion of methods by the children.

The position suggested in this paper is that rather than teach specific lessons about particular strategies children should explore and discuss a variety of strategies and adopt those that are suited to their needs at that particular time. The following mental mathematics activities have been provided to help teachers who wish to adopt a similar approach to developing mental strategies that relies on explanation and sharing of methods among the class. The aim of any session designed to develop mental strategies should be to develop flexibility in thinking by the children and for them to gain an insight into the structure and properties of number. Askew (1999) suggested that "Tasks that do not 'set ceilings' on the level of difficulty enable pupils to engage with the mathematics at a number of different levels of attainment" (p. 5). Any task given to a class should allow for participation by the whole class and at the same time match the mixed ability nature of the children present.

There are several formats contained in *Think Mathematically* (McIntosh, De Nardi & Swan, 1996) such as *Today's Number Is* and *How Did you Do it?* that encourage children to explore and discuss mental strategies. The *Today's Number Is* activity asks children to list all they know about a particular number. After children become familiar with the format of this type of activity the teacher can encourage children along particular paths.

The *How Did you Do It* activity involves presenting a calculation ($29 + 47$) to be performed mentally and then asking the children to explain how they went about

solving it. Note the use of a horizontal layout of the calculation if it is being shown to the class. This presentation allows for more open responses as children do not immediately equate the calculation with the traditional method.

A variation on the *How Did You do It?* theme used in the following activity.

1. How would you do it? – In your head, on paper or with a calculator.

A question is presented and each child decides the method they would most be inclined to use to solve it. Children are then asked to explain the method and why they chose it.

Another approach involves asking children to list a calculation they would perform in the head, on paper or with a calculator and to explain why they would do it in that particular form.

2. If I know ..., then I also know ...

Offer children a calculation for example $10 \times 5 = 50$. Then show that If I know 10×5 is 50 then I also know 9×5 , 11×5 , 5×5 , 10×50 10×0.5 and so on.

Present children with another calculation and ask them to decide what they also know and explain why they know these things (i.e. ask them to explain the connections). Ask them also to show how each calculation is related to the others.

3. I can see.

For example offer the sum 12×18 . Then tell the children

I can see $2 \times 6 \times 18$ and also

$$2 \times 6 \times 9 \times 2$$

$$4 \times 6 \times 9$$

$$4 \times 3 \times 2 \times 9$$

$$4 \times 3 \times 2 \times 3 \times 3 \text{ and so on.}$$

Ask them if some of the calculations above are easier to calculate than the original and to explain their reasons.

4. That's Easy!

Ask children to think of calculation that looks difficult but really is easy to do in the head. Have them explain why the calculation looks difficult but why for them it really is easy.

e.g. $3 \times 2 \times 7 \times 5 \times 5 \times 2$

An example of this activity could be as follows.

That looks hard because there are so many numbers.

Really it is easy because when you multiply it does not matter what order you do the multiplying so the question could be changed to look like this 2×5 (which is 10) multiplied by 2×5 or 10, 10×10 is 100. This only leaves the 3×7 part, which is 21, and this is multiplied by 100 to produce an answer of 2100.

5. Take it Easy

If you had one wish and could change one number in the following question which one would you change? Explain why.

$$17 \times 9$$

I would change the nine to ten because it makes it much easier to multiply.

How could you use 17×10 to help calculate 17×9 ?

By developing awareness of the method of calculation and the numbers involved, children are being helped to make sense of calculation. This awareness of calculation, sometimes referred to as *metacomputation*, is an important skill in making a sensible selection of not only calculation method, for example mental, written or calculator, but also calculation strategy.

The value of discussion

Clearly the most important aspect of any activity designed to improve knowledge of mental computation strategies is discussion. Jones (1988) noted that a

variety of mental methods stimulates conversation and can form the basis of instructive class or group discussion...A child has to think more carefully about his (sic) method to put it into words. Listening to a variety of approaches can inspire him to modify his own methods. The discussion of how methods are linked encourages him to think about the structure of number (p. 43).

Another benefit of discussing mental methods is the development of a mathematical vocabulary. As Jones continued:

The mathematical vocabulary needed to describe mental methods is extensive ...This use of appropriate vocabulary widens a child's conception of the range of situations that may lead to the use of a particular operation (p.43).

Encouraging talk by children about their methods allows a variety of ways to be demonstrated. It also keys further discussion about why a particular method was more suitable in this situation, how it works, and what other ways of thinking could be used.

Identified strategies

There are several lists of mental strategies (Rathmell, 1978; McIntosh, De Nardi & Swan, 1996). Some are more detailed than others and some use different terms to describe the same strategy but being able to give a strategy a specific name is not as important as understanding how and why it works. Children often adopt idiosyncratic methods of working, which may blend several different thinking strategies together. Teachers should not expect always to be able to categorise strategies under specific headings. Rather the teaching should focus on explanations by children of how they use the strategies. This can be used to determine the level of understanding they possess. The following list of mental strategies is neither

comprehensive nor exhaustive but simply provides an overview of the most common strategies used by children.

Addition and subtraction

Strategy	Example
Commutativity	$2 + 9$. $9 + 2$ is easier
Counting on or back	$9 + 2$, 9, 10, 11
Bridging ten	$8 + 5$: $8 + 2 = 10$, $10 + 3 = 13$
Doubles	$6 + 6 = 12$
Near Doubles	$6 + 7 = 12 + 1$
Changing subtraction to addition	$9 - 7$: $7 + ? = 9$

Multiplication and division

Commutativity	3×8 : $8 \times 3 = 24$
Skip counting	4×5 : 5, 10, 15, 20
Repeated addition	4×5 : 5 & 5 is 10 & 5 more is 15 and another 5 is 20
Splitting into parts	7×4 : 5×4 makes 20 and double 4 is 8 so the answer is 28
Convert to Multiplication	$18 \div 3$: 3×6 is 18
Repeated subtraction	$18 \div 3$: 18, 15, 12, 9, 6, 3, 0
Repeated addition	$18 \div 3$: 3, 6, 9, 12, 15, 18
Counting back	$6 \div 3$: 5, 4, 3, 2, 1, 0
Counting on	$6 \div 3$: 1, 2, 3, 4, 5, 6

The application of this style of mental mathematics develops children who are confident and competent in such situations. Generally, they have developed:

- a good bank of factual knowledge
- a wide range of mental strategies
- an ability to select from the range for appropriateness
- an ability to articulate their thinking
- an ability to answer quickly

Conclusion

Much of what is suggested in this paper is not new. French (1987) commented about the poor attitude children have toward mathematics and mental mathematics in particular:

The variety of methods that children and adults use in doing mental calculations is very great and discussion of these in the classroom is very valuable, not to produce a 'best method', but to encourage a flexible approach and make explicit the advantages and insights that come from considering alternatives (p. 39).

He also summed up the key thought when he suggested that the aim of developing mental strategies is to produce flexible thinkers. The ten quick questions approach of mental recall produces panic, fear and anxiety in many children and reduces flexibility of thinking. An approach to teaching mental computation whereby children are taught specific strategies which are practiced may not cause as much anxiety but still may reduce flexibility in thinking as children attempt to apply the teacher's strategy rather than their understood method. Developing mental strategies via discussion should help children gain more flexibility in their approach to solving problems and provide more insight into the properties of the number system. Children will also learn that there is more than one way to arrive at the solution to a problem.

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About the presenters

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Shaping Mathematical Ideas Visually

Steve Thornton

The well-known author of the Mathematical Recreations column in *Scientific American*, Martin Gardner, wrote that ‘a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance’ (Gardner 1973). However visual methods of problem solving or of illuminating mathematical results are all too rare occurrences in school mathematics. This paper argues that visual thinking should be an integral part of students’ mathematical experiences, and discusses its importance in developing algebraic understanding, in providing a powerful problem-solving tool, and in valuing a variety of learning styles. It includes examples of visual thinking from across the secondary school mathematics curriculum, and discusses some ways in which teachers can develop students’ capacity to think visually.

Why visual thinking?

Visual thinking has always been an important part of the thinking of mathematicians (Hadamard, 1945), but perhaps less so an integral part of school students’ mathematical experiences. It was the subject of some discussion in the mid 1980s, and again in the early 1990s as neuro-psychologists looked at the functioning of the brain. In the current educational climate there are at least three reasons to re-evaluate the role of visual thinking in school mathematics. The first is that the current trend that identifies mathematics with the study of patterns, together with the ready availability of hand-held technology that will easily develop a general rule for a given pattern, has the potential to devalue algebraic thinking. The second is that visualisation can often provide simple, elegant and powerful approaches to developing mathematical results and solving problems, in the process making connections between different areas of mathematics. The third is the importance of recognising and valuing different learning styles, and of helping students to develop a repertoire of techniques for looking at mathematical situations. This may well be a significant challenge to teachers of mathematics who, as successful students of mathematics at school and tertiary level, almost instinctively opt for a verbal-logical style of thinking, which may not always be the most effective in solving some mathematical problems.

Visual algebra: a study of patterns

Mathematics has been described as the study of patterns (Steen 1990). Nowhere is this emphasis on patterns more evident than in recent approaches to the learning of algebra. Texts and curriculum documents abound with examples of patterns involving matchsticks and various arrangements of dots or squares. The clear aim of these pattern-generalisation examples is to develop students’ algebraic thinking. Yet

there is a real danger that students may miss the point and fail to develop the generalised thinking these exercises attempt to develop.

Geometric patterns

I well remember a session on matchstick patterns at a previous conference. The presenter showed how he helped his students to observe and generalise a pattern of squares made from matchsticks (Figure 1).

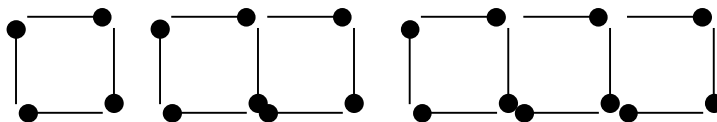


Figure 1: A matchstick pattern

As in most of these questions the students began by constructing a table of values (Table 1) showing the number of matchsticks required for varying numbers of squares.

Number of squares	1	2	3	
Number of matchsticks	4	7	10	

Table 1

They then adopted the problem-solving strategy of ‘look for a pattern’. Typically the students observed that the number of matchsticks required increased by 3 each time. As a well-meaning and helpful teacher, the presenter went on to explain how this observation could help students to induce that the relationship between the number of matches required and the number of squares to be constructed was linear, and of the form $3s + k$, where k was some number that made it work. The students then used substitution to find that $k = 1$ gave correct values for the number of matchsticks.

The advent of graphics calculators makes such a process even easier. It is now a very simple matter to generate a list showing the data collected, then plot a graph, and perform a regression to obtain the equation expressing the relationship.

In these inductive approaches the endpoint seems to be the development of an algebraic relationship, rather than the development of a sense of generality. Stacey (1989) identifies two problems with such an approach. The first is that of false proportionality where students see the construction as a whole and assume that, for example, the number of matchsticks required for the tenth pattern must be five times the number required for the second. The second, more subtle but perhaps more

insidious, is a focus on the recurrence relationship rather than the functional relationship. In such a pattern-spotting activity, algebra neither illuminates nor provides a means for validating the functional relationship generated (Noss, Healy & Hoyles, 1997). Furthermore there is a real danger that the very nature of the mathematical process itself may be misunderstood when numerical properties alone are used to construct general results.

... attention tends to become focused on the numeric attributes of the output. Worse still, school mathematics becomes constructed — by students and teachers alike — as a stereotypical data-driven ‘pattern-spotting’ activity in which it is acceptable to search for relationships by constructing tables of numeric data without appreciating any need to understand the structures underpinning them (Noss, Healy & Hoyles, 1997).

Rather than fixating on one variable and using some form of algorithm to generate a functional relationship, powerful algebraic thinking arises when students attach meaning to variables, and visualise the relationship in a number of different ways. The equivalence of several different algebraic expressions is an obvious outcome of such a visualisation activity.

Number patterns

My other vivid conference recollection is of a session at the recent National Council of Teachers of Mathematics Conference in Chicago, advertised as being about graphics calculators and number theory. The presenter had developed some ideas involving graphics calculators to help prospective secondary teachers to look at some ideas in number theory. He started with some number patterns involving square numbers and asked us to generalise the result:

$$1.3 + 1 = 4$$

$$2.4 + 1 = 9$$

$$3.5 + 1 = 16$$

He then asked us to prove the generalisation. As successful verbal-logical thinkers, the people in the group were quick to formulate an algebraic proof of the generalisation $n(n + 2) + 1 = (n + 1)^2$. However, to try to make the problem a little more mathematically interesting, I drew a dot picture representing $n(n + 2) + 1$, and showed how it could be rearranged into a pattern representing $(n + 1)^2$. The session leader asked whether participants felt that the dot picture was really a proof. Regrettably, many felt it did not, and believed that, for a proof to be valid, it had to conform to conventions of layout and notation. Eisenberg and Dreyfus (1991) report similar criticisms levelled at visual proofs during a session at the Sixth International Congress on Mathematics Education (ICME-6) in Budapest in 1988.

The mathematical power of visual thinking

If one accepts that the purpose of proof is to illuminate a mathematical result, then the dot picture certainly shows the result in a different light. It develops the

connection between the process of multiplication and the number of elements in a rectangular array, and shows that the array can be transformed by manipulating parts of it. To generate and understand such proofs the student needs to be able to see parts of the figure as entities in themselves that can be transformed and manipulated as necessary. In the dot picture proof, for example, students need to be able to take one row of dots and transform it into a column, in the process transforming a rectangle into a square with missing corner. The capacity to operate on mathematical entities as objects in their own right is the essence of algebraic thinking.

In his delightful book *Proof Without Words*, Nelsen (1990) provides some 100 visual proofs of results from number, algebra and geometry, some of which are also illustrated in my Dynamic Visual Algebra Web site. It is interesting that many of these proofs were originally published as space-fillers in the journal *Mathematical Magazine* and the *College Mathematics Journal*, which may, in itself, make a statement about their perceived mathematical significance.

Shear (1985) stresses the importance of studying trigonometric functions using visual methods on a unit circle, and of using visual as well as analytical methods to find elegant methods of solving problems. He uses as one example the proof of the identity $(\sec\theta - \cos\theta)^2 = \tan^2\theta - \sin^2\theta$. While this is not a difficult identity to verify analytically, it lends itself to a visual proof using a unit circle (Figure 2).

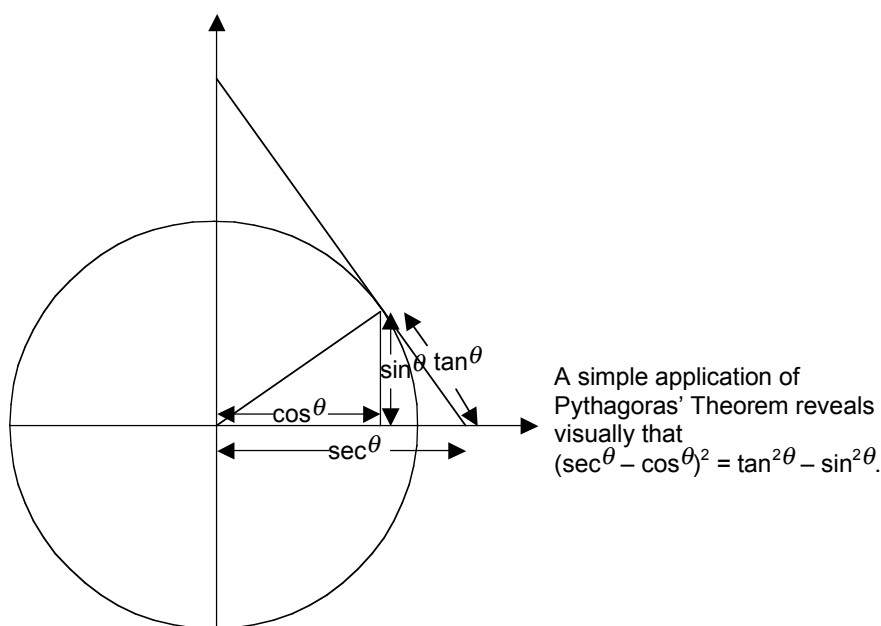


Figure 2: A visual approach to a trigonometric identity

The importance of visual thinking in mathematical discovery is graphically illustrated in the work of Hadamard (1945), himself a mathematician of some note, who surveyed many other mathematicians and scientists, asking them about their thought processes as they solved problems or investigated new ideas. He identified a

remarkable consistency in the way in which leading mathematicians used images to develop their thoughts, only resorting to more formal algebraic conventions when they wished to communicate their results with others. Among those interviewed by Hadamard were Henri Poincaré and Albert Einstein, who later wrote of their thought processes when engaged in mathematical and scientific discovery.

(Poincaré perceived) mathematical entities ...whose elements are harmoniously disposed so that the mind without effort can embrace the totality while realising their details (Poincaré, 1968).

(For Einstein,) words or...language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The physical entities which seem to serve as elements of thought are certain signs and more or less clear images...The above-mentioned elements are, in my case, of visual and some of muscular type (Einstein, 1979).

Valuing different learning styles

Krutetskii (1976) surveyed a large number of students aged 10 to 14, and identified two distinct types of problem solvers: verbal-logical and visual-pictorial. Verbal-logical thinkers had no need of diagrams, and attempted to solve all problems algebraically, even if a visual representation was available. On the other hand visual-pictorial thinkers tried to form a picture, even when it was unnecessary. Krutetskii further categorised some students as being harmonic thinkers, able to think both ways.

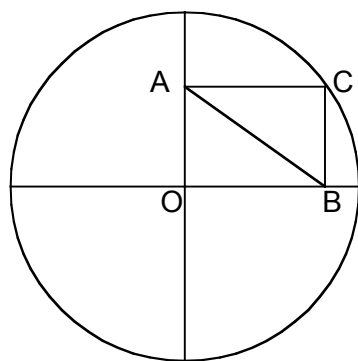
Presmeg (1992) identified five categories of imagery: concrete-pictorial imagery, pattern imagery, memory images of formulae, kinaesthetic imagery and dynamic imagery. She describes visualisation as being on a continuum from concrete to abstract, with pattern imagery and dynamic imagery requiring more abstract and conceptual thought processes than pictorial imagery.

Moses (1982) found that the problem-solving performance of fifth grade students improved significantly following a course in visualisation. She asked them to try to feel a part of the situation being considered, and to identify with the people or elements involved in the situation. She identified seeing, imagining and designing as three overlapping strategies that helped students to obtain a gestalt picture of the entire situation. The solutions developed through visualisation then made sense in the context of the original problem, rather than being the result of a formal, and potentially meaningless, mathematical operation. Campbell, Collis and Watson (1995) affirmed the role of concrete-pictorial imagery in motivating students, in helping them to clarify the structure of the problem, and in assessing the reasonableness of their results. This was particularly the case for students who struggled to understand mathematical concepts.

Presmeg (1992) described the importance of pattern imagery to expert chess players. When presented with an arrangement of chess pieces arising from an actual game, these experts were able to reproduce the situation from memory after a short exposure to the arrangement. Their performance was significantly better than non-

chess players. However, when presented with a random arrangement of chess pieces, experts were unable to reproduce the situation any better than non-chess players. The development of a similar mathematical imagery, in which students are able to focus on relationships and patterns, is surely one of the principal goals of mathematics education.

The larger the repertoire of strategies available to students, the more likely they are to be successful problem-solvers and to develop a deep understanding of mathematical concepts. Consider, for example, the problem illustrated in Figure 3, and used by Krutetskii (1976) as an evaluation item in his research.



AOBC is a rectangle, inscribed in one quadrant of the circle, centre O. The circle has diameter 2cm. Calculate the length of OC.

Figure 3: A problem from Krutetskii

Ira, a verbal-logical thinker, tried for a long time to solve the problem. She tried many different positions of AB and used analytical geometry to try to find a generalisation. It was only when the experimenter advised her to construct the radius OC that she found a simple visual solution.

In recent times a significant amount of research has been devoted to a study of how the brain functions. While historically such research was used to justify theories of innate ability differences between races and genders, it has great potential for informing the learning of mathematics. Sword (2000) describes the problems experienced by highly capable visual thinkers who may be 'at risk' in the school system because their learning style is not recognised. She maintains that traditional teaching techniques are designed for auditory-sequential learners, and hence disadvantage visual-spatial learners. Material introduced in a step by step manner, carefully graded from easy to difficult, with repetition to consolidate ideas, is not only unnecessary for the visual spatial learner, but, by failing to create links in a holistic picture, actively works against such students progressing to their potential. As a result gifted visual-spatial learners often exhibit characteristics such as lack of motivation, inattentiveness, weaknesses in basic calculations, and disorganisation.

Visual-spatial and verbal-logical learning styles are associated with 'right brain' versus 'left brain' thinking. The characteristics of these two learning styles are summarised in table 2.

Left brain thinking	Right brain thinking
Verbal	Visual-spatial
Analytical	Synthetic
Symbolic	Concrete-pictorial
Logical	Intuitive
Sequential	Multiple processing
Linear	Gestalt, holistic
Conceptual similarity	Structural similarity

Table 2. Left brain and right brain thinking (adapted from Tall 1991)

The common advice to students to read problems carefully, break them down into manageable steps, formulate algebraic expressions representing the situation, and then solve this formalised version of the original problem, could thus prove quite counter-productive for some students. Seldom do we ask our students to step back from the problem, to look at it holistically, and to try to visualise the situation in its entirety. As Leslie Hart vividly describes, the school curriculum is not designed for students who think in ‘right brain’ ways.

We have all been brainwashed by the undeserved respect given to Greek-type sequential logic. Almost automatically curriculum builders and teachers try to devise methods of instruction, assuming logical planning, ordering and presentation of content matter...They may have trouble conceiving alternative approaches that do not go step by step down a linear progression...It can be stated flatly, however, that the human brain is not organised or designed for linear, one-path thought (Hart, 1974).

Promoting visual thinking in the classroom

Zimmerman and Cunningham (1991) note that our use of the term visualisation in mathematics is not the same as the everyday use of the term. It does not equate to just forming a mental image. Rather it is about visualising a concept or problem more so than a physical situation. So the visualisation can be on paper, or using computer graphics. Nemirovsky and Noble (1997) describe visualisation as the means of travelling between external representations and the learner’s mind.

What is of crucial importance, then, is to promote flexibility of thinking, and to encourage students to look for the connections between alternative representations of mathematical entities. Noss (1997) described mathematical thought as being characterised by the capacity to move freely between the visual and the symbolic, the formal and the informal, the analytic and the perceptual and the rigorous and the intuitive. Brieske (1984) maintains that the transition from algebraic to geometric thinking and vice versa serves to significantly deepen students’ understanding of underlying concepts.

The following suggestions are intended to be starting points for teachers to help students to become more effective visual thinkers.

- Be sensitive to the possibility of finding visual solutions or representations of a given result.
- Encourage concrete-pictorial imagery by asking students to picture themselves as part of the situation.
- Encourage pattern imagery by connecting results in number and algebra with models such as area, length and arrays.
- Encourage dynamic imagery by using software such as *Cabri* or *Geometer's Sketchpad*. Noss (1995) describes the development of a dynamic algebra program in which students were asked to construct geometric patterns by focusing on their algebraic properties. My Dynamic Visual Algebra Web site provides a number of examples of dynamic imagery, which illustrate results from number and algebra.
- Promote discussion of alternative ways of thinking, and particularly of the transition from visual to symbolic.
- Encourage students to look at problems holistically instead of breaking them into parts.
- Draw three diagrams (a special case, a general case and a counter-example) when tackling geometric proofs. Ask why the result is true in the special case, whether it is still true in the more general case and why it is not true in the counter-example. Noss (1997) notes that diagrams have a tendency to take on a ritual character as mere appendages, particularly in geometric problems. The introduction of diagrams which illustrate a specific case and a counter-example can help focus attention on the key aspects of the geometric relationship, and make the diagram an integral part of the solution process. It may also help to avoid the pitfalls of metonymy (Presmeg 1992), in which students fail to recognise an object when it does not conform to their mental prototype, or in which students introduce extraneous properties by only considering a specific case.

Conclusion

Mathematical power involves the capacity to make connections, both between mathematical objects and concepts and between mathematics and the physical world. Visual thinking, whether in the form of concrete images, pattern images or dynamic images, has a key role to play in the development of students' mathematical power.

Visual thinking on the Internet

The outer angles of a polygon is a dynamic Java applet that beautifully illustrates the theorem that the exterior angles of a polygon sum to 360° .

<http://www.ies.co.jp/math/java/geo/gaikaku/gaikaku.html> (accessed 5 December 2000)

Galton's Board illustrates a binomial distribution via a Quincunx, in which marbles are dropped through an array of pegs.

<http://homes.dsl.nl/~berrie1/> (accessed 5 December 2000)

Proofs Without Words provides examples of visual proofs of numerical relationships.

<http://www.cut-the-knot.com/ctk/pww.html> (accessed 5 December 2000)

Virtual reality polyhedra allows the user to explore polyhedra from a variety of angles and to visualise what they would look like from inside.

<http://www.georgehart.com/virtual-polyhedra/vp.html> (accessed 5 December 2000)

My Dynamic Visual Algebra site illustrates many of the examples discussed in this paper, and others, using animated graphics.

<http://www.amt.canberra.edu.au/~sjt/dva.htm> (accessed 5 December 2000)

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About the presenter

Steve Thornton is Director of the Australian Mathematics Teacher Enrichment Project, a professional development program of the Australian Mathematics Trust. He has taught mathematics for over twenty years in both the government and private sector, most recently at Prince Alfred College in South Australia.

He is a committed conference 'junkie', having presented workshops or papers at almost every state conference of mathematics teachers during the past four years. He is putting his conference knowledge to good use in his role as chair of the committee responsible for organising this conference. His mathematics teaching interests include the development of algebraic thinking, particularly through visualisation, and mathematical humour as a stimulus for learning. He has recently been elected to the position of President-Elect of the Australian Association of Mathematics Teachers, a role he will take up on 1 February 2001.

What is the Value of Professional Journals for Shaping Teacher Development?

John Truran

In 1990 and 1996 the *Australian Mathematics Teacher* asked readers to assess its value for secondary school mathematics teachers. Neither of the results of these surveys has been publicly disseminated, but they have much to tell us about professional development. This paper summarises the results to underpin an analysis of the strengths and weaknesses of professional journals, and *AMT* in particular, in shaping teacher development — particularly relevance, applicability, and its links with research findings. This will be followed by a focused group discussion on how journal articles might have more classroom influence.

Teachers present themselves as professionals. One of the marks of a professional is a commitment to staying ‘up to date’ with new learning within an area of expertise. A standard way for new learning to reach practitioners is through professional journals. and the *Australian Mathematics Teacher (AMT)* has been an important medium of communication for Australian mathematics teachers, along with state journals and, more recently, *Australian Senior Mathematics Journal*, *Australian Primary Mathematics Classroom*, and *Mathematics Education Research Journal*. But there is a significant amount of evidence that these journals have little influence on most practising teachers.

Teachers do not use journals

In an extensive survey Jeffrey (1985) found that very few teachers even see *Set*, a research summary widely distributed by the Australian Council for Educational Research. In his survey only one *Set* article was being actually used, and that only because it was serendipitously relevant to a school’s plans at the time. Swinson (1993) found that Queensland teachers do not regularly read professional journals and that conferences ‘do not play a significant part in the professional life of mathematics teachers’, who saw ‘other teachers’ as their main source of new information. Haimes & Malone (1993) found that Western Australian teachers of that time largely rejected efforts to suggest appropriate methodologies for teaching new material and resisted using excessively detailed resources, preferring textbooks as their main source of support.

We have no systematic data for other states, but their situation must be much the same because teaching cultures across Australia are remarkably similar. Post-conference sales of the Proceedings of the very large Mathematical Association of Victoria conferences are negligible (David Tynan, pers. comm., Nov 1999). In

England a small survey of ten secondary schools and 60 teachers (Haggarty, 1992) found that only about 20% of the teachers subscribed to any mathematical association, and only two teachers had ever attended a full-scale conference. This is partial confirmation of the Australian results from a similar culture.

The increasingly common recent practice of publishing conference papers on the web has certainly increased accessibility, and the AAMT 1997 Conference site received about 20 visits a day for some three months after publication, though probably more than half of these were from non-Australian visitors (Tynan, pers. comm.) and nothing is known about what the visitors did with what they read.

Two surveys of *The Australian Mathematics Teacher*

The 1990 survey

In 1990 a survey of *AMT* was sent to 200 people, and completed by about 50% of them. The first 72 surveys returned were collated but not formally summarised and interpreted nor made public. The collation does not contain the actual questionnaire, which may be why it is confusing, and sometimes inconsistent. The sampling method is unknown, but 80% of respondents were secondary school teachers, with the majority holding positions of further responsibility.

The most striking aspect of the results is the great diversity of needs and interests which were listed. The most-read articles attracted only 19 readers, and at the other end five articles attracted only one reader each. The popular articles covered a variety of classroom topics, and also the section 'Diversions' as well as reviews of books and other new material. The least popular articles tended to be articles about mathematics education in general, wider uses of mathematics and the application of research to teaching. Respondents' lists of desired articles reflected strongly the direction of changes at the time — technology, probability and statistics, history of mathematics, applications and modelling, group work, problem solving, mixed ability teaching, and the National Curriculum.

These results are fairly predictable, but there are some interesting contradictions. Some articles which many readers saw as 'sources of professional information' were not in fact widely read, particularly those addressing curriculum or professional development. It was only 'sources of professional information' closely related to classroom practice (e.g., students' attitudes or drill and practice) which respondents tended to read.

Another contradiction is found in the 'Research for Teaching' section, mentioned above. Articles on research were more often seen as useful than actually read. Yet in the section where respondents were asked to indicate what results from research would be of interest to them, they listed a very wide set of possibilities, including classroom issues like the effectiveness of technology or the benefits of mixed ability classes as opposed to streamed ones. Eleven respondents specifically requested general summaries of research, albeit with a *caveat* that they should be practical and not esoteric. Similar comments about being practical appeared in other places as

well, and also comments on articles being presented in a way which made them appear attractive and accessible.

By the end of 1990 Jane Watson, acting on behalf of the Mathematics Education Research Group of Australasia (MERGA), had organised eight articles by leading researchers who summarised research findings about ideas as diverse as equity, parallelism, algebra and problem solving. It would be fair to say that the authors sometimes did not come to definitive conclusions, and were careful to claim no more than research findings would support. This may have led to respondents seeing the research as irrelevant. But the net result of the negative responses received was that the section was axed at the end of 1991, the semi-formal link between *AMT* and MERGA was lost, and the important issue of finding how research might have been made accessible to classroom teachers was shelved.

The respondents' general comments about *AMT* were positive, and when not, often a negative comment was balanced by a positive one on the same topic. There was little suggestion that readers saw the journal as being as irrelevant. This may well be because respondents were a self-selected group of the more committed teachers. But in spite of this general support, none of the respondents said anything about how or how often they actually used the articles in class.

For the purposes of this paper, the 1990 survey may be summarised as emphasising the importance of articles with practical application which were easy to read and not replete with theory, though there was no blanket objection to theory and research among most of the respondents. It indicated the issues which were seen as most relevant at the time, but found out very little about how the articles were actually being applied to the classroom. Unless this is found out, it will be impossible to answer fully the question being addressed by this paper.

The 1996 survey

During 1996, partly at my instigation as a member of the Editorial Board of *AMT*, a second, more carefully structured, survey was conducted to see the extent to which *AMT* was achieving its aims. A total of 29 people from all states and territories except Victoria agreed to fill out four detailed questionnaires commenting on each of the four issues of *AMT* produced during the year. There were 88 responses received, only 13 of them for the fourth issue. Every survey provided an opportunity for unstructured comments.

About half of the respondents were practising classroom teachers and most of the others were consultants or tertiary teachers of either mathematics or mathematics education. They read the journal principally to learn about teaching mathematics, rather than mathematics itself. About 15% saw the journal as something which they regularly passed on to others or a source of possible purchases of books or equipment, either for themselves or their institution's library. Another 65% occasionally passed on information they had read to others, but 20% rarely or never did. Some 33% occasionally used the journal as a guide for purchases, while about 55% rarely or never did. (Anderson, 1997)

Respondents were asked to rate the classroom applicability of every article on a scale from 1 to 5, with '1' indicating least satisfaction. This should provide a partial answer to the question of whether journal articles are being used in the classroom. However, for the vast majority of articles there was a full range of responses from 1 to 5 and every article received at least one '2'. This diversity may well have reflected the varying interests and responsibilities of the respondents. But although there were several very positive comments about articles, there was only one specific article which was actually reported as having been applied in the classroom although several articles from Vol. 52 No 2 were reported by a senior teacher in a secondary school to have been used by colleagues almost immediately. The article which was specifically mentioned (Lannen, 1996) was written by a curriculum consultant based in a country region and described a very simple 'Cat & Mouse' board game together with variations designed to encourage students to think about the nature of chance events and to illustrate for teachers some theoretical principles about good open-ended questioning. One secondary head of department tried the game with low-ability Year 8s, and reported that they 'seemed to enjoy it'. Another senior secondary teacher reported being a regular user of the game, and a third senior secondary teacher (the same one who commended Vol. 52 No 2) saw the article as very applicable.

So we may reasonably see the 'Cat & Mouse' game as a good example of what is seen as applicable in a classroom. Three senior teachers with some willingness to use such ideas in the classroom have commended it. Who were the respondents who did not see the game as applicable to their classrooms? There were only five people who gave a '2' or a '3' to this game. The '2' was given by a TAFE teacher, for whose environment the game may well not have been relevant; the '3' by two secondary classroom teachers, one curriculum consultant, and one primary education lecturer. Given that none of these saw the game as inapplicable, the most we can say is that it did not seem to fit strongly their style of teaching. And there is no reason why it should fit the needs or styles of everyone. Some variety of grading is to be expected, even for what several teachers saw as a particularly good game. Indeed, the level of diversity may be guessed at by three specific comments another article (Phillips, 1996) — 'something I can have my friends read', 'very interesting and commendable', 'boring and condescending'.

It is clear that there were some teachers who saw some of the content of each edition of the 1996 *AMT* as applicable in their classrooms, and that there was considerable enthusiasm about a small number of articles. This is a creditable achievement. But it still tells us very little about how many articles were actually used and how effective they were in enhancing students' learning.

Discussion

There is more which might be drawn out of these surveys, but I have chosen to focus on applicability, both potential and actual, and the influence of research on classroom practice. *AMT* seems to do quite well for producing potentially applicable material. Neither survey has told us much about how much is actually applied. Some teachers have expressed some interest in research findings, but they have

emphasised that it should be easily interpretable. I think that two comments are appropriate at this stage — one addresses what may be overlooked if research findings are neglected. The other is to make some comparisons between the teaching and medical professions.

Why might research be useful for teachers?

There is only space for one short example here to provide a partial answer to this question. Let us use a recent *AMT* article in which Quinn (2000), without any reference to any research of any sort, discusses how he has taught an important concept — the Law of Large Numbers. His article prompted a subsequent similar article (Fletcher, 2000). It is certainly appropriate to use *AMT* to report successful classroom practice, and Quinn helpfully discusses his attempts to address two common and important misconceptions about the Law of Large Numbers. One misconception is that students believe that empirical probabilities will be exactly equal to theoretical probabilities for a sufficiently large number of trials. This is a special case of what Kahneman & Tversky (1971) have called ‘a belief in the law of small numbers’ — a belief that two samples from the same population will resemble each other and the parent population more closely than theory predicts. Applications of this misconception are known as the ‘representativeness heuristic’ because a single sample is seen as being ‘representative’ of the whole population. This issue has been addressed by several researchers because it is known to be commonly used by adults when making important decisions (Tversky & Kahneman, 1974), and has also been found among pre-service teachers with strong mathematical background (Peard, 1992). Furthermore, Shaughnessy (1997), after careful research and experimentation, has suggested some ways of teaching which may help to eliminate the heuristic from students’ thinking. Quinn has suggested another which may well also be effective, but he presents less evidence for it than Shaughnessy does for his approach.

It seems to me that there is a need, which has not yet been fulfilled, for teachers to be provided with a readily accessible description of the heuristic (so that it may be readily identified when it crops up in the classroom) and also suggestions about a variety of methods which have been tried and found to work in certain circumstances. It also seems to me that Quinn and the *AMT* editors and referees have done teachers a disservice by not providing them with at least some ways of finding out more about the problem he is addressing. This is especially the case because Peard’s finding suggests that some senior teachers may well be unaware that they too use the heuristic.

Good teachers seem to use a variety of approaches. This is not surprising: their students are very diverse, and their problems often not easy to diagnose. Good teachers need to have a variety of proven approaches put in front of them so that they may select what is likely to work best with them and their classes. Such a situation is not dissimilar from that of a medical practitioner, and it seems to me to be constructive to compare the two professions.

Some comparisons between teaching and medicine

When we look at ways in which teaching and medicine provide in-service education for their practitioners, two differences seem to be especially relevant to this paper — the types of reference materials available to and used by practitioners, and the structure of the links between practitioners and researchers.

All medical practitioners receive journals as a consequence of their professional registration, to say nothing of the material which they receive from drug companies. Every fortnight journals like *Australian Medical Journal* bring reports of the latest findings in pure research, as well as broader discussions of aspects of clinical practice. Of course, doctors do not read everything they receive. But they regularly and frequently have put in front of them the expectation that they will keep up to date with research as much as possible. They also have available general handbooks of medicine which provide useful summaries in a readily accessible form.

On the other hand, teachers do not have to subscribe to professional associations, and, as we have already seen, many are largely uninfluenced by the material in professional journals. Such handbooks as do exist are very expensive, not easily accessible, and not always relevant. This has led to textbooks acquiring an authority for teachers which is not always justified by the quality of their content.

Medicine also has built-in mechanisms for in-service training which are used by all, and not merely by those who want to undertake further formal study. If a General practitioner (GP) is not sure about a condition — perhaps it is unusual, or requires more skills or equipment than he or she has — then there is a range of support services to call on. The principal one, of course, is that provided by specialists. Medical convention requires that the specialist reports back to the GP in writing, and frequently the management of the condition will be a joint undertaking between GP, specialist and patient. This structure provides a form of continuing education for GPs which can improve their own skills and efficiency. In my experience the two forms of medical practitioners usually operate as equals, where the narrower but deeper knowledge of ‘the expert’ is not seen as a reason for despising the wider but shallower knowledge of the GP. Nor is it seen as a reason for a GP to tell a specialist that he or she would not know what he or she was talking about, as classroom teachers sometimes say to university researchers.

The non-deterministic nature of medical treatment parallels that of pedagogic practice, but teachers usually work in far more difficult conditions and rarely have the luxury of dealing with only one student at a time. Experts in learning, psychology, course development, academic content, etc. are available, but there are few direct linkages on a person-to-person basis. Indeed, there is often a culture within schools which says that teachers should be able to handle all the problems which they encounter by themselves. In particular, although many parents seek outside help for their children, and presumably more would do so if they could afford it, there is little communication between classroom teachers and outside tutors. This contrasts with, for example, the common practice in medicine of inviting GPs to assist specialist surgeons at operations on their own patients.

None of this is trying to deny the great difficulties teachers work under today. But if teachers are to be seen as professionals, then they must conform with the behaviours and expectations which other professional groups expect of their members. Since it seems to be the case that journals are not currently having the influence on the classroom which we might reasonably hope, it is appropriate to discuss how this might be changed.

How might journals be made more effective?

In the Conference presentation I shall invite participants to consider this question, and suggest here some relevant supplementary questions which seem to arise out of the findings of the *AMT* surveys. It may well be possible to draw on members' experiences to prepare an article for *AMT* based on the answers to some of these questions.

- What type of articles are seen to be most applicable?
- How are journal articles actually applied to the classroom?
- What difficulties prevent teachers from implementing good ideas which they read in journals?
- What place do theory and research findings have in classroom practice?
- Are there needs which *AMT* could fulfil but is currently not doing so?
- What are the relative benefits and disadvantages of electronic journal production compared with traditional production?

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About the presenter

John Truran has worked as a secondary mathematics teacher, a tertiary statistics teacher, a tertiary lecturer in mathematics education, and as a private tutor. His research has investigated the teaching and learning of probability and statistics, and aspects of the history of science. He has written textbooks, has been on the editorial board of *The Australian Mathematics Teacher* and is currently Associate Editor of the *Statistical Education Research Newsletter*. He is deeply involved in the international stochastic learning research community, and is particularly concerned about linking research with practice in mathematics education.

WORKSHOPS

The Victorian Early Years Numeracy Strategy

Cathy Beesey and Kim Hamilton

This session focuses on the Early Years Numeracy Strategy and how the Early Numeracy Research Project (ENRP) and the Early Years Numeracy Materials will shape the future directions in Prep-Year 4 mathematics in Victoria. The key design elements of both the ENRP and the Victorian Early Years Numeracy Materials will be explored during this session. These include a structured classroom program, continuous monitoring and assessment, additional assistance for students requiring it, effective leadership and coordination, professional development with teachers working together in professional learning teams, and strong links between home and school.

For detailed information visit the following website:

<http://www.sofweb.vic.edu.au/eys/num/index.htm>

and follow the links to the various components of the Early Years Numeracy Strategy. Papers presented at the Early Years P–4 Conference in 2000, 1999, 1998 can be found on these web pages, including several on numeracy.

<http://www.sofweb.vic.edu.au/eys/conf/authNov.htm>

<http://www.sofweb.vic.edu.au/eys/pdf/proc99.pdf>

<http://www.sofweb.vic.edu.au/eys/pdf/proc98.pdf>

About the presenters

Cathy Beesey, Manager Numeracy Projects, Early and Middle Years of Schooling Branch, has had extensive experience in mathematics education as a primary teacher, consultant, author, lecturer in universities and now in her work with the Department of Education, Employment and Training, Victoria. The focus of her work has been on assessment, improving student learning through focused teaching and professional development.

Kim Hamilton is a Senior Project Officer for the Early and Middle Years of Schooling Branch. Her main responsibility is coordinating the Early and Middle Years of Schooling Conferences, including the Early Years Numeracy Conference in May 2001, she is also involved in the development of the Early Years Numeracy Materials.

Mental Computation: Shaping Our Children's Success in Mathematics

Janette Bobis

The ability of children to compute mentally is a far better indicator of their true competence for dealing with number than their ability to correctly apply rote learnt pencil and paper procedures. This paper examines the rationale for the current emphasis on mental computation in curricula and provides examples of activities and teaching strategies useful for shaping children's success in mental computation.

During interviews with Year 3 and 4 students to determine their mental computation strategies, I posed the problem $53 - 38$. One child, added 2 to 38 to make 40 and then added 13 more to reach 53. She then added 13 and 2 to get her final answer of 15. Another child started from 53 and counted back to 38, using his fingers and toes to keep track of the count, thus arriving at 15 also. A third child visualised a pencil and paper procedure. He remembered to trade a ten from 50 to allow 8 to be taken from 13 and then took 3 from 4 to get his final answer of 15.

While each of these children got the correct answer, a close examination of the strategies they used reveals significant differences between the students' strategies. The counting-back strategy adopted by the second child, is typical of many children applying simplistic mental strategies that work for less complex problems to a problem requiring more sophisticated strategies. While the strategy still worked, it was cumbersome and prone to error.

The third child simply visualised a pencil and paper procedure. While the procedure was done 'mentally' — without the assistance of an external calculating device—it is not an efficient mental strategy. The decision to use a mental image of an algorithmic procedure in such an inappropriate manner demonstrates the child's lack of number sense and lack of mental computation skills. Such a method is typically a consequence of emphasising 'pencil and paper procedures to the exclusion of other methods' (Australian Education Council, 1991, p.109) and demonstrates how some of us become bound by tedious algorithms even when mental computation is more efficient.

It is obvious that the first child used a more sophisticated mental computation strategy. The use of a variety of non-count-by-ones strategies such as this is not only typical of an individual efficient at mental computation, but of one who possesses a good sense of number.

Mental computation and number sense

Curricula documents emphasise the importance of number sense and consider mental computation to be a key feature of its successful development (Curriculum Corporation, 1991; National Council of Teachers of Mathematics, 2000). While mental computation and number sense are related, they are also different. This is illustrated by the varying degrees of number sense evident in the strategies described earlier in this paper. While all three students used some form of mental computation to solve $53 - 18$, the first student demonstrated a superior sense of number compared to the other two students. Hence, mental computation refers to computational procedures performed without the use of an external calculating device such as pencil and paper or a calculator. However, number sense is much broader in scope. A range of number concepts and processes have been identified as being integral to the development of number sense (National Council of Teachers of Mathematics, 2000). These include:

- the ability to decompose numbers naturally;
- use particular numbers like 100 or $1/2$ as referents;
- use numbers flexibly when mentally computing and estimating;
- judge the magnitude of numbers and reasonableness of results;
- find links between new information and previously acquired knowledge; and
- use relationships among numbers and arithmetic operations

The focus of the activities presented in this paper is on the development of efficient mental strategies that utilise children's knowledge of number relationships, particularly their ability to decompose numbers. The rationale for this emphasis is based on the premise that the more relationships a child 'sees', the more flexible they can be in their mental strategies. In addition, a heightened awareness of number relationships enables unfamiliar mathematics to make more sense.

It must be emphasised that simply drilling mental strategies or the concepts affiliated with the development of number sense is not enough. Children learn to apply more efficient strategies within a rich context of skills and knowledge. Together, they are able to build a deeper understanding of the relationships among numbers and operations. Therefore, to promote number sense we need to provide opportunities for students to explore number concepts and operations, patterns and relationships in interesting and meaningful ways.

Exploring number relationships

Ten-frame Target (K–2)

What you need

A set of ten frames, dot cards, dominoes or similar.

What you do

Place the ten frames face down. Children take turns selecting a frame and tell ‘everything’ they know about the number represented on the ten frame. For example, for the ten frame representing the number 8 (see Figure 2), a child might say: ‘8 is the same as 5 and 3; 6 and 2; 4, 2 and 2; and I need 2 more to make 10’ etc. Encourage children to ‘see’ as many different parts as possible.

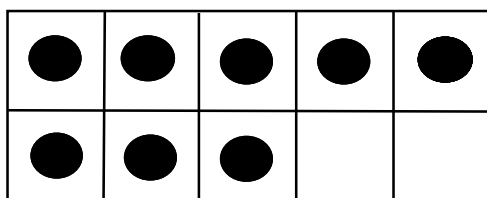


Figure 2. Ten frame for 8

Flip tiles to 5, 10 or 20 (K–2)**What you need**

Flip tiles (but coins or bread tags are suitable)

What you do

Select a target number that is appropriate to the child’s ability. Place that number of tiles, say 5, and place them in a paper cup. The child shakes the cup before tipping the contents onto the table. Ask children to record the number of each colour tile. Repeat this activity until various combinations of the target number have been recorded.

Extension (3–6)

For older children who still need practice with combining and partitioning strategies. Select an appropriate target number (beyond 20 it becomes too tedious) and ask the children to predict and then find the most frequent combination of numbers that make that number in a given number of throws.

Two-part numbers (K–2)**What you need**

Any type of dot card (dominoes can be used for extension)

What you do

Spread all the dot cards on the floor face down. Specify a target number appropriate to children's abilities. Have students work together to find pairs of cards (and later three or four) that total the target number.

Variation

Give each child one, two or three cards that total fewer dots than the target number. Each child must then race to find another card that will join with the initial card(s) to equal the target number.

Finding Connections

(McIntosh, Reys, Reys & Hope, 1997)

What you need

A game board or worksheet with about 15–20 numbers displayed (e.g. 600, 90, 120, 60, 800, 2, 20 etc.). The actual numbers on the board may vary depending on children's abilities. Transparent counters and a calculator.

What you do

Children take turns to find three numbers that are related in some way. Once they have found three numbers, they cover each number with a transparent counter. Transparent counters allow children to still see the numbers underneath while describing the relationship(s) between the numbers. For example, if a counter is placed over a 2, 60 and 120, a child might say that the numbers are related because $2 \times 60 = 120$ or because $120 \div 60 = 2$. The child's partner might need to check the relationships cited by the first child using the calculator to confirm that they are correct. If it is correct, the first child scores a point. The more relationships a child can say about the three numbers the more points they score for that turn.

Compatible numbers for 5, 10, 20, 50, 100 and 1000 (K–6)**What you need**

A game board/stencil containing numbers that are compatible randomly scattered around the board/page (e.g. compatible numbers for making 10 are 6 & 4, 5 & 5, 8 & 2; and compatible numbers for making 100 are 75 & 25, 60 & 40 etc.). Counters (transparent ones are more effective). Ten frames for children still developing part-whole knowledge to 5 or 10; and/or a calculator for children with more efficient strategies and can deal with numbers to 20, 50, 100 or 1000.

What you do

Children take turns to cover pairs of numbers that sum to 10 (or whatever the focus number is). Transparent counters enable students to keep track of pairs already

selected. Ten frames can be used by young children to visually reinforce the part-whole relationships. For example, if a child covers the numeral 7, they can use the ten-frame for 7 to help them find the remaining part to make 10. As children make a successful choice they score a point. The player with the most points when all the numbers are covered is the winner.

Extension

Children must find two or more numbers with a sum of 20 (50, 100 etc.).

Make 10 (K–3)

What you need

Two overhead projector blank ten frame, transparent counters.

What you do

Using the overhead projector, present the two ten frames. Construct representations for numbers using the transparent counters. Start by showing one 10 and other numbers. Discuss strategies for adding a number to 10. Then represent numbers such as 8 on one of the ten frames. Ask children to 'build-up' the ten frame with the highest number to make ten by physically manipulating counters from the other ten frame so that a 10 is now one of the numbers to be added (see Figure 3). Later, present ten-frames for say 7 and 4, but do not allow children to physically move the counters to make 10. Encourage visualisation of the moving counters. Eventually children will be able to visualise the entire process.

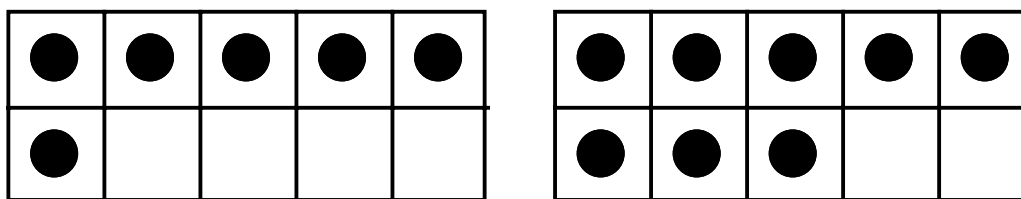


Figure 3. Take 2 dots from the 6 dot arrangement to make 10 with 4 remaining.

Break Apart (3–6)

What you do

List familiar table facts on the board and ask children for good ways to 'break apart' each fact. For example, 4×5 , might be broken apart to make 2×5 plus 2×5 . Move to unfamiliar facts, say 6×8 , and ask the same question. Children might answer: five 8s (40) and one more 8 (8) is 48.

For more advanced students encourage the breaking apart of more difficult computations, say: 16×25 . Children might respond 16 is 4×4 , so 4×25 is 100, $\times 4$ is 400.

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About the presenter

Janette Bobis was a primary school teacher for more than 8 years and is now a Senior Lecturer in Mathematics Education at the University of Sydney. She has worked closely with the NSW Department of Education and Training's early numeracy professional development project (Count Me In Too), having conducted formal evaluations of the project since 1996. Her recent research has focused on the professional development of teachers, school/university partnerships, and the development of number sense in young children.

Shaping Assessment for Effective Intervention

Rosemary Callingham and Patrick Griffin

Formal assessment in mathematics has traditionally focussed on achievement, providing summative information for reporting and grading purposes, or diagnosis, providing detailed information about an individual student. Both forms of assessment may be of limited value to teachers in planning their teaching programs.

Linking assessment to learning and teaching through the use of rich tasks, together with a carefully structured scoring rubric, in a way that allows teachers to target teaching for maximum effectiveness has the potential to transform teaching in an outcomes-based environment.

This paper outlines a process for developing rich mathematical tasks and scoring rubrics linked to a generalised learning sequence.

Introduction

Increasing emphasis on numeracy development with a focus on using ‘some mathematics to achieve some purpose in a particular context’ (AAMT 1997) requires a different approach to assessment. Tests of mathematical skills alone are not sufficient to provide teachers with the information they need to plan and implement appropriate and timely intervention for numeracy development. The mathematics curriculum, in line with the numeracy focus, is increasingly emphasising problem solving and mathematical applications. In order to assess this adequately we need to develop assessment tools that will allow us to make judgements about the kinds of strategies that students are using as well as the results of applying these; assessment tools that are sensitive to process as well as product (Clarke, 1988).

The most effective way to do this is through the person closest to the teaching and learning process — the classroom teacher. It has been argued for many years that assessment and teaching should be seamless, and that assessment should support learning (e.g. Wiggins, 1990; Pandey, 1990; Griffin & Nix, 1991; Shepard, 2000). Producing these tools is, however, more difficult than it seems. There are many texts devoted to assessment suggesting techniques as varied as observation, portfolios, learning logs or clinical interviews. All of these strategies have their place in a teacher’s assessment arsenal. None of these methods, however, is free of potential variation in interpretation and the challenge is to provide worthwhile learning tasks that can be used for assessment in consistent and equitable ways, and that assess what we need to assess in the 21st century. This requires a reshaping of our thinking about and approach to assessment (Griffin, 2000a).

Performance assessment, in which students create a product or produce a response that demonstrates their knowledge and skill (Airasian, 1994), has been advocated as

one way of addressing this problem. More recently, this has led to the idea of ‘rich tasks’ that allow for a range of approaches and outcomes, are motivating to students and are learning activities in themselves (Clarke & Clarke, 1999; Beesey et al., 1998).

Developing rich tasks

A rich task used for assessment can be almost any open-ended teaching activity that is used in the classroom (Griffin 2000a). It should match the kinds of activities that would be used in a teaching and learning unit to facilitate development, incorporating any usually available materials including manipulatives and technology.

There are three aspects to developing rich tasks. One is the task itself — the activity that the students will engage with. The second is the basis for judgements about the students’ performance on the task — the scoring scale or rubric. Finally the reporting process, whether to teacher, student or parent, should be considered. Each of these aspects needs to be considered when a new task is developed. Processes used to develop tasks that allow each of these considerations to be addressed are outlined below.

Step 1: Describe the underpinning knowledge and the levels of targeted concept development

The first step is to identify the intended underpinning knowledge and understanding, and the different levels of development of the target concepts likely to be found in the classroom. This step can be informed by national or state documents such as the Victorian Curriculum Standards Framework (CSF) (Board of Studies 1995) or the National Mathematics Profile (Curriculum Corporation 1994). This provides some information about the levels of development that could be expected. However, it should not be limited only to the expected levels. One feature of using these kinds of assessment activities is that students frequently surprise teachers with unexpected levels of understanding. The targeted development should, therefore, allow for a greater range of understanding than might normally be anticipated, at both ends of the scale.

Step 2: Develop a stimulus for the assessment and identify the task components

Will it be a take-home task, a classroom task, able to be done in groups and so on. What time allocation is required? What level of support can be provided? What materials are needed? How will students demonstrate their understanding?

These kinds of questions are the same as those required when planning a unit of work, emphasising the links between assessment and teaching. The task may be relatively unscaffolded, with few indications of how the students should approach the task. One example is the Jumper task from early childhood shown in figure 1 (Neal, D., Personal Communication, November 2, 2000). Alternatively, it may be quite structured with a series of sub-tasks that address particular aspects of the

underlying construct, such as the Street Party task shown in figure 2 (Callingham 1999; Callingham & Griffin 2000; Griffin 2000b).

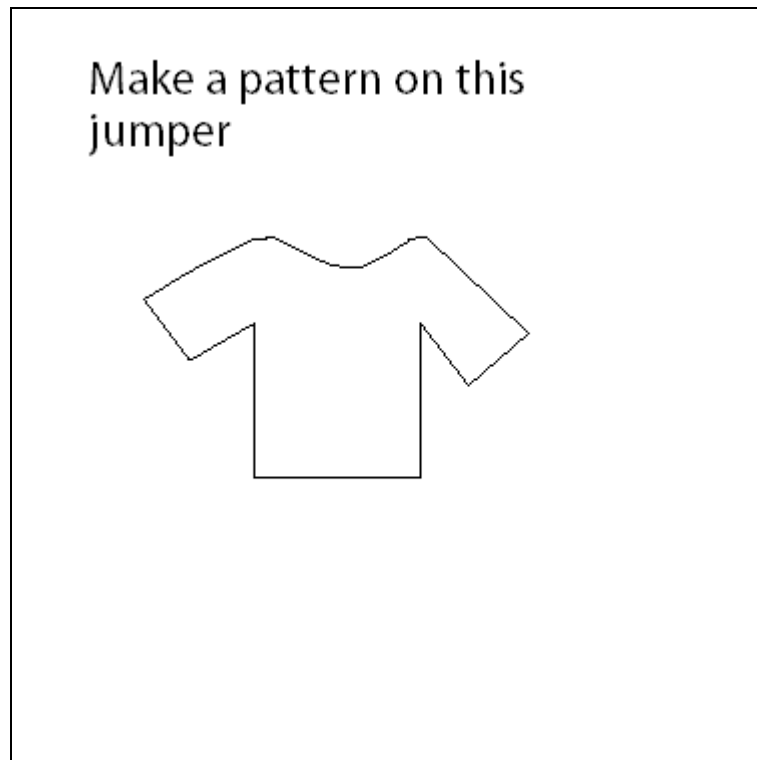


Figure 1: Rich task from an early childhood classroom

Dean's community is planning a street party to celebrate the year 2000. They have lots of small square tables. Each table seats 4 like this:



The community decides to put the tables in an end-to-end line along the street to make one big table.

1. Make a line with 2 tables. How many people will be able to sit at it?
2. Make a line of 4 tables. How many people will be able to sit at it?
3. Make a line of tables that would seat
 - (a.) 8 people.
 - (b.) 12 people.
 - (c.) 20 people.
4. Find two ways of showing all your results so far.

The community can borrow 99 tables.

- 5a. How many people could they seat?
- 5b. Explain how you got your answer and how your answer would change if the tables were arranged differently.
6. Make a rectangular shape for a small table.

Draw your small table showing the people sitting round it.
- 7a. Use your rectangular table to make some bigger tables by putting tables together. Draw three diagrams showing how your bigger table grows, and also show the number of people that can sit around it as it grows.
- 7b. Explain what happens to the number of people as your table grows bigger. Show your findings in at least two ways.
- 8a. How many of your rectangular tables using your arrangement would you need for 200 people?
- 8b. Explain how you got your answer and how your answer would change if the tables were arranged differently.
9. Find at least two rules to work out how many of your rectangular tables you would need for any number of people at the party.

Explain how each of your rules describes a relationship between a table arrangement and the people sitting around it.

10. Describe any general relationships you find between your rules.

Figure 2: Street Party task

Step 3: Anticipate different quality performances for each sub-task and define codes for these

This aspect is crucial to the development of these tasks. The different performances anticipated should recognise differences in quality, and should form a sequence of response. This should be based on the targeted construct, rather than extraneous information. In the examples given above the underlying concepts are those related to patterns and relationships, and the evidence required to judge the quality of the students' performances should address these substantive mathematical issues, rather than neatness or some other inappropriate attribute.

In the Jumper task, the targeted idea is that of element recognition and repetition. The anticipated responses of 5 year-olds to this task are shown in table 1. This projected performance provides the *scoring rubric*. Each rubric is then assigned a *code* that provides a numerical way of recording the responses.

Step	Rubric	Codes
Pattern creation	Pattern created contains repeated elements and is consistent across jumper (e.g. sleeves are symmetrical)	3
	Pattern created contains some repeated elements but is not consistent	2
	Different elements included but pattern does not repeat	1
	No recognition of pattern understanding, e.g. jumper is one colour only	0

Table 1: Coding performance quality on Jumper task

The Jumper task has only one step. This is appropriate with young children as they have difficulty following a series of complex instructions. However, the coding allows for four levels of quality of performance that indicates growing understanding of the underlying ideas.

In contrast, the Street Party task has a number of sub-tasks, each of which has a separate rubric. Examples of these rubrics, with the appropriate numerical code, are shown in table 2.

Step	Rubric	Codes
5a.	Correct answer for the arrangement shown	1
	All other answers	0
5b.	Detailed explanation includes symbols or equations that relate the table arrangement to the symbolic expression	4
	Explanation relies on relationships, e.g. the number of people is double the number of tables plus two for the ends	3
	Explanation relies on patterns, e.g. goes up by 2s	2
	Explanation relies on guess and check or lists	1
	No explanation or irrelevant	0

Table 2: Examples of coding performance quality for questions from Street Party

Each sub-task of Street Party has a separate coding scheme associated with it. Each code represents a level of quality of performance *within that sub-task*, rather than on

the task as a whole. This is a contrast to other scoring systems for rich tasks, which use holistic scoring systems based on responses to the whole task.

The principles underlying the development of analytical scoring rubrics of this type for rich tasks are summarised in figure 3.

<p>Rubrics should</p> <ol style="list-style-type: none"> 1. reflect the quality of performance of some cognitive, affective or psychomotor learning. 2. discriminate between levels of quality learning. 3. be developmental in nature so that each successive code implies a higher level of performance quality. 4. be descriptive and allow inferences to be made, rather than be a count of things right and wrong. 5. be based on anticipated responses and confirmed by an analysis of a variety of work samples. 6. be written in clear and unambiguous language that is easily understood by all stakeholders, including students. 7. consistently describe performance within the same developmental sequence. 8. lead to reliable and consistent judgements across judges.
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Figure 3: Principles for developing scoring rubrics (after Griffin 2000a)

One advantage of using a more detailed scale is that it allows for the placement of students on a profile of development rather than simply describing what they did on the particular task or sub-task. Relating this coding to a continuum of development and profile levels is the next step.

Step 4: Developing a continuum of development and profile levels

This is the process of combining all the information from each sub-task onto one scale, and is the most difficult part of the process.

A grid system allows for translation of the coding onto an overall sequence that addresses the underlying concepts. One such sequence that has been demonstrated as useful in assessing numeracy concepts (Callingham & Griffin 2000) is shown in table 3, with the Street Party and Jumper codes mapped onto it. The placement of each code is based initially on the teacher's judgement and may be confirmed by considering the actual responses provided by students.

Level	Street Party sub-tasks														Jumper sub-task	
	1	2	3a	3b	3c	4	5a	5b	6	7a	7b	8a	8b	9	10	
Hypothesis formation														5	2	
Relationship generalisation								4		4		4	4	1		
Rule extension								3		3		3	3			
Rule or process use						4		2		2	1	2	2			
Rule or process recognition						3	1	1		1		1	1			
Pattern or structure use			1	1	1	2									3	
Pattern or structure recognition		1				1				1					2	
Element identification	1									1					1	
No apparent understanding	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3: Generalised learning sequence showing Street Party and Jumper codes

In placing these two tasks on the same scale we are assuming that the tasks are in the same domain of cognitive development — that is that the same kind of thinking is required for pattern recognition in the Jumper task as the Street Party. This is justified if the same steps occur and the same kinds of quality of performance are anticipated for each task.

We can test this assumption by asking what would be the next logical development of the Jumper task on the learning sequence, and see whether this fits with the kind of thinking required for the next step on the Street Party task. From the learning sequence, the next step would appear to be asking children to explain the rules they have used to make their Jumper pattern. Various answers could be anticipated such as ‘it has blue and red stripes’; or ‘two red stripes, one blue stripe’ or, at a more sophisticated level, using a numerical pattern — ‘this is a 2-1 pattern because it has two red stripes and one blue one’. Children at this level could go on to recognise that this is the same pattern even when some attribute of the element changes such as ‘the pattern has two green stripes and one gold stripe but is the same as before’. This response could be coded at the higher performance level of rule use. This quality of thinking is similar to that required to gain a code of 2 in the Street Party question 5b shown in table 2. This suggests that the tasks are addressing the same domain and that the Jumper task could be extended in the same way as Street Party.

Despite the apparent difference between these tasks, the quality of students’ responses can be mapped onto the overall sequence that describes increasing sophistication of understanding of patterns and relationships. Young children who can make a pattern that contains consistent repeating elements are at a similar level of development in respect of understanding of pattern to older students who need to use concrete materials to make patterns of repeating elements to solve simple problems. The observable responses provide explicit evidence of the level of

cognitive functioning in a particular domain — in this instance one concerning patterns and relationships.

Why then are the responses at each level given a different numerical value? This is because these are *codes* only — not a traditional mark. The numerical value is for convenience, to show the development *within* each step of each sub-task. Through an analysis of the kind of thinking required to answer each step, such as that described above, each *code* is then placed on the overall scale. Only then can we look at the whole task and decide if it adequately addresses the whole of the domain that we want to describe. In Street Party, all the sub-tasks are contained within the one task, but other sub-tasks would be needed to combine with the information from the Jumper task if we were to be able to support fully the argument that it was measuring the same domain as Street Party.

Step 5: Placing a student on the continuum and interpreting the information

Sally's responses to the Street Party task were coded as shown in table 4.

Student	Street Party sub-tasks														
	1	2	3a	3b	3c	4	5a	5b	6	7a	7b	8a	8b	9	10
Sally	1	1	1	1	1	2	1	0	1	1	0	0	0	0	0

Table 4: Coded responses to Street Party

What does all this mean? Looking at the general sequence shown in table 3, Sally's responses are mainly at the Pattern Use level. She has managed to get a correct answer to question 5a but cannot explain how she did this (see table 2). She can do the easier questions of 6 and 7a, both of which require drawing diagrams of the small tables, but does not demonstrate any understanding in response to later questions that require more complex cognitive functioning.

Are there any advantages in going through what may seem to be a complicated process? One key benefit is that this process can provide both a description of what a student is ready to learn, and also a plan for action in developing the skills.

What did Sally's answers to the sub-tasks appear to demonstrate to be coded in the way that they were? The parts of Street Party sub-task 3 are all simple inverse questions that could be solved by recognising the underpinning pattern and using it consistently. Sub-task 4 asks students to record their findings in two ways and code 2 says that 'two systematic correct methods, e.g. diagram *and* table' have been used. The higher level codes for this question require explicit acknowledgement of a relationship, either in words (code 3) or symbols (code 4). Sally cannot as yet demonstrate these higher levels. She can create and draw a sequence of small tables showing the people sitting around them (sub-tasks 6 and 7a). The common theme is consistency of recognising a pattern, recording this systematically and using it in

straightforward related situations. Sally, however, cannot as yet express these ideas as rules or relationships. Overall this suggests that Sally is operating around the Pattern Use level of the sequence.

Students at this level need experiences that allow them to consolidate their understanding of pattern and opportunities to verbalise patterns and develop the mathematical language of pattern. They should be asked to make up rules for patterns, both of their own creation and those provided by the teacher. Repeating the Street Party task, or using a similar task after targeted teaching will allow a judgement to be made about the Sally's subsequent progress.

Thus, clustering the information gained from a rich task onto a common sequence, and identifying the common elements of each sub-task, allows interpretation of each level as the points where targeted teaching is likely to be most effective. The level and mode of that teaching can be determined by the nature of the common themes at each level.

Conclusion

Rich assessment tasks are appropriate at all levels of schooling. The nature and complexity of the task should be appropriate to the target group of students and allow for all levels of understanding to be demonstrated. In many instances teachers have reported that these 'assessment' tasks have led to further rich learning activities, further linking assessment and teaching.

This is different from the traditional approach to administering assessment and interpreting assessment information. Generally students are assumed to have mastered the thinking required and to have demonstrated this by getting the questions correct. The more questions that students get correct, the more they demonstrate their mastery of the subject. The approach presented here instead recognises quality of response. A student may, for example, make a simple arithmetical error and incorrectly answer a question such as, 'How many people could they seat?' If, however, that student can explain the reasoning used, this can be acknowledged, and the quality of that response, whether it is explained using diagrams, words or formulae for example, can provide information about the most productive approach to improving that student's understanding. This explicitly and intentionally makes the links between teaching, learning and assessment, and allows the full benefits of outcomes-based approaches to be realised.

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About the presenters

Rosemary Callingham has worked for the Department of Education in Tasmania in a variety of roles, including teaching maths in grades 7–10. Currently she is teaching pre-service primary and secondary maths teachers at the University of Tasmania.

Professor Patrick Griffin is Director of the Assessment Research Centre at Melbourne University.

Using Literacy Strategies in Mathematics Lessons

Leslie Casey

The workshop will discuss how we can Shape Australia through greater understanding in Mathematics. Many aspects of a mathematics lesson can be classified as literacy activities. Literacy activities help students make meaning from the world around them and shape our students' views of Mathematics. By working through examples of activities used in secondary classrooms and from NSW Syllabus Support Documents participants will be more confident in using literacy activities and in adapting their lessons to incorporate effective literacy activities.

Activity 1

Participants (as a group) to brainstorm and record as a mind map what constitutes a literacy aspect or consideration in Mathematics.

Activity 2

Some constructivist ideas presented and group to contribute how this might influence what teaching and learning activities are presented in a Mathematics lesson.

Activity 3

Looking at the book *Teaching Literacy in Mathematics in Year 7*, in particular pages 6–9; talking, listening, reading and writing in mathematics.

Consideration of the language of mathematics and looking at some activities which may be used to support students in this area:

- square-saws
- concept maps or mind maps
- definitions and where our words came from
- plurals
- be the expert and report to your group.

Activity 4

What are the different text types? Participants will complete a table of text types including the purpose, structure and language features of each text type.

When can they be used in Mathematics lessons and how do we do this?

Activity 5

The notation, conventions and language of Mathematics. Participants will brainstorm their understandings of conventions and common notations and how these might be taught to students in the middle years.

Modelling, joint construction, independent construction.

Activity 6

Reading factual text and recognising the mathematical concepts contained therein.

Strategies for teaching students to handle factual texts.

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About the presenter

Leslie Casey has worked for the NSW state system of education for more than twenty years. She spent nearly fifteen years in metropolitan high schools as a classroom teacher, then moved to rural NSW as a head teacher. As the Griffith District Mathematics Consultant she thoroughly enjoys working with students from Kindergarten to Year 12 and supporting staff with teaching numeracy and Mathematics. Leslie is an active member of the Riverina Mathematics Association. She is a Vice President and the editor of their journal. She has presented workshops at several of the RMA Annual Conferences. Leslie enjoys patchwork and quilting, with pieced geometrical quilts her favourite style. She is an active member of two quilting groups.

Practical Activities in the Middle School Classroom

Leslie Casey

'Mathematics – Shaping Australia' prompts one to think of space, geometry and measurement, with maybe a little problem solving mixed in as well. This workshop presents a series of practical activities which can be used in upper primary and junior secondary classes. Some activities will be new to all, but other activities are old favourites which may be new for some, or may be forgotten and not used by others. Participants will receive several practical but inexpensive ideas for activities which relate to the NSW Mathematics K–6 syllabus and the NSW Years 7 and 8 Mathematics syllabus. Some of these activities will use group work and problem solving strategies. Activities will include pentominoes, tetrominoes, tangrams, plane figure activities, straight-line patterns in circles and envelope curves.

Activity 1

Paper folding to produce cubes, activity from *Curriculum Ideas SM 14* from the NSW Board of Studies.

Activity 2

Tetrominoes and pentominoes — sheets developed by presenter for use in her own classes. Ideas will be given for production and storage of pentominoes packs.

Activity 3

Tangrams — presenter has drawn up activities which the participants will complete to gain ideas for managing tangrams within the class. Ideas for production and storage of tangram sets and the production of new tangram sheets will be demonstrated.

Tangrams Activities

This is an ancient Chinese puzzle which helps spatial understanding. It also involves problem solving and some calculations.

1. Take out all of the tangram pieces. How many are there? The Chinese called it the *Seven Board of Cunning*.
2. Can you make up a rule and sort the pieces into two groups using your rule? (Discuss this as a whole workshop group and find out the types of groups. This is a literacy aspect of Mathematics - stating a rule and then justifying it.)

3. Take all of the triangles. They form a group, but they are different sizes. What mathematical idea do these demonstrate? (Similar figures — they have the same shape but are different sizes).
4. Take, the two smallest triangles. Can you use them to completely cover the square?
5. Take the two smallest triangles. Can you use them to completely cover the parallelogram?
6. Take the two smallest triangles. Can you use them to completely cover the middle-sized triangle?
7. What can you say about the area of each of the square, parallelogram and medium triangle?
8. Can you cover the largest triangle using the two smallest triangles and the square?
9. Can you cover the largest triangle using the two smallest triangles and the parallelogram?
10. Can you cover the largest triangle using the two smallest triangles and the medium triangle?
11. Can you now compare the large triangle to the:
 - smallest triangle?
 - parallelogram?
 - medium triangle?
 - square?
12. Now looking at the card you have been given. Can you completely cover the shape using all of the tangram pieces? Show this completed puzzle to the presenter and receive another puzzle.

Activity 4

Colouring plane figures — students show their understanding of plane figures by colouring shapes. Two sheets drawn by the presenter will be discussed.

Activity 5

Counting triangles in a given figure.

Activity 6

Straight line patterns in circles

Just a couple of ideas on how the presenter has linked algebra and geometrical patterns or constructions.

Activity 7

Soma Cubes — some ideas on how to produce these and use them in the classroom.

Activity 8

Link a Cube

Found at the website

<http://www.nrich.maths.org.uk/primary/oct00/magazine.htm>

(accessed on 24/10/00)

These are my instructions written for use with a class. It does not have any graphics here, but diagrams are on the website.

Link a cube

This puzzle comes from Japan.

Steps:

1. Cut the three strips out very carefully.
2. Fold along the lines of each strip. Again do this carefully as the squares must be squares and not wonky quadrilaterals.
3. Take the strip with the two pictures on it and form it into a cube using sticky tape.
4. Take the strip with one picture and link it through the cube formed in Step 3. Using sticky tape form a second cube.
5. Take the strip with three pictures and link it through the cube formed in Step 3 and use sticky tape to form the third cube.
6. You should now have a chain of cubes, the first one has one picture, the second has two pictures and the third one has three pictures.
7. By turning and gently pushing the links inside each other, make a cube. Try to make a cube which has a picture showing on every face. Try to make a cube which has no picture on each face.

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About the presenter

Leslie Casey has worked for the NSW state system of education for more than twenty years. She spent nearly fifteen years in metropolitan high schools as a classroom

teacher, then moved to rural NSW as a head teacher. As the Griffith District Mathematics Consultant she thoroughly enjoys working with students from Kindergarten to Year 12 and supporting staff with teaching numeracy and Mathematics. Leslie is an active member of the Riverina Mathematics Association. She is a Vice President and the editor of their journal. She has presented workshops at several of the RMA Annual Conferences. Leslie enjoys patchwork and quilting, with pieced geometrical quilts her favourite style. She is an active member of two quilting groups.

How Much Algebra Do We Need to Teach Our Students?

David Driver

At Brisbane School of Distance Education, we have been using graphical calculators in years 11 and 12 extensively for 6 years. We have recently introduced them into year 10 for extension students. We are also trialling algebraic calculators in years 11 and 12 Mathematics B. This paper looks at what is being done to enable students with limited access to a teacher to develop skills in the use of both graphical and algebraic calculators. A tool is useless and potentially dangerous in the hands of someone who does not know how to use it. In mathematics, students need to know: what the calculator is doing; how to achieve the outcome of a particular mathematical procedure; and how to interpret the calculator's output.

The current Queensland senior secondary syllabuses in Mathematics B and C (the courses used as prerequisites for tertiary study in mathematics based faculties such as science and engineering) include a statement on the use of instruments in mathematics, which say, in part:

In order that the emphasis of the mathematics learned is on concepts and techniques rather than tedious and / or involved calculations it is appropriate that students are confident and competent in using a calculator and a computer. Other instruments encountered by students may include graphics calculators... It is important that students become competent in the appropriate selection and use of instruments [current internal syllabus].

or

In order that the emphasis of the mathematics learned is on concepts and techniques rather than tedious and / or involved calculations candidates should be confident and competent in using a graphing calculator ... It is important that students become competent in selecting and use instruments [current external syllabus].

or

A range of technological tools must be used in the learning and assessment experiences offered in this course. This ranges from, for example, pen and paper, measuring instruments and tables through to higher technologies such as graphing calculators and computers. The minimum level of higher technology appropriate for the teaching of this course is a graphing calculator [draft internal syllabus].

Our system of internal assessment allows individual schools considerable freedom to choose which advanced technologies they provide students to support their learning.

This ranges from the use of computers only, through the use of both computers and graphical calculators, the provision of class sets of graphical calculators, or individual student purchase of graphical calculators. The degree of sophistication of graphical calculators being used includes basic models (such as the Casio fx 7400); 'top of the range' graphical calculators such as the Texas Instruments TI-89, Hewlett Packard HP-39, Sharp 9650 or Casio cfx 9850PLUS; and algebraic calculators such as the Texas Instruments TI-89 and TI-92.

At Brisbane S.D.E. we have used graphical calculators (Sharp EL 9300 and Casio fx 7400) in years 11 & 12 extensively for 6 years. We have recently introduced the Casio fx 7400 into year 10 for extension students. We are also trialling algebraic calculators (Texas Instruments TI-89) in years 11 and 12 Mathematics B. Since many of our students, however, are school based (i.e. they studying only one or two subjects through distance education and their other subjects in a regular school) they may be using other makes and models of calculator, as determined by their base school.

A tool is useless and potentially dangerous in the hands of someone who does not know how to use it. In mathematics, students need to know: what the calculator is doing; how to achieve the outcome of a particular mathematical procedure; and how to interpret the calculator's output.

There has always existed a healthy scepticism to the introduction of new technology into mathematics classrooms.

In spite of this, tables of logarithms were replaced by scientific calculators. Scientific calculators are quickly being replaced by graphics calculators. I believe that in due course, graphics calculators will be replaced by algebraic calculators — hand held devices which include a computer algebra system.

Rather than sitting back and waiting for this to happen, as young teachers who have grown up with computers join the teaching force, we need to actively plan for the effective integration of this technology into our existing pedagogy.

One aspect of this planning is to determine what algebraic manipulation skills need to be retained and the level to which these skills should be developed by students if they are to subsequently use CAS capable calculators effectively; i.e. students know when why and how to use the technology and how to mentally check the calculator's output and interpret the output.

I do not envisage that lower secondary students will be using this technology in the foreseeable future. Upper secondary students, however, are already using it in a number of schools in Queensland. So the question is: given that lower secondary students may use this technology at some time in their future, how much algebra do they need to know now?

I do not purport to have any definitive answers to this question. However I have some experience in teaching students using this technology and have given some thought to the question.

A National Statement on Mathematics for Australian Schools (AEC 1990) lists the following outcomes for Algebra in the various bands.

Bands A /B

- AB1 use, verbal expressions (oral or written) to describe and summarise spatial or numerical patterns
- AB2 make and use arithmetic generalisations
- AB3 represent (verbally, graphically, in writing and physically) and interpret relationships between quantities
- AB4 construct and solve simple statements of equality between quantities

Band C

- C1 express a generalisation verbally (orally and in writing) and with standard algebraic conventions
- C2 generate elements of a pattern from a verbal or - algebraic expression of its rule
- C3 develop and apply algebraic identities involving the use of the distributive property of multiplication over addition and index laws for integral powers
- C4 manipulate, algebraic expressions for a purpose, making use of notational conventions in algebra, the distributive property of multiplication over addition, and inverses of addition and multiplication
- C5 identify variation in situations and use the idea of variable
- C6 draw freehand sketches of and interpret which model real phenomena qualitatively
- C7 use graphs to model real situations and make predictions including those based on interpolation, extrapolation, slope and turning points
- C8 recognise algebraic expressions of linear, reciprocal, quadratic and exponential functions and the graphs which represent them
- C9 use algebraic expressions (formulae) to model situations and make predictions based on the general characteristics of the formulae
- C10 formulate equations and inequalities in a range of situations
- C11 solve equations choosing an appropriate technique from 'guess, check and improve', successive approximation, graphical iteration, 'backtracking' and 'do the same thing to both sides', and interpret solutions in the original context
- C12 solve simple inequalities, choosing an appropriate technique from 'backtracking', graphing and properties of inequalities

Band D

- D1 express generalisations, functions and equations algebraically using one or more variables
- D2 manipulate algebraic expressions to generate more convenient forms
- D3 identify and express recursion and periodicity in various contexts

- D4 recognise and determine important features of families of functions
- D5 recognise different situations which can be modelled by the same function and fit curves to data sets
- D6 formulate equations and systems of equations and solve to appropriate levels of accuracy, making use of graphical, computational and analytical methods
- D7 determine feasible regions under sets of constraints

The Algebra menu of the TI-89 is illustrated in Figure 1, and the analysis following looks at how the TI-89 can be used to address the outcomes in *A National Statement on Mathematics for Australian Schools*.



Figure 1

National Statement Objective	Skill	Manual Manipulation Limit	TI
AB4 construct and solve simple statements of equality between quantities.	Collect like terms	$x^2 + 2x + 3 + 4x$	$\frac{2 \cdot x + 5 + 3 \cdot x - x}{2x+5+3x-x} \quad 4 \cdot x + 5$
	Simplify expressions	$3abc \times 2a \times 5b^2$	$\frac{2 \cdot y^3 \cdot \left(\frac{x}{y} + \frac{y}{x}\right)}{x^2 y^3 (x/y+y/x)} \cdot \left(x^2 + y^2\right) \cdot y^2$
C2 generate elements of a pattern from a verbal or - algebraic expression of its rule.	Substitution into a formula	Find I if $I = Prt$ and $P = \$250$; $r = 6\%$ and $t = 3$ years	$\frac{x^2 + 3 \cdot x + 5}{x^2+3x+5} \Big _{x=2.5} \quad 18.75$
C3 develop and apply algebraic identities involving the use of the distributive property of multiplication over addition and index laws for integral powers.	Expand brackets	$(x + 4)(x - 1)$ $(x + 4)(y + 3)$ $(3x + 5)^2$	$\frac{\text{expand}((x + 1) \cdot (x + 2) \cdot (x + 3))}{x^3 + 6 \cdot x^2 + 11 \cdot x + 6}$ $\frac{\text{expand}((x^2 - 9) \cdot (x^2 - 4 \cdot xy))}{x^4 - 4 \cdot xy \cdot x^2 - 9 \cdot x^2 + 36 \cdot xy}$ $\frac{\text{Expand}((x^2-9)(x^2-4xy))}{\text{factor}(x^2 - x - 12)}$ $\frac{\text{factor}(x^2 - x - 12)}{(x - 4) \cdot (x + 3)}$ $\frac{x^2 - 9}{x^2 - 5 \cdot x + 6} \cdot \frac{x^2 + x - 6}{x^2 - 4}$ $\frac{(x + 3)^2}{(x - 2) \cdot (x + 2)}$ $\frac{((x^2-9)/(x^2-5x+6)) * ((x^2-9)/(x^2-4))}{((x^2-9)/(x^2-5x+6)) * ((x^2-9)/(x^2-4))}$
	Factorise	$x^3 + 2x^2 + 3x$ $x^2 + 9x + 20$ $2x + 4y + ax + 2ay$ $9a^2 - 4b^2$ $\frac{x^2 - 4}{x^2 - 5x + 6}$	$\frac{5\sqrt{2} + 6\sqrt{8} + \sqrt{32}}{5\sqrt{6}(4\sqrt{2}) + 2\sqrt{3}}$
	Simplify expressions involving indices	$5\sqrt{2} + 6\sqrt{8} + \sqrt{32}$	$\frac{5\sqrt{6}(4\sqrt{2}) + 2\sqrt{3}}{40 \cdot \sqrt{3} + 30 \cdot \sqrt{2}}$

<p>C11 solve equations choosing an appropriate technique from 'guess, check and improve', successive approximation, graphical iteration, 'backtracking' and 'do the same thing to both sides', and interpret solutions in the original context.</p>	<p>Solve linear equations</p>	$3(x + 1) = x + 7$ $\frac{3m - 5}{7} = \frac{2 - 3m}{5}$	<p>■ solve(4 · x - 1 = x + 11, x) x = 4</p> <p>■ solve(4x - 1 = x + 11, x)</p> <p>■ solve(3 · x² - 5 · x = 2, x) x = 2 or x = -1/3</p> <p>■ solve(3 * x² - 5 * x = 2, x)</p>
	<p>Solve quadratic equations</p>	$3x^2 - 4 = x^2 + 2x$	<p>■ solve(x + 2 = 1 + 1/x, x) x = $\frac{-(\sqrt{5} + 1)}{2}$ or x = $\frac{\sqrt{5} - 1}{2}$</p> <p>■ solve(x + 2 = 1 + 1/x, x)</p>
	<p>Solve simultaneous equations</p>	$2x + 5y = 16$ $5x - 3y = 9$ $y = 2x + 1$ $y = x^2$	<p>■ solve(2 · x + 3 · y = 12 and 3 · x = 3 and y = 2 ... 12 and (3x - 5y = -1), (x, y))</p> <p>■ solve(y = x² - 4 and y = 2 · x = 3 and y = 5 or x = -1 a) ... ^2 - 4 and (y = 2x - 1), (x, y))</p>
<p>D2 manipulate algebraic expressions to generate more convenient forms</p>	<p>Transpose a formula</p>	<p>If $v = u + at$, find a formula for a in terms of v, u & t.</p>	<p>■ solve(t = 2 · π · $\sqrt{\frac{1}{g}}$, 1)</p> <p>1 = $\frac{g \cdot t^2}{4 \cdot \pi^2}$ and $t \geq 0$</p> <p>■ solve(t = 2 * π * sqrt(1/g), 1)</p>

Of course, the use of computer algebra makes many questions traditionally asked of lower secondary students pointless. The following examples are taken from a current, popular year 10 textbook.

1. Simplify these expressions:

a. $7t - 4t^3 + 6t \times 4t - 8t \times 6$

b. $\frac{7x^2 - 13x + 6}{9x^2 - 19x + 10}$

c. $\frac{a^2 - ab}{a + b} \times \frac{2a - b}{b^2 - ab} \div \frac{2a^2 - 5ab + 2b^2}{a^2 - 4b^2}$

d. $\frac{1}{a^2 - 1} - \frac{1}{a^2 + 2a - 3} - \frac{1}{(a^2 - 1)(a + 2)}$

e. $4\sqrt{63} + 3\sqrt{20} - 2\sqrt{28} + 3\sqrt{80}$

2. Evaluate the following:

a. $\frac{(a - b)^2 + 11}{cd + 2}$ (if $a = 10$, $b = 3$, $c = 5$ and $d = 2$)

b. $h(-3, 4, -5)$ (if $h(1, j, k) = 1 + 4(i - j)^2 + 2(j - k)^3$)

3. Simplify, expressing with positive indices $\left(\frac{m^4 n^{-2}}{p^2}\right)^3 \div \frac{(m^{-2} n^3)^5}{p^3}$

4. Expand $(8a - 5b)(7b - 2a)$

5. Factorise the following expressions:

a. $-2xy^2z - 3x^3y^2 + 6x^2yz$

b.

c. $4m^2np + 8mn^2 - 10n^2p$

d. $12a^2c(3b - 2)^2 + 4a(3b - 2)$

e. $36g^2 - 108g + 81$

f. $2(a + b)^2 - 2(a - 2b)^2$

g. $30m^2 - 31m - 44$

h. $6a^3 - 12a^2 - 4a + 8$

6. Transpose the formula $(R - a)^2 + c = d^2$ so that R is the subject

7. Find the roots of $\frac{2(x + 5)}{5} - \frac{3x + 1}{4} = \frac{7}{5}$

8. Solve $8(3 + x)^2 - 18(3 + x) + 9 = 0$

9. Solve the equation $5x^2 - 18x + 3 = 0$, using the quadratic formula. (Leave your answer in surd form.)

10. Solve the following pairs of simultaneous equations simultaneously:

a. $7x - 3y = 37$

$3x - 5y = 27$

b. $4x = 8 + y$

$y = 2x^2 - 4x$

Summary

My current belief is that we still need to teach all of the algebraic manipulation skills that we currently teach. However, we do not need to expect students to manipulate expressions as complicated as we may have done in the past. In most instances, these more difficult manipulation exercises were not realistic anyway.

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Polar Bears and Popcorn: Building Confidence in Low Achievers

Alison Flannery

Students are constantly put under pressure to achieve in mathematics, 'because they'll never get a good job without it.' They come to fear the fact that mathematics is shaping Australia. Participants will receive handouts about and experience a number of activities I have used in class. These activities were used to build up the confidence and self esteem of the students in a low level Year 10 maths class to the point where they enjoyed coming to maths classes and don't feel daunted by fact that mathematics will shape their future.

Activities

1. Strings

We had ploughed through all of our properties of polygons, and now I was faced with getting the students to revise and remember them for the exam. Thanks to Rosemary Veitch and her session at the local QAMT conference in Townsville this was an effective and interesting way to do it.

First I made four circles from pieces of string between 4 and 5 metres long. You need one per group of students. The ideal size of each group is 4, but you can manage with 3. A group of 5 has too many hands to do interesting things with the string.

Once each group had their string, I got them to make;

- a triangle
- an isosceles triangle
- an equilateral triangle
- a square
- a rectangle
- a parallelogram
- a tetrahedron
- a cube.

The most important part of the activity was going around to each group, when they said they had made the shape, and asking them to prove that it really was that shape. They really had to think about the rectangle (how did they know it was 90 degrees) and the parallelogram.

2. Popcorn

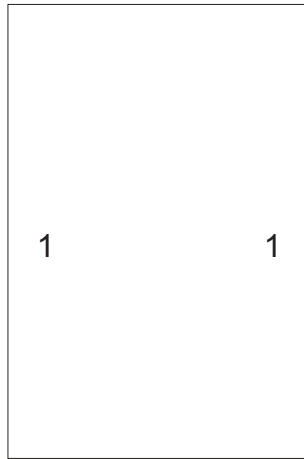
So many students cannot see any relationship between surface area and volume. This activity constructs various cylinders out of the same size piece of paper, and compares their volumes (worksheet attached).

Materials required; 5 sheets of A4 paper per group, scissors, sticky tape, popcorn (or substitute)

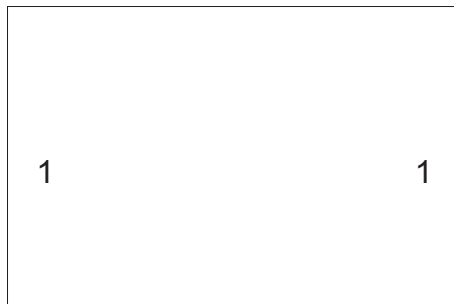
Divide students into groups of 4, and tell them each student in the group will make one of the following cylinders.

1. Use sticky tape to join the two edges of the sheet of paper, labelled 1, to make a cylinder open at both ends.

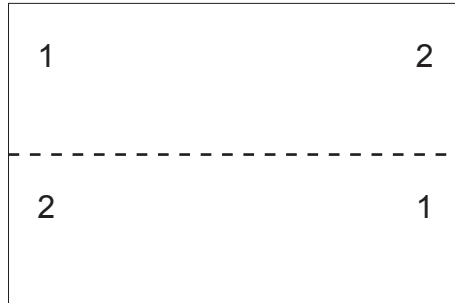
I found I had to tell my students not to have any overlap.



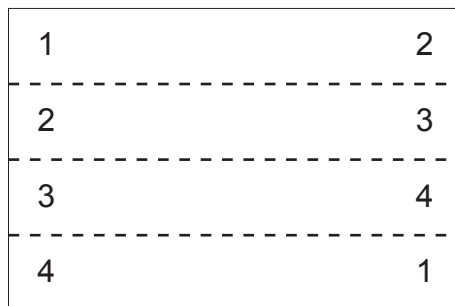
2. Use the tape to join the two edges marked 1, to make a cylinder open at both ends.



3. Take a sheet of paper and cut it along the dotted line. Join the edges marked 2, to make one long strip and then join the edges marked 1 to make a large short cylinder.



4. Take a sheet of paper and cut it along the dotted lines. Join the edges marked 2, then the edges marked 3, then the edges marked 4 to make one long strip of paper. Join the edges marked 1 to make a cylinder.



As a group answer the following questions.

Q1 Look at cylinders 1 and 2. Which cylinder do you think is bigger and why?

Q2 Put cylinder 1 inside cylinder 2, fill 1 with popcorn and then remove it, letting the popcorn fill up cylinder 2. Which cylinder, 1 or 2 is bigger. Estimate how much bigger.

Q3 Consider cylinders 2 and 3, and repeat the experiment, just with enough popcorn to fill cylinder 2. What fraction of 3 does the popcorn fill?

Q4 Comment on how much bigger 4 is than 3.

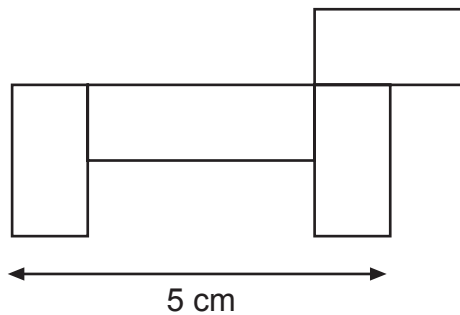
Q5 Can you explain your answers mathematically.

Polar Bears

A model of a 'dog' is made from 6 rectangular prisms, 5 of which are $1\text{ cm} \times 1\text{ cm} \times 2\text{ cm}$ (4 legs and a head) and 1 which is $1\text{ cm} \times 1\text{ cm} \times 5\text{ cm}$ (body). The Surface Area and Volume were calculated. We then calculated the ratio of SA to Volume. Groups of students were organised to construct enlargements of the original dog, and do similar calculations for these models. (See work sheet) We then discuss what the final model actually looks like, in the animal world and why they may be able to cope with or need the low SA to Volume ratio.

Materials: Lots of sheets of 1 cm graph paper, scissors, sticky tape, glue.

Take in a dog made out of rectangular prisms. The body is a $5\text{ cm} \times 1\text{ cm} \times 1\text{ cm}$ prism while the legs and head are $2\text{ cm} \times 1\text{ cm} \times 1\text{ cm}$ prisms (Cuisinaire rods are good for this).



This is a side view of the dog.

Divide the students into groups of 3 or 4. I had three groups of four and set them the following tasks.

Make an enlargement of the dog. Everyone is to help calculate the sizes of the nets required. Three people in the group are to make the scale model while the fourth calculates the volume and surface area. Write this in the table on the board.

All students are to fill in the following table from the board.

Group	Scale Factor	Surface Area	Volume	Surface Area/ Volume
1	$\times 1$	62 cm^2	15 cm^3	
2	$\times 2$			
3	$\times 3$			
4	$\times 4$			
5	$\times 5$			

Now answer the following questions.

Q1 What do you notice about the relationship between Surface Area and Volume as the scale factor increases.

Q2 The original model looks like a dog. What does the biggest model look like?

Q3 Can you suggest reasons why these two animals would need the SA to V ratio that they have?

Q4 Investigate the function of the largest organ in the body and explain why it needs this size.

About the presenter

Alison Flannery is the Head of Mathematics at The Cathedral School of St Anne and St James, an Anglican coeducational day and boarding school in Townsville, Queensland. She has been at the school for 13 years and spent 13 years prior to that teaching in the public school system in Queensland. As well as teaching she enjoys science fiction and fireworks.

The Rule of 3 Comes Alive! Graphic, Numeric, and Analytic Representations in Real Time

Mark Howell

Brand new software running on portable hand held devices allows students to interact directly with mathematical objects. Students and teachers can see immediately the consequences of their actions in graphic, numeric, and symbolic forms, affording amazing opportunities to engage students deeply in concepts without the overhead of a cryptic user interface. Connections abound in this session, which will illustrate uses of this new technology in algebra, geometry, and calculus!

Introduction

This session focusses on emerging hand-held technologies that support dynamic interactive learning of mathematics. New types of opportunities for dynamic exploration of algebraic, geometric, and calculus concepts are possible with an environment that supports direct interactions between the learner and mathematical objects.

When graphing calculator technology penetrated schools in the United States during the 1990s, educators were forced to re-examine content and pedagogy. Significant reform took place, particularly in the teaching of calculus. Nonetheless, technology continues to impose barriers that impede students and teachers alike from the task of exploring mathematics. Complex command syntax, designs that lose students in multiple layers of interface, and unrealistic static representations of mathematical objects all conspire to render immediate interactions between the learner and mathematical objects all but impossible.

As available computing power increases and, more importantly, as system interfaces are designed with the needs of the learner as the primary guides, exciting and innovative technologies should emerge that enhance the teaching and learning of mathematics.

The session will explore new learning paradigms facilitated by direct stylus interactions between the user/learner and graphical, numeric, and symbolic representations of functions and relations. The notion of representational plasticity, implemented as far as existing hardware permits, brings mathematics alive for students in ways that were previously not possible. We will take a look at software with a simple user interface, designed from the bottom up by schoolteachers to be a mathematical learning environment.

The Workbook

Math Xpander consists of several major operational environments. The one that ties them all together is the Workbook. In the Workbook, calculations are performed and visits to other major environments are recorded (see figure 1).

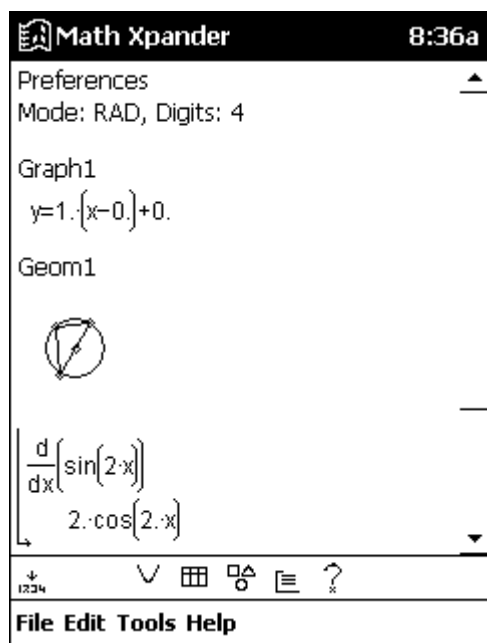


Figure 1. The Workbook.

There are currently two major interactive environments that afford interesting explorations: the Graph Xpander and Geometry Xpander. We will take a look at a few examples in each. An important part of the software design, however, is that an historic record is kept of all the interactions that occur during any number of visits to various environments. Users can save an entire workbook in the memory of the host device. Notes can also appear in the workbook to offer instruction or otherwise provide guidance to students.

Graph Xpander

Graph Xpander is a synthetic environment in three parts: symbolic, graphic, and numeric. Each occupies its own section of the screen, but the amount of space allocated to any one representation can be adjusted by movable “sashes” that divide the different areas (see figure 2).

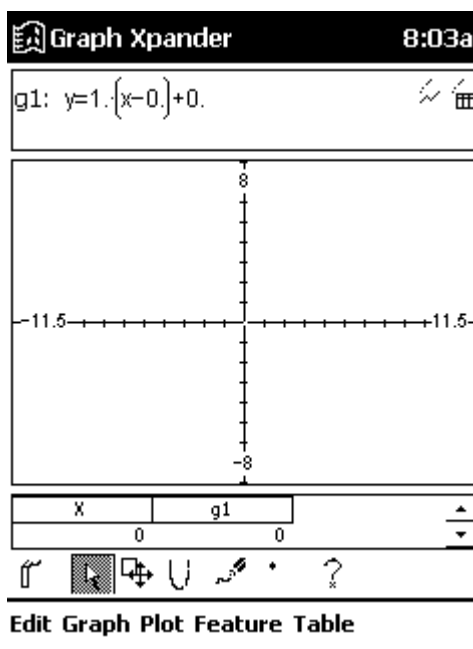


Figure 2. Graph Xpander layout.

Note the simple fact that all three representations are visible on the screen at the same time. More importantly, when looking at a particular relation, the underlying mathematical object is alive in the sense that the user can interact directly with it. Tools to translate and dilate relations using a stylus are available (see figure 3).

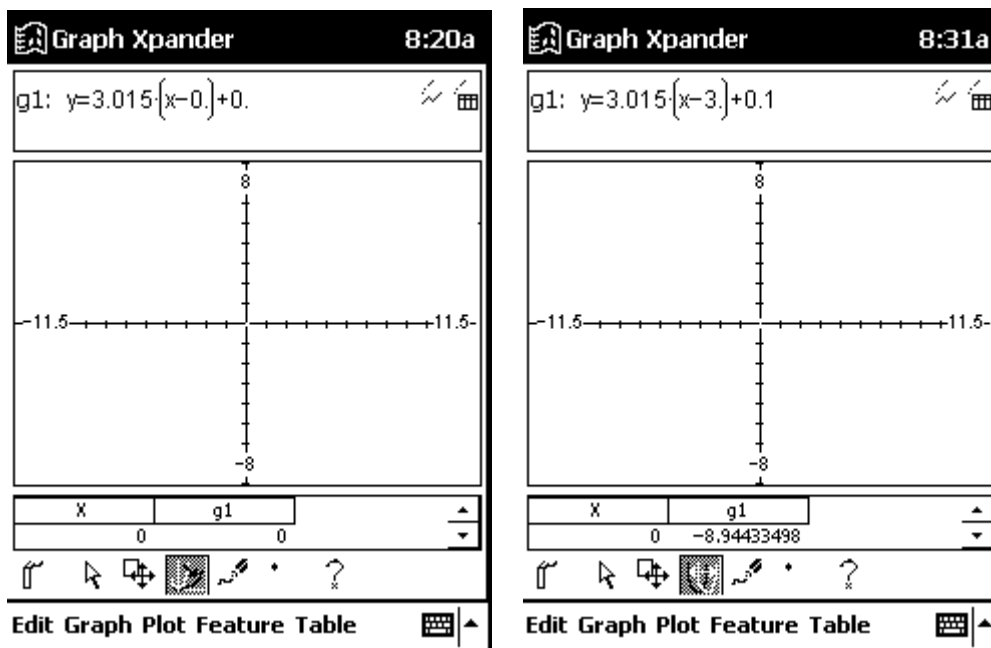


Figure 3. Dilating and translating a linear relation.

The user can of course edit the symbolic form of a relation directly, and adjust the viewing window by panning, zooming, or entering a new boundary value directly onto the graph screen. The table can be shifted by entering a new value into the independent variable column. In most cases, wherever user configurable output appears on the display, the user can input new data right there.

Another unique feature of the software allows users to sketch a graph and convert that sketch to a bona fide mathematical object, selecting from a library of standard relation types (see figure 4).

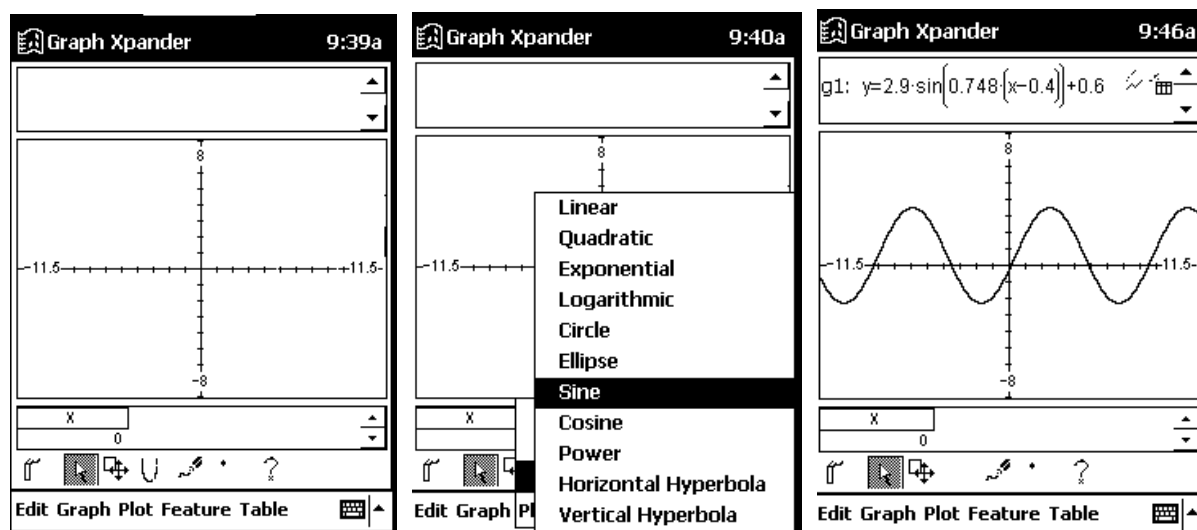


Figure 4. Fitting a sketch.

This sketch-fitting feature can be used together with some attribute constraints to investigate linear functions. We create two legs of a right triangle, a horizontal segment of length 5 and a vertical segment of length 3. Nominate one of the free endpoints to leave a point trail, and ask students to translate the construction so that the trail point follows through the path of the lead point. Then fit a linear function to the sketch to see how well the student did (see figure 5).

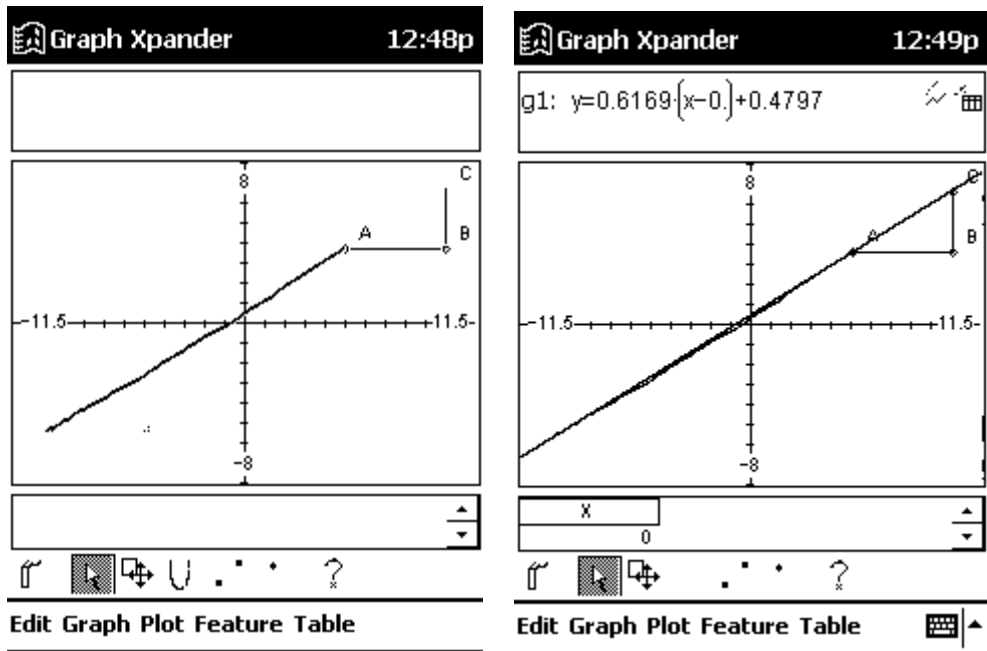


Figure 5. A slope investigation.

Another feature of the Graph Xpander environment allows the user to investigate the linked behaviour between a function and its derivative (see figure 6).

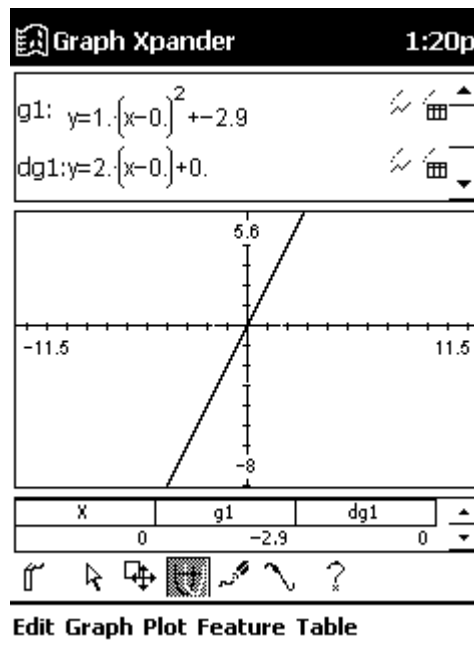


Figure 6. Function and derivative.

There are many instructive investigations for students to pursue in this rich scenario. Can I translate the original function in such a way that the derivative remains unchanged? What feature(s) of the two graphs always have the same relationship?

Is there a function you can sketch whose derivative is exactly the same as the original function? Is there a class of function you could transform somehow so that its derivative coincides with the function? What happens to the derivative of a sinusoid when it is dilated horizontally? Vertically?

Geometry Xpander

The geometry environment in the *Math Xpander* application is equally rich and interactive. In some ways, it is similar to *Geometer's SketchPad*. One major difference is that constructions in *Math Xpander* are constraint based rather than construction based. Particular constraints can be imposed and relaxed at will. In this way, students can observe how particular mathematical properties depend on hypotheses.

The geometry environment consists of a construction pane, a message area, and a numeric pane. The relative sizes of the construction and numeric panes can be adjusted with horizontal sashes, just as in Graph Xpander.

In this first example, a segment AB is drawn and its endpoints fixed. Segments AC and BC are constructed and constrained to be congruent. Then point C is translated. Of course, C must always lie on the perpendicular bisector of AB. We can nominate point C to leave a trail of its former locations behind as well (see figure 7).

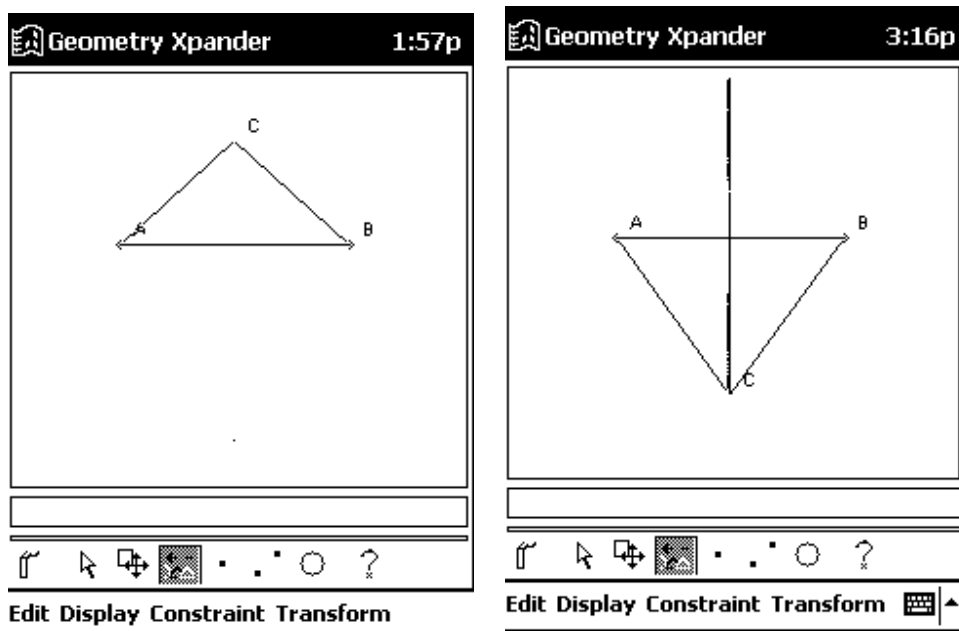


Figure 7. Congruent constraint with point trail.

Here is how to construct a parabola. Constrain point A to be at (0,1), constrain line BC to be horizontal along $y=-1$. Construct segment AD and DE, constrain them to be congruent, and then constrain point E to be on line BC. Last, constrain DE and BC to be perpendicular. Translate point E while leaving a point trail behind point D. Cool! Having created this point trail, it is converted to a sketch and can be copied and pasted into the analytic graph environment. There, you could fit a quadratic to this sketch (see Figure 8).

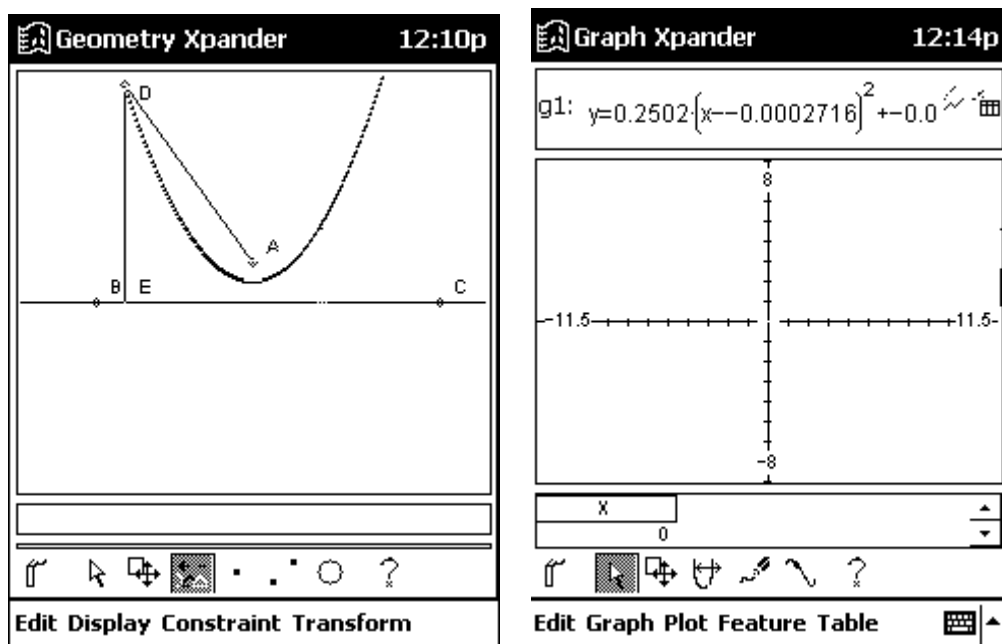


Figure 8. Constructing a parabola, and fitting a curve to the construction.

In the Geometry Xpander, you can create calculations based on the measurements of geometric objects. For example, you could create one calculation, called t_1 , given by the sum of the squares of two sides of a triangle. Then create a second calculation given by the square of the third side of the triangle. An activity for students in this setting could be to translate a vertex of the triangle so that $t_1=t_2$. You could constrain the first two sides to be perpendicular, transform the triangle somehow, and observe the calculations (see figure 9).

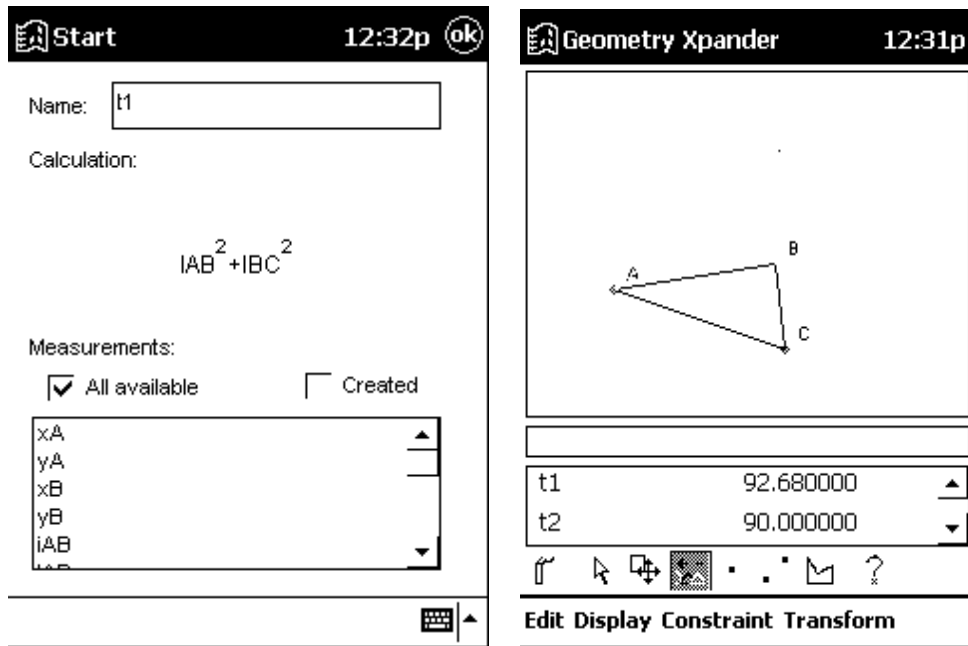
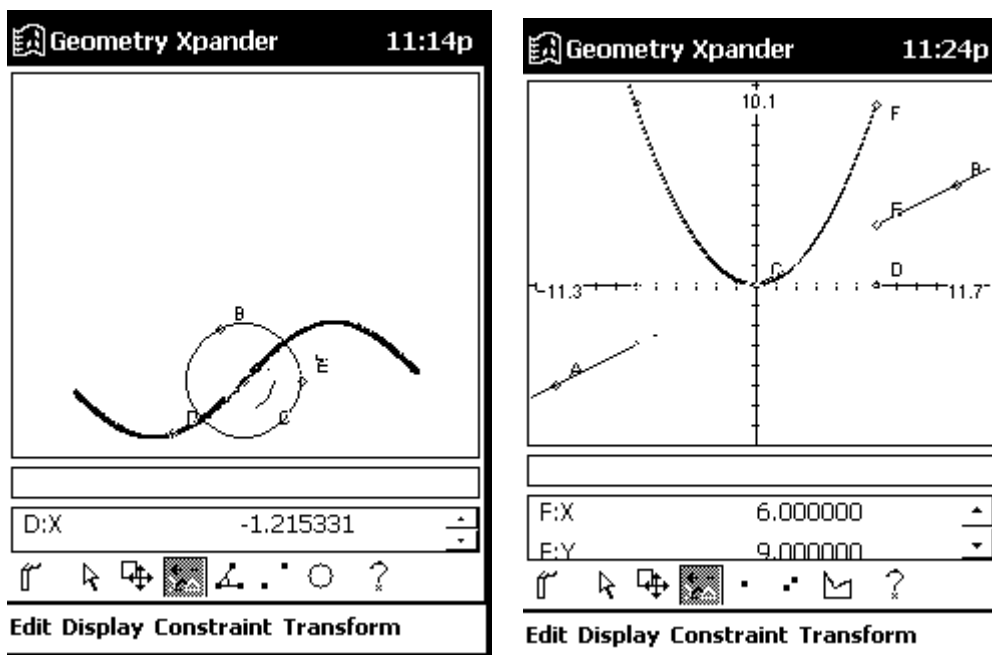


Figure 9. Calculations with triangle sides.

A powerful and unique feature of Math Xpander allows the user to specify a defining attribute of a geometric object by a formula. We could constrain a point's coordinates to follow the angle of inclination and y-coordinate of another point that lies on a unit circle. We could constrain a point's coordinates to represent the area under the graph of a line between two selected points. We could constrain the sum of the lengths of two segments to be a constant (see figure 10).



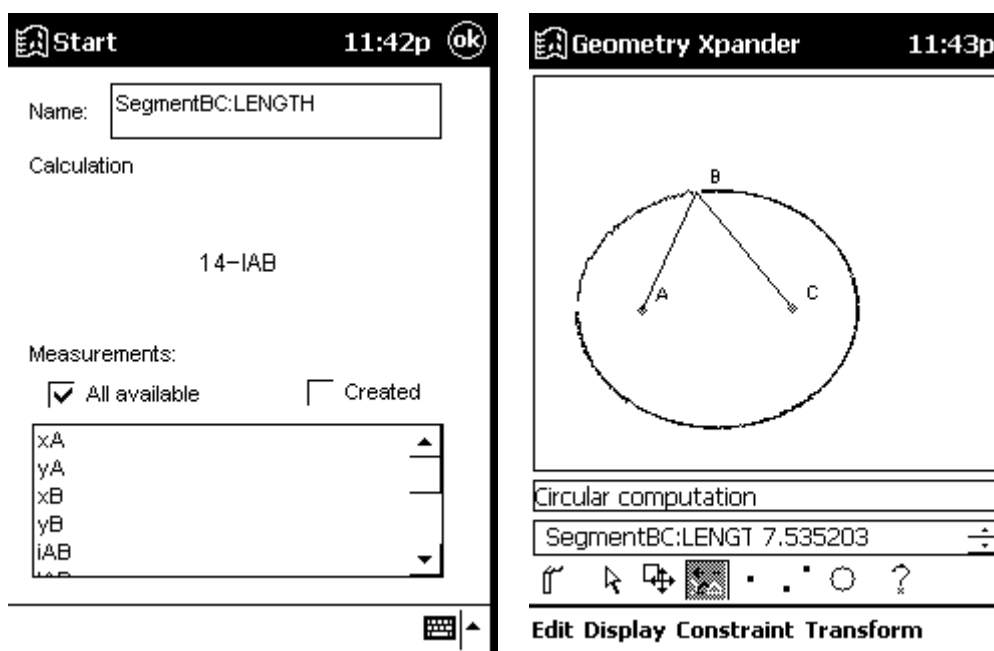


Figure 10. Dynamic attributes.

These examples offer only an appetizer. There is a feast of mathematical goodies on the table. This feature of *Math Xpander* has particularly significant implications for the teaching of analytic geometry.

Platforms and availability

The future of the *Math Xpander* software is uncertain. At the time of this writing, *Math Xpander* runs on a variety of Pocket PC devices, including the Compaq Aero 1550 and the HP Jornada 54x. Details regarding the availability of the software will be forthcoming.

About the presenter

Mark Howell received his BA in mathematics in 1976 and MAT in 1981, both from the University of Chicago. He taught Mathematics and Computer Science at Gonzaga High School in Washington, DC from 1977 through 1999. Mark has been active in the Advanced Placement Calculus program for many years, serving as a reader, table leader, and question leader at the AP Calculus Reading from 1989 to 2000. He served on the AP Calculus Test Development Committee from 1997 to 2000. Mark won the Presidential Award for Excellence in Science and mathematics Teaching for the District of Columbia in 1993, the Tandy Technology prize in 1999, and the Siemen's Award for Advanced Placement Teaching in 1999.

After spending the 1999-200 year on sabbatical teaching at Iolani School in Honolulu, Hawaii, Mark is currently on leave of absence from his teaching job and working as Curriculum Advisor with the Hewlett Packard Australian Calculator Operation.

Asia Counts — Studies of Asia and Numeracy

Jan Kiernan and Howard Reeves

This 90 minute workshop presentation will provide primary, middle school and secondary teacher participants with the opportunity to explore some of the activities developed by the presenters in a Curriculum Corporation project highlighting the numeracy content and processes in aspects of Studies of Asia. The activities are designed to shape students' knowledge, beliefs and understandings of Asia and Australia's position in the Asia region. Participants will be invited to engage in some typical classroom activities from the collection and examine how the materials support teachers by identifying and clarifying those aspects of the activities that make demands on, and enhance students' numeracy.

Introduction

Our interest in Studies of Asia and numeracy comes from two completely different but complementary perspectives. One is from the perspective of a Studies of Society and Environment teacher and curriculum officer acknowledging the importance 'mathematical' techniques to students being able to understand and describe their environment; the other from the perspective of a Mathematics teacher and curriculum officer who believes that there are numeracy demands on students in all their learning area experiences.

As with all learning areas, studies in the areas of society and environment demand a range of understandings, skills and competencies of students. Some of these demands are of a mathematical nature. Whether explicitly stated or implied, students are expected to draw on and use a variety of mathematical ideas, skills and techniques in order to undertake and fully engage in learning experiences in SOSE.

These numeracy demands are often not explicitly described nor intentionally planned for in the teaching program. In the context of Studies of Asia themes and perspectives, we have developed a collection of materials to provide opportunities for teachers to be explicit and intentional about the development and application of mathematical ideas and concepts that are fundamental to students becoming more numerate, while enriching their learning experiences in the SOSE. However, it is not intended that the mathematical ideas and concepts be developed 'exclusively' through SOSE activities. Introduction to, and development of the mathematical ideas are very much in the domain of mathematics classrooms.

The workshop for which this background paper has been written is based on the classroom materials developed by the authors and published by the Curriculum Corporation under the two titles *Asia Counts — Primary* and *Asia Counts — Secondary*.

What are the SOSE outcomes?

With respect to the SOSE focus, the tasks and classroom ideas presented in collections provide opportunities for students to explore issues and concepts about the countries of Asia and to investigate links between Australia and Asian countries.

Students have the opportunity to undertake a range of activities to develop generic skills which include:

- reading, viewing, listening and talking about a range of subject matter;
- selecting appropriate information from a range of sources and employing a range of strategies;
- developing skills for communicating ideas and discoveries in a variety of ways; and
- developing cooperation and participation skills through purposeful interactions with the teacher and other students.

These sound like Key Competencies!

In the language of the Key Competencies [Mayer, 1992] these generic skills relate to:

- collecting, analysing and organising information;
- communicating ideas and information;
- planning and organising activities;
- working with others in teams.

What are the Studies of Asia emphases?

From a SOSE perspective, students have the opportunity to explore issues with specific reference to the Studies of Asia emphases:

- Developing concepts of Asia
- Challenging stereotypes
- Contemporary issues
- World contributions by the peoples of Asia
- Likely implications of closer Asia-Australian relationships.

These opportunities assist students to develop cultural awareness and understanding of the countries and peoples of Asia.

What are the numeracy outcomes?

In being explicit and intentional about numeracy demands and outcomes, the materials provide opportunities for students further develop their

- sense of number;
- spatial sense;

- ability to handle data and interpret situations involving chance;
- knowledge and use of measurement in a variety of practical contexts;
- ability to recognise and describe patterns and trends leading to conjecturing and decision-making.

In other words, opportunities are presented for students to further develop and demonstrate their ability to use mathematical ideas and techniques to describe and make sense of their world.

The use of the word *sense* as in number sense and measurement sense is deliberate. The word is used to describe the mix of mathematical skills, knowledge, understanding, previous experience and intuition that an individual brings to a situation which makes demands on, or requires some numeracy competence to resolve the situation. With such *sense*, individuals often have the greater confidence and the disposition to bring quantitative techniques to situations which require or benefit from such action ... greater confidence and disposition than individuals simply possessing some mathematical skills.

The materials in *Asia Counts* aim to establish and support discussions between the teacher and students and enable a sharing of ideas, techniques and strategies. It is vital that the teacher shares his / her strategies for dealing with the quantitative and mathematical demands of the tasks and activities. Equally important is the opportunity for students to hear about the strategies and approaches of each other. Students' development of the techniques, development of confidence as well as disposition to use mathematical ideas and techniques in SOSE contexts should not be left to chance. Intent on the part of the teacher is critical and is supported by the numeracy notes which accompany each of the activities of the collections.

It is not the intention that the numeracy aspects of the collections of activities would replace the study of the related underpinning mathematical ideas in the mathematics curriculum. Rather, the activities should be seen to complement the mathematics curriculum by providing some relevant and rich contexts to reinforce students' mathematical and numeracy development. At the same time, there is 'value-adding' for the SOSE curriculum... data collection and interpretation techniques (data sense) can, for example, bring increased objectivity to decision making situations.

References

- Kiernan, J. and Reeves, H. (In print). *Asia Counts – Primary*. Carlton: Curriculum Corporation
- Kiernan, J. and Reeves, H. (In print). *Asia Counts – Secondary*. Carlton: Curriculum Corporation
- Australian Education Council (1992). *Key Competencies*. Canberra. AEC

About the presenters

Jan Kiernan is the Tasmanian State Advisor for the Asia Education Foundation. Coming from a teaching, SOSE curriculum and publishing background her involvement with the AEF has given her unique opportunities to further Studies of

Asia in schools. She is currently part of a unique partnership between the University of Tasmania and Flinders University that is developing and delivering post-graduate units in Studies of Asia using CD-ROM and the Internet. Jan is author of the *60°–170° East* kit and co-author, with Howard Reeves, of two books *Asia Counts Primary* and *Asia Counts Secondary*.

Howard Reeves is the Principal Education Officer (Mathematics) in Professional Learning Services Branch of the Department of Education, Tasmania, a position he has held since 1985. During this time he has been involved in a number of AAMT projects and activities and national Mathematics and numeracy initiatives. At the Tasmanian state level he has been involved in the writing of Departmental numeracy policy documents, background papers (*Numerate students – Numerate adults*) and outcome statements (*Key Intended Numeracy Outcomes*). He is a Past President of the Australian Association of Mathematics Teachers, a Life Member of the Mathematical Association of Tasmania.

Activities to Support the Shaping of Number Sense

Brian J. Lannen

This paper presents a 'grab-bag' of good number activities and Internet references to more of a similar nature. Teachers are encouraged to use these activities as part of carefully directed programs in the development of number sense. The differences between number skills, number sense and numeracy are discussed and some reference is also made to the national emphasis on shaping numeracy and the various numeracy programs that have been adopted by the States and Territories of Australia.

A shapely activity

'I'd like you all to hold your hands above your ears and pretend that they are rabbit's ears', the kindergarten teacher explained to his eager class of budding mathematicians. 'Now, see if you can hold up the right number of fingers to make the number *four*', he continued. 'OK, now bring them down and have a look to check. Did you make four? What about the person next to you? Did they make four? Did they make it the same way?'

Well, this is a neat activity that would probably shape up well to most kindergarten classes. But really, what is it shaping? According to the list of suggested activities that many New South Wales teachers now have as part of their *Count Me In Too* program support materials, the activity, called 'Rabbit's Ears', is one that may assist in moving a student from the *perceptual* counting stage to the *figurative* stage. The terms 'figurative' and 'perceptual' are taken here from the Learning Framework in Number (Wright, 1991), which is a key element of the *Count Me In Too* early numeracy program. However, the notion of using carefully selected activities (especially in the light of quality assessment data) to assist students along a learning framework, is not unique to this numeracy program. Each state education system across our nation now has some form of early numeracy program in place. As discussion continues as to how numeracy is actually more than just number skills and number sense, it is hoped that the number-rich activities presented here may be effectively transplanted into classrooms across the country to help students progress along their numeracy learning path — however it may be defined.

Shaping numeracy programs across the nation

The report from the 1997 Federally funded conference on Numeracy Education Strategy Development, *Numeracy = Everyone's Business* (AAMT, 1997), lists nineteen research or developmental programs in numeracy across the country. The complete report is available on the AAMT website at (http://www.aamt.edu/AAMT/Attic_3.html). A set of seven source papers can also be found at <http://www.aamt.edu.au/AAMT/ctxintro.html>.

The main early numeracy programs that have been adopted by education systems across the nation are summarised in Table 1.

State/Territory	Program	Web reference
ACT	Count Me In Too (and others)	http://www.decs.act.gov.au
NSW	Count Me In Too	http://www.oten.edu.au/countmein
Northern Territory	Numeracy in Schools, Numeracy in the Early Years, First Steps, Maths? No Fear!	http://www.ntde.nt.gov.au
Queensland	Support a Maths Learner	http://education.qld.gov.au/learning_ent/ldf/schools/maths/index.html
Tasmania	Flying Start	http://www.discover.tased.edu.au/literacy/flyingstart/default.asp
South Australia	The Early Years Strategy	http://www.learnsa.net/learnsa
Victoria	Early Years Numeracy	http://www.sofweb.vic.edu.au/eys/num/numclass.htm
Western Australia	First Steps Mathematics	http://www.eddept.wa.au/centoff/ece/inn2.htm

Table 1. Numeracy programs across Australia.

Common to all of these programs is the idea of helping students to progress along a developmental framework. Indeed many early numeracy frameworks could be viewed as a fine focus amplification of the more general curriculum framework, whether that general framework be called a Curriculum Standards Framework, Number Developmental Continuum, Benchmarks, Staged Outcomes Statements, or whatever. One ingredient is essential to all of it, and that is teachers in classrooms facilitating appropriately targeted, rich learning activities, most often against the background of well informed assessment knowledge.

Figure 1 shows the foreground of classroom activities set against the background of assessment data, which in turn have been gathered against key stages of a developmental framework.

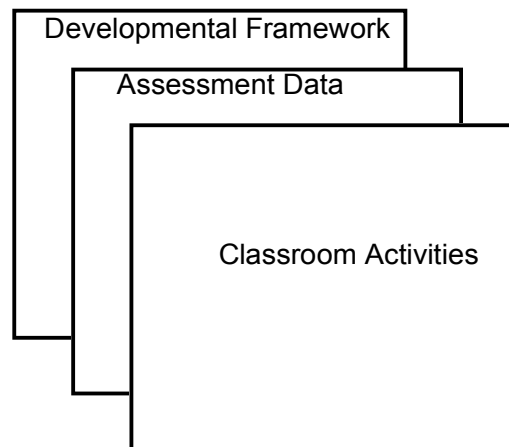


Figure 1. Connections between learning framework, assessment data and learning activities.

Some comment on government policy on numeracy education is made under the heading of 'Policies that count' in *Mathematics at home with a three-year old* (Lannen, 2000). The following goals agreed between all of the State and Territory Education Ministers are cited in *Numeracy = Everyone's Business* (AAMT, 1997):

That every child leaving primary school be numerate, and be able to read, write and spell at an appropriate level.

That every child commencing school from 1998 will achieve a minimum acceptable literacy and numeracy standard within four years (p. 3).

Yet the report also acknowledges,

Whilst the Commonwealth government and most state and territory authorities have undertaken substantial work in the area of literacy, attention to numeracy as an educational issue has been much more recent (p. 7).

To some degree this is because 'numeracy' as a defined area of educational development is relatively new and more involved than 'mathematics' education. That is not to stop us, however, from taking tried and proven number-rich learning activities and using them to help support the grander goals of numeracy development.

Number skills, number sense and numeracy

The classroom activities selected for presentation in this paper are probably best described as those that help shape students' number sense. Many of them involve the practise and development of number skills, but there is a richness that takes them further.

Mathematics = Everyone's Business (AAMT, 1997) describes number sense as incorporating 'both an ability to use numbers and an appreciation of number and number relationships.' (p. 11). Numeracy, it seems, is a quality of still higher order and the report defines 'mathematical aspects of numeracy' as including 'data sense', 'spatial sense' and 'formula sense' in addition to number sense. (p. 11).

All this is not to say that we should abandon our teaching of number skills. To the contrary, number skills form an essential part of number sense, which in turn forms an essential part of numeracy and the general intellectual demands of our society. The nested nature of these elements is illustrated in Figure 2.

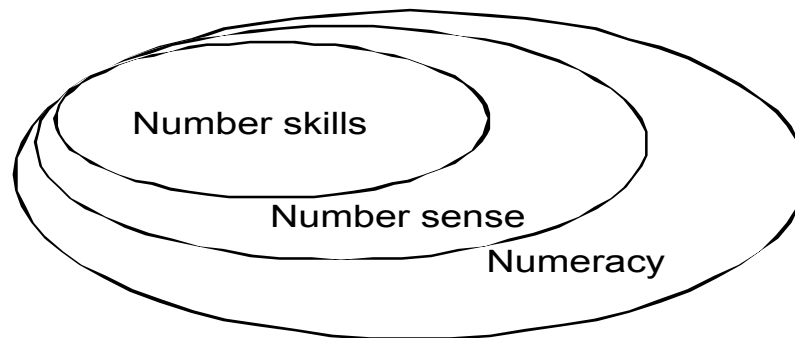


Figure 2. Nested number domains of development.

Swan and Sparrow (MAV, 1999) make a particular point in their work on calculator activities to define the difference between number skills and number sense. They assert that non-sensible use of calculators take away an element children's thinking about the numbers involved in the activity, yet carefully directed calculator activities will actually encourage students to explore number relationships and thus build their shaping of number sense. 'Using calculators more, in the described ways, children will develop number sense, and, in fact, need to use the calculator less for everyday calculations' (p. 364).

Swan and Sparrow refer to number sense as having 'an underlying notion that children will be able to make sense out of numbers and interactions with them' (p. 359). They also cite a definition by McIntosh, Reys and Reys (1997), who referred to number sense as:

a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful and efficient strategies for managing numerical situations (p. 358).

Number *sense* is then clearly more than just number *skills* and a significant step towards the greater goal of numeracy.

There has also been much discussion concerning the cross-curricular nature of numeracy, and its relationships to school mathematics and literacy:

Numeracy is not a synonym for school mathematics, but the two are clearly interrelated. All numeracy is underpinned by some mathematics; hence school mathematics has an important role in the development of young people's numeracy (AAMT, 1997. p. 11).

Further, while knowledge of mathematics is necessary of numeracy, having that knowledge is not in itself sufficient to ensure that learners become numerate. An

immediate implication of this thinking is that, for schooling, numeracy is a cross-curricular issue... Like literacy, numeracy is therefore *everyone's business* (AAMT, 1997. p. 12).

There are clearly common attributes in numeracy and literacy. For example, one aspect of numeracy involves being mathematically literate, that is understanding and using the language of mathematics to interpret a real world problem. Depending on one's understanding of numeracy, there can also be distinct differences between literacy and numeracy; the ability to use cognitive mathematical processes to analyse a real world situation is an example of where the skills of numeracy can be seen as distinct from and different to those of literacy (Coombes, 1996, p. 5).

Figure 3 presents one way of viewing the cross-curricular nature of numeracy and literacy and the interwoven relationship between the two.

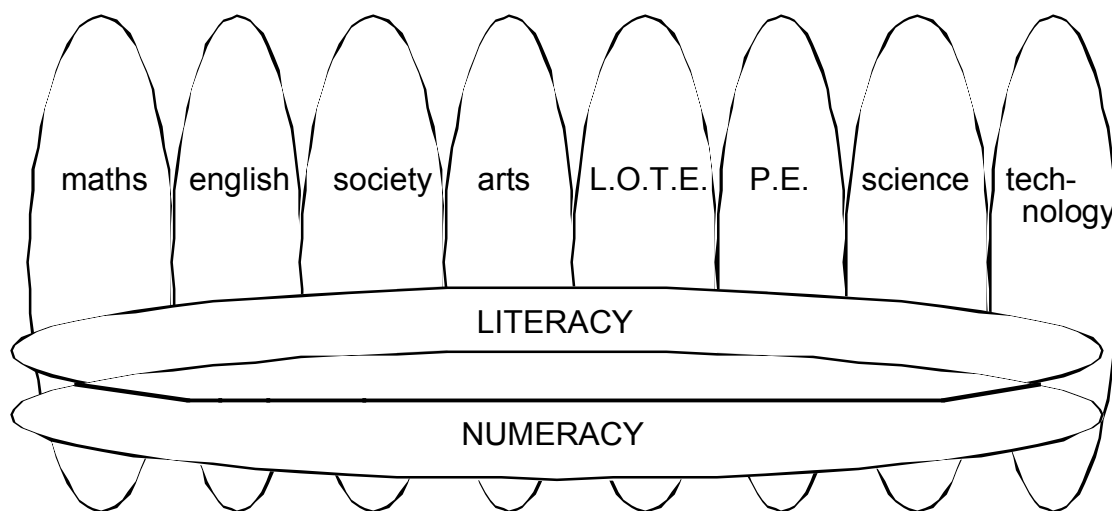


Figure 3. The cross-curricular and interrelated nature of literacy and numeracy.

Sources of good activities

The activities presented in this paper have been selected for their richness in potential for the development of students' number sense, easy access and support from the World Wide Web and are taken as a cross-section from the states and territories of Australia. They do not necessarily represent the particular approach taken by State education systems to numeracy education or reflect the research and policies of State Universities and Mathematical Associations. The selection of activity sources and web references is summarised in Table 2.

State/Terr.	Activity source	Web reference
ACT	Numeracy — a Classroom Perspective (paper by Paulene Kibble, 1999)	http://www.occ.act.edu.au/home/numeracy/classroom.htm
NSW	Count Me In Too (online support)	http://www.oten.edu.au/countmein
NT	Maths? No Fear!	http://www.schools.nt.edu.au/olsu/mnf/index.htm
Qld	Secondary Mathematics Assessment and Resource Database (SMARD)	http://smard.cqu.edu.au
Tas.	Chance and Data in the News	http://www.themercury.com.au/nie/mathguys/introduc.htm
SA	The History of Mathematics	http://www.roma.unisa.edu.au/07305/home.htm
Vic.	Maths300 (Curriculum Corporation)	http://www.curriculum.edu.au/maths300
WA	Graphics Calculators	http://www.eddept.wa.edu.au/centoff/graphcalc/index.htm

Table 2. A selection of web references and sources of rich number activities from across the States and Territories of Australia.

ACT — Numeracy, a classroom perspective

In this paper, Kibble (1999) discusses the classroom response to various pattern games and sees number sense outcomes in what might initially appear to be space activities.

Noughts and Crosses

This is a game where players take turns writing a '0' or a 'X' in spaces on a 3×3 grid. The first player to write three of his symbols in a row, column or diagonal is the winner.

Kibble's observations are recorded in her paper:

I wanted the children to consider if there were any positions which lead to a greater likelihood of winning. The discussion provided an opportunity to reinforce the meaning of vertical, horizontal and diagonal and encouraged the use of appropriate terminology. As children played this game I was able to observe those children who were as yet unable to see three as a single unit.

From my understanding of the Learning Framework in Number (*Count Me In Too*, NSW DET 1999) I knew that seeing a number as a composite unit was a

critical understanding for children to develop. The children who could not see three as a composite unit had difficulty winning a game as they were unable to get an overall picture of their position...

The 3×3 grid connected with work done in first term on exploring square numbers (p. 2).

Finding Squares

Students were presented with a diagram (Figure 4) and asked 'How many squares can you see in this shape?'

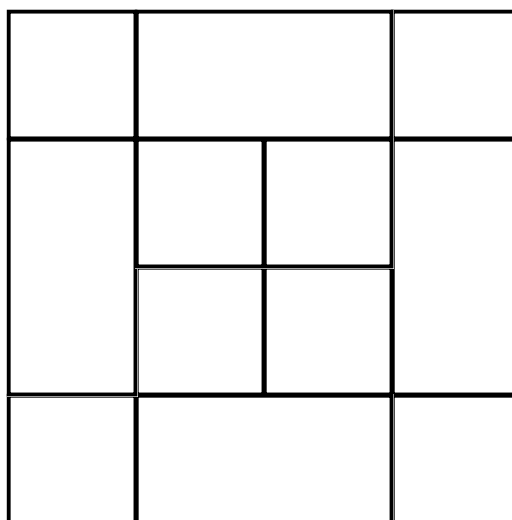


Figure 4. Finding Squares.

NSW — Count Me In Too

This site is part of the system support for the NSW Department of Education and Training's *Count Me In Too* (CMIT) early numeracy program. An overview of the program is given on the site along with advice on the professional development perspective, information for parents, and some great JAVA-enhanced student activities. Many CMIT support activities have been produced by the State's team of mathematics consultants and two such activities are reproduced here by permission of consultant, Garry Stanger. For a description of key aspects and stages of early number learning, see *Count Me In Too — In the Bush* (Stanger, 1998)

Bus Stop

This activity uses a die and picture card of a bus made from two ten-frames (Figure 5). Players take turn throwing the die and placing the indicated number of counters on the bus. The 'seats' are to be filled bottom row first, from the front. The game proceeds until one player fills his or her bus. An optional rule is that the players need to throw the exact number to fill the last spaces.

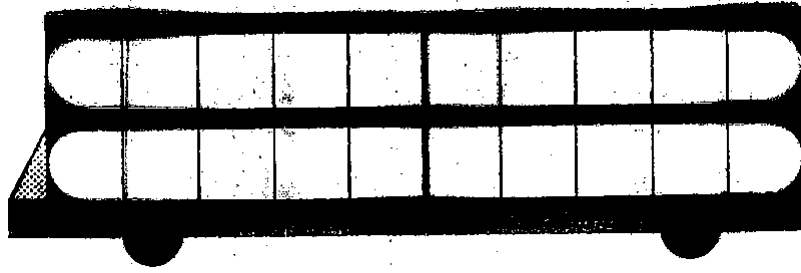


Figure 5. Bus card made from two ten-frames.

Five Plus

This activity uses a set of domino cards that all have five dots on one side and one, two, three, four or five dots on the other side. Figure 6 shows a stencil for making the domino cards.

A player takes the top card from the stack, places a cover over the five section and then tries to work out the total number of dots on the domino. Uncover the five section to check and, if correct, take that many counters from a pile and add them to your scorecard. A variation of this game is also presented in the form of an interactive JAVA applet in the children's section of the CMIT website.

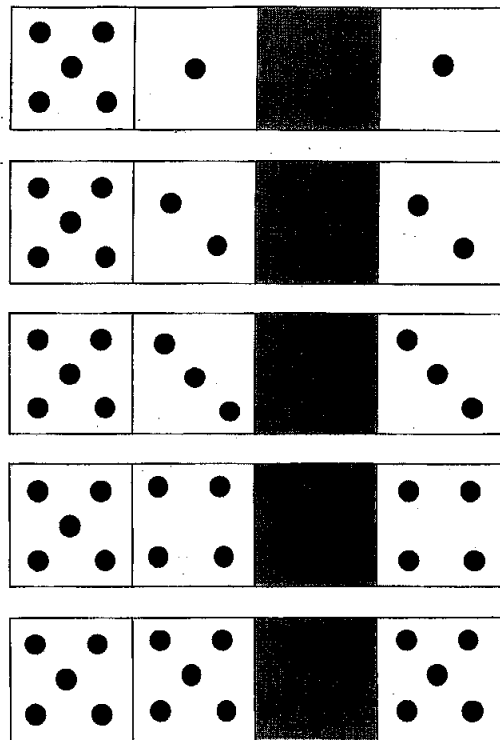


Figure 6. Photocopy stencil for making Five Plus dominoes.

Northern Territory — Maths? No Fear!

This program is a cooperative professional development project between the Northern Territory Education Department and Curriculum Corporation. It was funded under the Commonwealth's Indigenous Education Strategic Initiative Program and has adapted many of the activities from Curriculum Corporation's Mathematics Task Centre.

Number Tiles

Arrange the digit cards 1 to 9 to make three 3-digit numbers such that the first two numbers sum to make the third. Can you find more than one solution? How many solutions are possible? Can you *prove* that this is the maximum number of possible solutions?

Fay's Nines

Arrange the digit cards 1 to 9 to make three 3-digit numbers that sum to make 999. Can you find more than one solution? How many solutions are possible? Can you *prove* that this is the maximum number of possible solutions?

Both these activities are classic examples of extended investigations with multiple levels of access. To find one solution is usually not too difficult and will allow all students to experience success at that level. As the investigations progress, they have a richness that will challenge the thinking of all students and thus facilitate number sense outcomes for all students. As for the practice of number skills, well most students will need to execute twenty or more mental calculations before finding their first solution. However, the activity is carried out using manipulative cards, without the threat of committing incorrect attempts to pen and paper, and is certainly more interesting than completing a worksheet of twenty arithmetic drill exercises.

Queensland — Secondary Mathematics Assessment and Resource Database (SMARD)

This is an online database of classroom activities owned and operated by the Queensland Association of Mathematics Teachers. It invites teachers of mathematics to share ideas for classroom activities and assessment resources. If you follow the links to the *SMARD Database*, *Junior Database*, then *Whole Numbers* section of *Number*, you will find these two activities among those presented.

Triangle Problem

How many triangles can you make with whole number side lengths that add up to 15?

This problem encourages students to explore whole number partitions of 15, while being mindful of the 'triangle inequality' that states the two shorter sides must sum to be greater than the length of the longest side.

Digital Roots

This activity involves a number trick that asks students to think of any number and multiply it by 9. They are then asked to take the number that is produced and sum the digits, take that number and sum its digits and thus continue until they have reduced it to a single digit number. The single digit number will of course be nine. The teacher can then have students perform more operations on this number resulting in the illusion that the teacher can ‘magically’ predict the number that all students have made.

South Australia — The History of Mathematics

This Web page is a compilation of student contributions in the subject History of Mathematics conducted at the University of South Australia in 1996, and a Graduate Certificate in Mathematics Education project presented in 1997. Topics include a chronological study of the development of number from a European and Eastern perspective, as well as the development of other specific branches of mathematics and the contributions of particular mathematicians.

The number activities here are taken from the section on ‘Ancient Chinese Mathematics’ written by Stapleton and Tripodi (1996).

Magic Squares

The challenge in this activity is to arrange the digits 1 to 9 in a 3×3 grid such that the sum in each horizontal, vertical and major diagonal will be a constant.

Stapleton and Tripodi describe this as the ‘legend of Lo Shu’ in which the Emperor Yu the Great discovered the magic square (Figure 7) in the back of a tortoise in approximately 2000 BC.

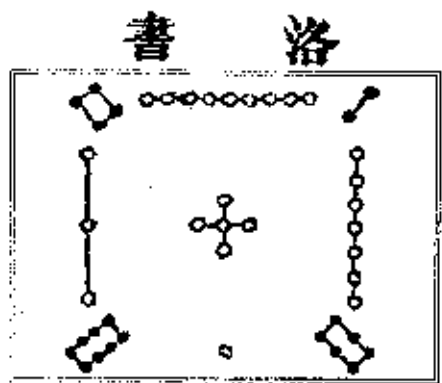


Figure 7. The Lo Shu magic square

Magic Circles

In this activity the challenge is to draw a number of concentric circles that are cut by the same number of diameters. This will produce a number of nodes where the circles and diameters intersect. The challenge is to number all the nodes, such that the sum of numbers on all circles will be a constant and the sum of numbers along all diagonals will also be a constant (but not necessarily the same constant).

Figure 8 shows a solution for two circles and two diameters.

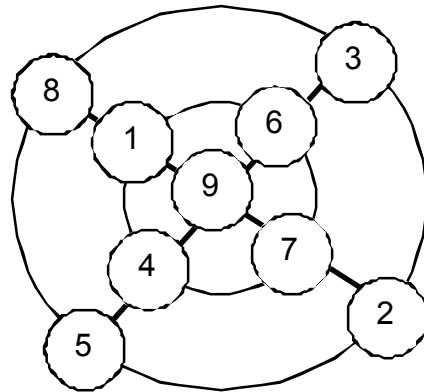


Figure 8. Magic Circle of order two

Stapleton and Tripodi include a solution to a magic circle of order four that was developed by the 13th century mathematician, Yang Hui (Figure 9).

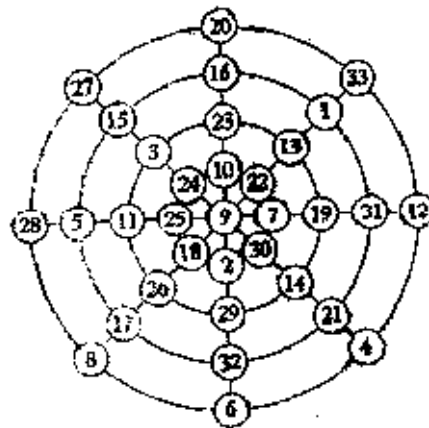


Figure 9. Magic Circle of order four

Tasmania — Chance and Data in the News

This site is a personal favourite of the author and an interesting one to include in a discussion on number sense and numeracy. Watson, who writes the teacher discussion and student questions component of the project, is an advocate for ‘statistical literacy’ (MANSW, 1998). The website is a cooperative production between the University of Tasmania, Hobart Mercury newspaper and the AAMT. It presents a selection of newspaper articles with mathematical comments suggested classroom applications from Watson. The reader may search for articles either relating to a particular social topic or to a particular component of chance and data or numeracy curriculum. A section of articles are presented under the specific heading of ‘numeracy’, although it could be argued that the all of the ‘statistical literacy’ content is in itself a vital component of numeracy development.

Article 1

Decriminalise drug use: poll

SOME 96 percent of callers to youth radio station Triple J have said marijuana use should be decriminalised in Australia.

The phone-in listener poll, which closed yesterday, showed 9924 — out of the 10,000-plus callers — favoured decriminalisation, the station said.

Only 389 believed possession of the drug should remain a criminal offence.

Many callers stressed they did not smoke marijuana but still believed in decriminalising its use, a Triple J statement said.

The poll followed a recent decriminalisation ruling in the Australian Capital Territory.

Figure 10. Article from *The Mercury*, 26 September, 1992, p. 3.

The article in Figure 10 is listed under the Data Collection and Sampling part of the website and Watson has used it to raise questions about the degree of representation that is presented by a sample. Given that in this case the sample was taken from the listeners of a youth radio station and that participation was by way of voluntary phone-in, the result is not necessarily a valid reflection of general public attitude.

Article 2

Another article presents a Calvin and Hobbs cartoon (Watterson, 1997) with the following dialogue:

Calvin: Psst.. Susie! What's $12 + 7$?

Susie: A billion.

Calvin: Thanks! Wait a minute. That can't be right... That's what she said $3 + 4$ was.

Watson uses the cartoon to discuss what aspects of numeracy Calvin does exhibit and what aspects are missing.

Victoria — Maths300

This is a bank of lessons available for a subscription fee from Curriculum Corporation. Many of the lessons are adaptations of Curriculum Corporation's highly successful *Mathematics Task Centre* activities, which are also related to the *Mathematics Curriculum and Teaching Program (MCTP)* and *Maths? No Fear! Program*.

Highest Number

This is a game where each player has a set of cards numbered 1 to 6 and a place value board with columns of hundreds, tens and units. Players take turns rolling a die and placing the number card indicated onto their place value board. The winner is the player who produces the highest number.

At its simplest level, this game is useful for promoting a sense of place value. Lesson notes accompanying the game also give suggestions for extending the investigation and exploring the related probabilities.

Truth Tiles

The challenge in this task is to place the digits 1 to 9 into three equations of the form:

$$_ + _ = _$$

$$_ - _ = _$$

$$_ \times _ = _ \text{ so that all equations are true.}$$

As with 'Number Tiles' and 'Faye's Nines', this task also gives students the potential to explore various sets of solutions, while exercising number skills in an interesting and motivating way.

Western Australia — Calculator Activities

A lot of great calculator activities have been coming from Western Australia. The work of Swan and Sparrow (1999) is exemplary in showing how sensible use of a basic calculator can greatly enhance students' development of number sense. Two such activities are reproduced here. Note, however, that the Western Australian web reference in Table 2, leads not to the work of Swan and Sparrow, but to another

calculator activity resource site. The site, called *Graphics Calculators*, is owned and operated by the Education Department of Western Australia. It is primarily a resource activity bank for secondary teachers, although a lot of the activities that have been contributed could also be used effectively in primary classes. This website, along with the work of Swan and Sparrow, shows strong leadership from Western Australia in this developing field and are valuable contributions to the educational pursuit of *number sense* as opposed to basic *number skills*.

Constant Counting

In a conference workshop paper, Swan (1998) discussed how the constant counting feature of most simple calculators can be used to assist students learn skip counting patterns and multiplication tables.

Most calculators can be set by a fixed amount using the following keystrokes sequence $+ 2 = = = .$ (Note: Calculators vary.)

Children can count along with the calculator or can be encouraged to predict the value after a particular number of key pushes. The calculator may also be set to subtract a constant amount (p. 39).

Predicting the Digits

In another conference paper, Swan and Sparrow (MAV, 1999), presented this problem:

When a two-digit number multiplies a three-digit number, how many digits would you expect to find in the answer?

They explained in their discussion how the calculator is used to take a lot of the 'hackwork' out of the investigation and that it also provides the opportunity for students to pose their own problems such as, 'How many digits would I expect the answer to contain if a three-digit number is multiplied by a three-digit number?'

Eventually the limitations of the calculator to display large numbers will be revealed, at which stage the investigation may be extended to the use of a computer.

Conclusion

It is hoped that readers will find these activities useful. More importantly it is hoped that they find the links useful; not just Internet links (although the sites referenced are all good quality), but the links between number skills, number sense and numeracy, as well as the linking of approaches, frameworks, programs and activities from across our nation. Development of number sense leading to numeracy is an educational pursuit deserving of our attention. However, it is not such a daunting task when so many good programs and resources are already in place. Move forward with this and enjoy.

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About the presenter

Brian Lannen has taught in NSW and Victorian schools and worked for several years as a District Mathematics Consultant for the NSW Department of Education and Training. Currently on leave from the Department, Brian enjoys the sunshine and

scenery of Yackandandah, north-east Victoria, while being 'Mr. Mum' to his arithmetic progression of children ($a = 1$, $d = 2$ years $n = 4$). He makes his own beer and bread and plays off a golf handicap of 36. Brian also operates his own consultancy business and, in the past year, has been involved in the writing of several text books and classroom support materials.

Numerical Methods in Senior Secondary Mathematics

David Leigh-Lancaster and Michael Evans

Mathematics provides tools for modelling and solving problems in practical and theoretical contexts. In many situations powerful numerical methods, as well as analytical and graphical approaches can be used, in other situations only numerical methods may be available. We will consider aspects of numerical methods in the areas of study and outcomes for the Victorian Certificate of Education (VCE) senior secondary mathematics courses Mathematical Methods and Specialist Mathematics. Graphics calculators, spreadsheets and computer algebra systems will be used to illustrate and explore applications involving numerical equation solving, differentiation and integration, as well as the numerical solution of simple differential equations by Euler's method.

Introduction

Numerical approaches to solving certain types of problems have been part of senior mathematics curricula for some time.

The extent to which they have been an *integral* rather than *incidental* part of these curricula has varied both with the mathematical *values* of curriculum designers, and the *extent of access* to suitable technology for implementing related algorithms. For example, readers may recall 'by hand' methods for the 'extraction' of square roots, or the application of methods for finding approximate values for the root of an equation. When technology such as graphics calculators, spreadsheets or CAS is available, the emphasis of mathematical activity can be shifted from *computation* of numerical values within a sequence of algorithm outputs to *exploring the behaviour* of the set of values generated by an algorithm. This use of technology provides an opportunity for numerical, graphical and analytical approaches to be considered together. Graphics calculators, spreadsheets and CAS are powerful tools that enable tabular and graphical data to be readily displayed, and can carry out computations accurately and efficiently. CAS can also be used to link such representations directly to symbolic forms.

Analytic, numeric and graphic work often goes hand in hand — the conditions under which certain numerical processes are valid, such as convergence to a root, requires careful mathematical analysis. Numerical and graphical analysis of differential equations, such as those involved in chaotic systems, can produce qualitative data about the behaviour of the system that is not so clearly evident from its defining equations.

Consider a simple task such as finding the roots of the quadratic equation:

$$x^2 - 4x - 4 = 0$$

An analytic approach such as completing the square produces the two exact values:

$$x_l = 2 - \sqrt{8} \text{ and } x_r = 2 + \sqrt{8}$$

The location of these roots can be represented graphically by the horizontal axis intercepts x_l and x_r of the relation $y = x^2 - 4x - 4$ as shown in Figure 1:

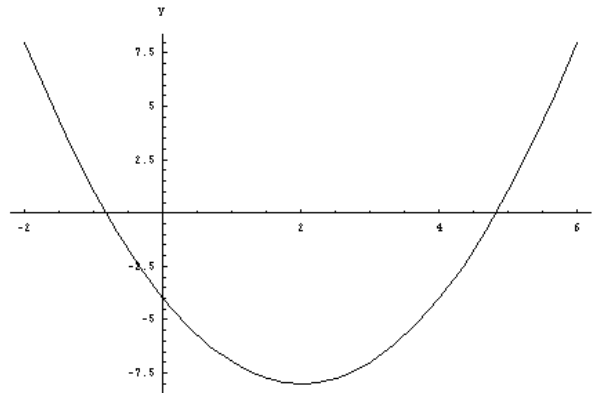


Figure 1: Graph of $y = x^2 - 4x - 4$

Re-writing the equation $x^2 - 4x - 4 = 0$ as a recurrence relation gives:

$$x_{n+1} = 4 + \frac{4}{x_n} \text{ where, for example, } x_0 \text{ the initial estimate value, could be } x_0 = 5$$

This recurrence relation produces a sequence of rational approximations that converges to $x_r = 2 + \sqrt{8}$. The following is an implementation of such an algorithm using the computer algebra system *Mathematica*:

```
x[n_] := 4 +  $\frac{4}{n}$ 
NestList[x, 5, 15]
{5,  $\frac{24}{5}$ ,  $\frac{29}{6}$ ,  $\frac{140}{29}$ ,  $\frac{169}{35}$ ,  $\frac{816}{169}$ ,  $\frac{985}{204}$ ,  $\frac{4756}{985}$ ,  $\frac{5741}{1189}$ ,  $\frac{27720}{5741}$ ,  $\frac{33461}{6930}$ ,  $\frac{161564}{33461}$ ,
 $\frac{195025}{40391}$ ,  $\frac{941664}{195025}$ ,  $\frac{1136689}{235416}$ ,  $\frac{5488420}{1136689}$ }
```

The corresponding sequence of numerical values, correct to 10 decimal places is:

```
{5., 4.8, 4.8333333333, 4.827586207, 4.828571429, 4.828402367, 4.828431373,
4.828426396, 4.82842725, 4.828427103, 4.828427128, 4.828427124, 4.828427125,
4.828427125, 4.828427125, 4.828427125}
```

This sequence has converged to a fixed value, correct to 10 decimal places, after 12 evaluations. Clearly for the given value of $x_r \cong 2 + \sqrt{8}$ then we can similarly find the corresponding value for $x_l = 4 - x_r$ of -0.828427125. An important consideration is how convergence of the sequence of values depends on the choice of the initial value x_0 . Two aspects of this are the *rate* of convergence to a limiting value, that is, is the algorithm an *efficient* one, and for what interval of initial values of x_0 centred on x_r does the sequence converge on x_r . This is called the *radius of convergence*. For example, what happens when a quadratic function with linear factors over \mathbb{Q} is used? A general root finding procedure is the Newton-Raphson method which is based on successive tangent root approximations to the root of a given function f . This may be readily implemented in *Mathematica* by the functions $\mathbf{g}[\mathbf{x}_-] := \mathbf{x} - \frac{\mathbf{f}[\mathbf{x}]}{\mathbf{f}'[\mathbf{x}]}$ and `NestList[g,a,n]` where \mathbf{a} is an initial value estimate for the root and \mathbf{n} is the number of iterations required.

Some numerical algorithms generate sequences of *values*, other produce sequences of *intervals*, such as the bisection algorithm for determining the location of a known root in a given interval, or sequences of *coordinates* of points, such as Euler's methods for numerical solution of a differential equation. Each situation has a set of conditions under which the numerical process can be validly applied. This provides an ideal context for structured investigation, including both theoretical work in terms of exploration of the process used, and applied work in terms of the use of these approaches to solve certain classes of practical problems.

Numerical methods in the VCE Mathematics study

The use of numerical methods is specified by content from the areas of study for the course and by key knowledge and key skills for the course outcomes.

Mathematical Methods includes 'numerical evaluation of derivatives' and requires students to 'demonstrate the ability to apply a range of analytical and numerical to obtain solutions (exact or approximate) to equations over a given domain, and to be able to verify solutions to a particular equation or equations over a given domain'. Specialist Mathematics includes 'evaluation of definite integrals numerically using technology' and 'numerical solution of differential equations by Euler's methods' and requires students to 'demonstrate knowledge of standard modelling contexts for setting up differential equations and associated solution techniques, including numerical approaches'. For 2000 such material has been incorporated in mathematics examinations. Students are assumed to have access to a Board of Studies approved graphics calculator for these examinations and are permitted to have stored additional programs in their calculator memories. For coursework assessment, other technologies such as spreadsheets, dynamic geometry software, statistical software or computer algebra systems may also be used, as appropriate.

Linear approximations

The gradient function can be used to provide an approximation for the value of a function for a given value of x . for example, the graph of $f(x) = x^2+5$ is shown below

and the graph of the tangent at $x = 3$ which has equation $y = 6x - 4$. Table 1 compares values of the two functions:

x	x^2+5	$6x-4$
2.7	12.29	12.2
2.8	12.84	12.8
2.9	13.41	13.4
3.0	14	14
3.1	14.61	14.6
3.2	15.24	15.2

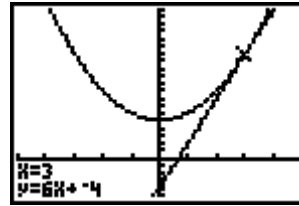


Table 1 values of a function and a tangent to the function

In earlier courses where linear approximations were included, the technique was mainly used to find values of functions, which could not be otherwise easily evaluated. The importance of this technique has been diminished by the use of calculators, however the emphasis can be shifted to consideration of the behaviour of the tangent line as an approximating function. What is the difference in value of the function and its approximation? For a function g and distance h from $x = a$, this difference will be given by the expression $g(a + h) - (g(a) + h g'(a))$.

For the function f this gives us $a^2 + 2ah + h^2 + 5 - (a^2 + 5 + h \times (2a)) = h^2$. The difference does not depend on the value of x but purely on the increment h . An investigation of other polynomials is of interest. For example, if $f(x) = x^3 + 5$ then the expression for the difference becomes difference is $3h^2 a + h^3$. Further investigation could lead to series approximations, as in the advice for applications tasks in the December 1999 edition of the *VCE Bulletin*.

First order differential equations with a graphics calculator

Certain types of differential equation can be solved numerically by use of the definite integral to define a function by $y = \int_a^x f(t) dt$. If $\frac{dy}{dx} = f(x)$ with $f(a) = b$ then

$y = \int_a^x f(t) dt$. For example, if $f(x) = x^2$ and $f(3) = 10$, then:

$$y = \int_3^x t^2 dt + 10 = \left[\frac{t^3}{3} \right]_3^x + 10 = \frac{x^3}{3} - 9 + 10 = \frac{x^3}{3} + 1.$$

Some graphics calculators can be used to plot an anti-derivative function of a function over a given domain. For example, if $\frac{dy}{dx} = \sin(x^2)$ and $y = 2$ when $x = 0$, then

$y = \int_0^x \sin(t^2)dt + 2$. Figure 2 shows how the function $y = \int_0^x \sin(t^2)dt$ is entered and its graph produced for the interval $[0, 2]$, using a TI-83:

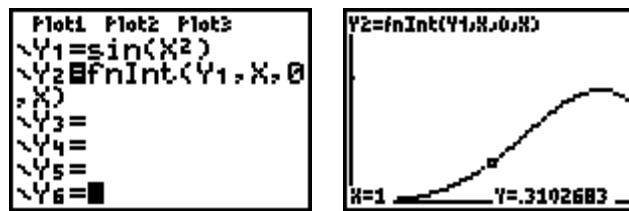


Figure 2: Graph from numerical integration of y

The table facility of the calculator can be used to find a value of y for a given value of x or it can be evaluated directly in the 'home' screen . For example to find the value of y when $x = 2$ the integral $\int_0^2 \sin(t^2)dt$ can be evaluated, then 2 added, as in Figure 3:

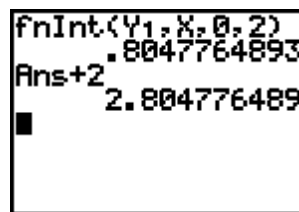


Figure 3: Evaluation of $y = \int_0^x \sin(t^2)dt + 2$

Jovanoski and McIntyre (2000) describe how their students have used a graphics calculator program to produce a slope field for a given differential equation and suggest how this may be incorporated into an introduction to Euler’s method for the solution of first order differential equations.

Euler’s method for numerical solution of first order differential equations

Euler’s method is based on repeated applications of linear approximations using line segments from information about the gradient function of a function f , and generates a *sequence of ordered pairs* (x_i, y_i) . This method provides students with access to a picture of an anti-derivative function without having to firstly find explicitly an antiderivative expression . This sequence is generated in the following way, let $x_1 = a$ and $y_1 = b$ where (a,b) are the coordinates of the starting point for the solution function, then: $x_2 = x_1 + h$, $y_2 = y_1 + f'(x_1) \cdot h$, $x_3 = x_2 + h$, $y_3 = y_2 + f'(x_2) \cdot h$ and in general, $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + f'(x_n) \cdot h$.

Graphics calculator implementation

This process can be carried out by graphics calculator program where the initial condition is $y = b$ when $x = a$ and $h = 0.1$:

```

ClrList L1, L2, L3
Prompt Y1
Prompt A : Prompt B
Prompt C
A→X
X→L1(1)
B→L2(1)
0→N
While X < C
N+1→N
L2(N)+ 0.1Y1(X)→ L2(N+1)
L1(N)+0.1→ L1(N+1)
X+0.1→ X
End
Disp"SEE L1, L2"
Plot1(Scatter, L1, L2, .)
FnOff
ZoomStat
    
```

For example, consider the differential equation $\frac{dy}{dx} = \sin(x^2)$ with $y = 1$ when $x = 0$. In

Figure 4 the screen to the left shows the responses to the prompts from the program to find values of y for x between 1 and 3 inclusive. The centre screen shows the sequence $\{(x_i, y_i)\}$ plotted and the screen to the right shows a section of lists 1 and 2, which contain the x and y , values respectively.

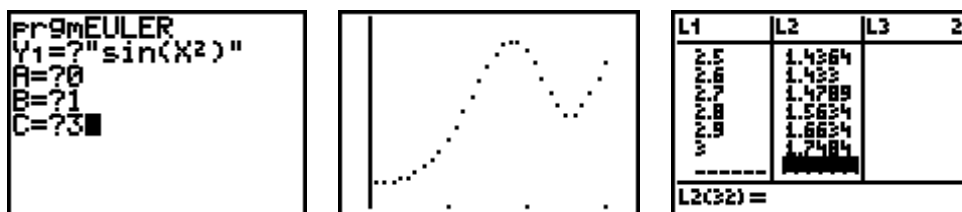


Figure 4: Euler's method program on a graphics calculator

When the increment is $h = 0.1$, $x = 3$, gives $y = 1.7464$, when $h = 0.01$, $x = 3$ gives $y = 1.7715$. The values and graph for this differential equation can also be determined by considering $y = \int_0^x \sin(t^2)dx + 1$. The result and the corresponding table of values are shown in Figure 5, where $x = 3$ gives $y = 1.7736$.

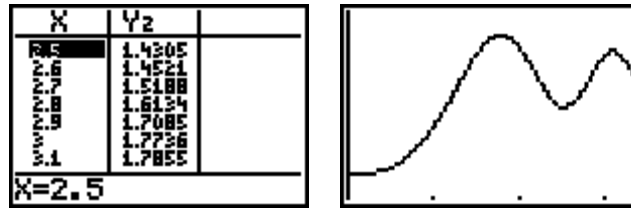


Figure 5: Graph of $y = \int_0^x \sin(t^2) dx + 1$.

Spreadsheet implementation

Euler’s method can also be implemented using a spreadsheet, as illustrated in Figure 6 for the differential equation $\frac{dy}{dx} = \sin(x^2)$. The spreadsheet can also be easily altered to consider the effect of transformations or starting points, with the corresponding graph appearing on the same page as the calculations.

x	y
0	1
=A11+0.01	=B11+0.01*SIN(A12^2)
=A12+0.01	=B12+0.01*SIN(A13^2)
=A17+0.01	=B17+0.01*SIN(A18^2)

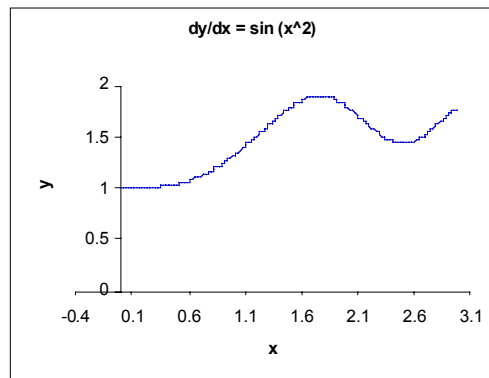
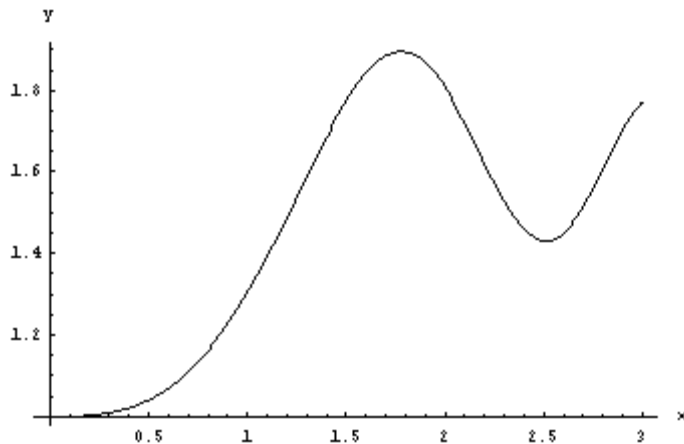


Figure 6: Euler’s method using a spreadsheet

CAS implementation

A computer algebra system can also be used, this implementation illustrates the recursive nature of the process:

```
f[x_] := Sin[x^2]
Y[{u_, v_}] := N[{u + 0.01, v + 0.01 f[u]}]
values = NestList[Y, {0, 1}, 300]
```

It is not possible to find an analytic anti-derivative for this function in terms of elementary functions:

$$\int \sin[x^2] dx = \sqrt{\frac{\pi}{2}} \operatorname{FresnelS} \sqrt{\frac{2}{\pi}} x.$$

A similar function with important probability applications can be based on $\frac{dy}{dx} = e^{-x^2}$.

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About the presenters

David is Manager of Mathematics at the Board of Studies, Victoria, and previously Head of Mathematics P–12 at Kingswood College, Victoria. He has been involved extensively in curriculum development, professional development, examinations and verification of school based assessment. His background is in pure mathematics, in particular mathematical logic and the foundations of mathematics. David has a special interest in the use of computer algebra systems (CAS) and their effect on pedagogy, curriculum and assessment in mathematics education, and has used the CAS *Mathematica* with his students from years 9–12 at Kingswood College during 1994–1998.

Michael is Head of Mathematics at Scotch College in Victoria. Scotch College introduced the use of graphics calculators in years 9–11 in 1995 and soon after in year 12. Their use has been integrated into the curriculum at all these levels. His background is in pure mathematics, receiving a Doctor or Philosophy in this discipline from Monash University. Michael has been actively involved with the Australian Mathematical Olympiad program, has worked extensively with the Board of Studies in Victoria and is currently the convenor for VCE mathematics.

BoxCars Hands-On Maths Games

Sharyn Livy

BoxCars is a Canadian maths games resource enabling teachers to teach maths concepts using specific card and dice games.

It requires nothing for teachers to make or do and can be immediately implemented in the classroom.

BoxCars cover skills and concepts taught from K–12 and allows students to learn in a relaxed, motivating way — REPETITION WITHOUT BOREDOM.

Using games as a teaching strategy has been documented as beneficial in breaking down learning barriers and developing positive attitudes. Games are non-threatening and children learn without fear of failure.

Games allow peer tutoring, socialisation, co-operation and sharing of maths strategies. All children participate at their ability level.

More information on *BoxCars* resources can be found at www.boxcarsandoneeyedjacks.com

Session 1 Outline: Primary K–7

This workshop focuses on the best of *BoxCars* primary games. Come and play with cards, dice and multi-sided dice and learn tips, tricks and strategies for mastering primary maths concepts and skills. These games appeal to all learning styles and complement any maths program. Experience the power of games for developing understanding, proficiency and confidence in your students.

Outline

The following topics will be addressed during the workshop session:

- What is *BoxCars and One-eyed Jacks*? — Games suitable for regular classrooms, Ed. Support and home schooling.
- What can you do with the cards and dice? — Describes types of cards and dice available and what you can do with them.
- What does a *BoxCars* classroom look like? — How you can make the games come to life in your classroom.
- How can I manage my students during *BoxCars* sessions? — Classroom management techniques to keep the 'dice rolling'.

- Demonstration and playing of games covering:
 - Number recognition and awareness
 - Tables
 - Probability
 - Problem solving
 - Patterns
 - Fractions
 - Place Value
 - Doubling
 - Operations
 - Time
 - Graphing
- What are the advantages of using maths games as a teaching strategy?
 - Multi-sensory experiences
 - Compliment any maths program
 - Rich in problem solving
 - Chances to invent and create games
 - Opportunities for written reflections and journal entries

Session 2 Outline: High School Yrs 8–12

An interactive games workshop using cards, regular dice and multi-sided dice with a high school slant. By combining games and problem solving, teachers are able to maintain high interest even from the low-level students.

Outline

The following topics will be addressed during the workshop session:

- Why use games?
 - Student-centred
 - Highly motivational
 - Uses all learning channels
 - Cards and dice are a more acceptable alternative than concrete objects in upper grades
 - Adapts easily to all class ability levels
- What can you do with the dice? — Suggestions on using dice for different ability levels.

- Useful warm-up games
- Play games, games and more games, covering:
 - Application of combined operations
 - Problem solving
 - Integers
 - Exponents
 - Chance and Probability
 - Place Value
 - Polynomials
 - Linear equations
 - Trigonometry
 - Fractions
 - Tables
 - Thinking skills and strategies

About the presenter

Sharyn Livy has been teaching in Victorian schools for fourteen years in a variety of classroom settings from Prep. to Year 12. Mathematics has always been an area of focus for Sharyn, completing her Bachelor of Education with a major in mathematics. Sharyn is currently working as an educational consultant providing workshops on a weekly basis for teachers in Victoria. By presenting meaningful and challenging activities Sharyn encourages enjoyment and enthusiasm for the teaching of mathematics.

Using Algebra in the Early Stages of its Learning

Ken Milton and Howard Reeves

Algebra is the language of mathematical structure, patterning and generalisation. When they first learn arithmetic, students can become aware of relationships, and those patterns which can be generalised. This paper (and the Conference workshop) presents examples of where and how arithmetic and algebra can be meshed. It provides a forum for the discussion of factors related to the teaching and learning involved.

Introduction

This paper articulates some of the basic underlying beliefs which we hold relative to the nature and purpose of arithmetic curricula encountered first in primary school and developed further in high school. Basically, our approach endeavours to link arithmetic experiences to algebraic generalisation and to reveal something of the mathematical 'use' of algebra and algebraic thinking. Of necessity the paper is brief. Our workshop presentation is more expansive and has a greater focus on classroom activity examples than is the case in this paper.

Arithmetic in the school program

The arithmetic curricula of today should have the features of 'down playing' the over-learning of written algorithms for calculations and be emphasising arithmetic as a 'study of number and its operations'. Viewed in this light, the teaching and learning of arithmetic has a focus on 'number for its own sake', on the one hand, and number as a helpful way of 'ordering and arranging the world around us', on the other. This is not to say that there is any clear and sharp distinctive schism between these two elements of focus. Rather, it is to highlight that both the intellectual (and truly mathematical) and utilitarian aspects of arithmetic can, and should, be catered for. And this can be so even from the early days in primary school! Put another way, arithmetic should be presented both as a fundamental branch of Mathematics and also as a vital utilitarian tool of Numeracy.

Historically, algebra grew out of arithmetic. To highlight this, many would contend that algebra can be defined as 'a generalisation of arithmetic' (Smith, 1997, p. 71). This is to recognise that elementary algebra adopts the generalisable structural properties of arithmetic as the 'rules' which govern the 'behaviour' of algebraic 'letters'.

We certainly do not advocate restricting the foundation arithmetic curriculum to only those kinds of experiences which are directly linked to the needs of beginning 'formal' algebra. Far from it. What should happen is that students in the primary school be given a very broad and intellectually engaging experience with, and study of, number, its operations, and its applications. Children should have the

opportunity to explore situations and be excited by the findings resulting from such exploration. They should have the chance to experience the intellectual excitement which can come from playing with numbers and getting to know their behaviour and properties.

A rich and diverse arithmetic background, together with an ongoing involvement with arithmetic, greatly aids and abets the realisation of the goals of algebra education. With the relationship between arithmetic and algebra in mind, any involvement with the concept of variable concerns the generalising of arithmetic pattern (Usiskin, 1988). Buxton (1984) put this nicely when he declares that there is no mystery in algebra for a learner who is truly at ease with arithmetic.

Davis (1985) declares that children can be learning some of the inherent features of algebra as early as Grade 2 or 3, through the way in which the arithmetic is approached. We agree with Davis (1985) about the teaching of arithmetic, for what may be called 'algebraic purposes', in three senses:

- (i) there should always be a concentration on 'structure' and 'representation'; this applies right from the time when the number operations are introduced;
- (ii) children should be encouraged to make and create equivalent number expressions; through such activity children can come to tolerate 'non closure' and become aware of properties of equality;
- (iii) children should be involved in applying their arithmetic to 'reality' situations and recording the 'structure links' between the two realms. Booth (1984) calls this making the 'method' explicit. For example, consider this' problem':

'A dog has 4 legs and a bird has 2 legs. How many legs would 5 dogs and 4 birds have together?'

By whatever representation a child may use in the process of understanding the situation and tackling the problem, the number should be finally recorded as

$$(5 \times 4) + (4 \times 2).$$

The intention ought be to make the path from the 'specific to the general', and thus from 'arithmetic to algebra', clear. Or, put another way, the intention is to have students experience arithmetic in such ways that structure and relationships are identified and can be meaningfully expressed. In this regard, Collis (1972, 1975) maintains that students make sense of arithmetic in relation to their perceived 'reality' of the situation. Basically, the 'reality' is linked to 'number size' or 'number value'. This is the frame within which the arithmetic experiences are developed in a staged progression. Specifically, the 'arithmetic' numbers involved cover the categories of small and larger 'number values'. And, in that algebra can be viewed as the action of generalising arithmetic, the properties and relationships considered are to be represented using 'pronumerals' when algebraic (generalised) numbers are brought into symbolic representation.

Preliminary ‘discussion’

Children first become aware of ‘numbers’ through counting. It does not take children long to realise some important features of the ‘numbers used for counting’, albeit that their ‘conception’, understandably, lacks maturity. It is important, within the context of the quest that we have set ourselves, that these realised features are made explicit, can be ‘built on’, and represented appropriately. The identified features would include:

- (i) ‘you can count forever’; that is, the set of counting numbers is infinite;
- (ii) there is a definite agreed order in counting:
 - every counting number has a definite ‘unique’ *successor*;
 - every counting number has a definite ‘unique’ *predecessor*;

put another way:

- except for the number one, every counting number is the successor to some counting number,
- every counting number is the predecessor of some counting number;
- (iii) for each counting number:
 - the successor number is ‘one more than the number’;
 - the predecessor number is ‘one less than the number’

Considering (ii) and (iii), for example, we can express the relationship between 4 and 3 as: 4 is the successor of 3; 3 is the predecessor of 4; and we can represent this as:

$$4 = (3 + 1) \text{ and } 3 = (4 - 1)$$

Several points are worth making here:

- This approach to the ‘naming of numbers’ can make a valuable contribution to having students not want to attempt ‘closure’ when confronted with algebraic symbols such as $(a + b)$;
- This form of ‘number naming’ can be extended; so, for example: $5 = (3 + 2)$ says that 5 is the number that is ‘two more than three’ says that 3 is the number that is ‘two less than five’
- It is possible to give meaning to the number ‘zero’ by ‘creating’ a predecessor for the number 1. So, $0 = (1 - 1)$ and $1 = (0 + 1)$.
- With the inclusion of 0 we have the set of whole numbers $\{0, 1, 2, 3, 4, \dots\}$
- Negative numbers (negative integers or, as some children call them, ‘negative whole numbers’) can be ‘created’ by devising firstly a predecessor for 0. So, $-1 = (0 - 1)$ and $0 = (-1 + 1)$. It is important to realise that we are ‘naming numbers’ in these kinds of situations, not carrying out calculation or computation.

Some examples to consider

Example 1

Children in Grade 3 can generate number expressions using beads on a string. Suppose there are 20 beads on the string. By arranging and rearranging the beads, children can generate a ‘great’ collection of equivalent expressions for 20 and, as a result, have a collection of expressions equivalent to one another. No calculation is carried out.

So, it may be derived that, for example:

$$(i) \quad 15 + 5 = 16 + 4 = 17 + 3 = 18 + 2 = 14 + 6 = 13 + 7 = 12 + 8 = 11 + 9 = 10 + 10$$

$$(ii) \quad (2 \times 8) + 4 = (10 \times 2) = (8 \times 2) + (2 \times 2) = (5 \times 4) = (2 \times 2) + (8 \times 2) \text{ and so on.}$$

The outcomes could be arranged so that patterns can ‘emerge’, if the children are able to discern such patterns. Familiarity, arising from the freedom and flexibility inherent in this kind of activity, together with appropriate ‘guided’ discussion, can enhance pattern recognition.

Consider the series of generated equivalences (i).

$$15 + 5 = 14 + 6 = 13 + 7 = 12 + 8 = 11 + 9 = 10 + 10 = 9 + 11 = 8 + 12 = 7 + 13 \\ = 6 + 14 = 5 + 15$$

The equivalences had been generated by a child moving one bead at a time from one ‘sub-collection’ to the complementary ‘sub-collection’. Through discussion with the teacher the child is firstly made consciously aware of the action which he or she has taken and secondly made to record the chain of representational outcomes arising from the action. The patterned action, or relationship, can be viewed in two different ways:

- (a) ‘You can take one from one “sub-collection” and add one to the complementary “sub-collection” and you’ve still got the same number’.

This is the first stage recognition of an inherent relationship. This can be extended to become:

‘You can take any number from one “sub-collection” and add that number to the complementary “sub-collection” and you still have the same sum’.

- (b) ‘You have the sum of two numbers, say, a first number and a second number. That sum is the same as for another two numbers, the first number being one less than before and the second number being one more than before’.

This can be extended to become:

‘You have the sum of two numbers, say a first number and a second number. That sum is the same as for another two numbers, the first being “any number less” than before and the second being that same number more than before’.

For the extended versions of both (a) and (b) we can record, say:

$$15 + 5 = (15 - 1) + (5 + 1) = (15 - 2) + (5 + 2) = (15 - 3) + (5 + 3) = \text{etc.}$$

As the student's experience broadens and his or her 'number horizons' expand, there is an acceptance that the pattern applies to 'all' number, irrespective of size. This is surely 'algebraic thinking' which can ultimately become symbolised as:

if a , b and c are whole numbers, then $a + b = (a - c) + (b + c)$.

It is also clear from similar chains of 'naming the same number' that we have equality statements of the form:

$$8 + 7 = 7 + 8$$

$$5 + 4 = 4 + 5$$

$$7 + 3 = 3 + 7$$

etc.

Again, with appropriate discussion and experience involving a concentration on 'structure', students can accept that it is 'reasonable' to believe this 'reversibility' property concerning (+) should apply to all numbers. The ultimate algebraic form symbolism to 'capture' this thinking could be:

if a and b are whole numbers, then $a + b = b + a$ for any and all a and b .

The approach of providing experiences where the concentration is on arithmetic operations and related 'structure' involving mathematical equality (as meaning 'names the same number') can, and should, be a part of Number programs from about Grade 3 onwards. Through such approaches students see that it is meaningful and acceptable to make number 'statements' which are unclosed' and to consider (=) to mean other than a 'do so something signal' (Denmark, Barco and Voran (1976)). Moreover, the 'results' can be applied in many settings and used to enhance meaning and understanding in those settings. Two examples may illustrate the point:

Example 1.1

What is the sum $398 + 175$? A pencil and paper algorithm could be applied. However, using the example treated earlier in this paper we have:

$$398 + 175 = (398 + 2) + (175 - 2) = 400 + 173 = 573, \text{ arrived at 'mentally'.$$

Example 1.2

Solve the equation: $y + 3 = 8$

It must be that $y + 3 = (5 + 3)$, since $8 = (5 + 3)$

It follows that $y = 5$

Or, using a knowledge of the relationship between (+) and (-):

If $y + 3 = 8$, it must be that $(8 - 3) = y$. That is, $5 = y$.

Or, using the 'reversibility' property of (+):

Since $y + 3 = 8$

$y + 3 = (3 + 5)$, leading to the conclusion that $y = 5$.

Example 2

Some, say, red markers (#) are put down in a line.

#

Some, say, green markers (*) are placed in between the red ones, like so:

* # * # * # * # * # *

We are interested to find the total number of markers in specific situations, to discuss and analyse the 'methods' used to arrive at the various totals, and to use such analyses to capture and record any generalisable 'method patterns'.

It is a good idea initially to have students describe, in words, the 'structure' of the situation as it is 'built up'. The students actually do the building up. The following sort of thing could be said:

- (i) When I put out two red markers I need one green marker and this gives me three markers.

When I have three red markers I need two green and so have five.

For four red I need three green to give seven altogether. And so forth.

This phase of 'building up' and describing the actions being taken is important for students.

Variations in build up 'styles' involve the emergence of different pattern identification. For example a student has given the following description of 'what she did':

- (ii) I put out the two red ones with a green one in between. This made three of them. Then every time I put out another red one I put a green one with it; that's another two 'things' each time.

With a 'building up' as in (i) the following table can be constructed:

Number of red markers	Number of green markers	Total number of markers
2	1	3
3	2	5
4	3	7

And with a recognition of number relationships and 'number naming' alternatives:

5	$(5 - 1)$	$(2 \times 5) - 1$
6	$(6 - 1)$	$(2 \times 6) - 1$
7	$(7 - 1)$	$2 \times (7 - 1) + 1$

And with the description of relationship through the use of a 'pronomeral':

n	$(n - 1)$	$(2 \times n) - 1$
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It is clear from the 'build up' approach that:

- the total number of markers = $n + (n - 1)$, where n is the number of red markers;
- the total number of markers = $(2 \times n) - 1$;
- the total number of markers = $2 \times (n - 1) + 1$

With the 'building up' as in (ii) we have the following:

Number of red markers	Total number of markers
2	3
3	$3 + (1 \times 2)$
4	$3 + (2 \times 2)$
5	$3 + (3 \times 2)$

And with recognition of number relationships and 'number naming' alternatives:

6	$3 + [(6 - 2) \times 2]$
7	$3 + [(7 - 2) \times 2]$

And with the description of relationship through the use of a 'pronomeral':

n	$3 + [(n - 2) \times 2]$
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In the light of discussion concerning (ii), another student suggested the following:

Start with a red one and keep adding on a green and red together.

This analysis led to the following table:

Number of red markers	Total number of markers
1	1
2	$1 + (1 \times 2)$
3	$1 + (2 \times 2)$
4	$1 + (3 \times 2)$

And with the recognition of number relationships and 'number naming' alternatives:

5	$1 + [(5 - 1) \times 2]$
6	$1 + [(6 - 1) \times 2]$

And with the description of relationship through the use of a 'pronomeral':

n	$1 + [(n - 1) \times 2]$
-----	--------------------------

We have named the number of markers in 'equivalent ways'. Thus it can be stated that:

$$n + (n - 1) = (2 \times n) - 1 = 2 \times (n - 1) + 1 = 3 + [(n - 2)] \times 2 = 1 + [(n - 1) \times 2]$$

Example 3

I have a belief that: 'The sum of any whole number and its square is always an even number'. Do you support this belief? Show how you reached your 'verdict'.

- (i) A starting point would be to make a number of 'case studies'. In every case investigated the claim 'holds up'. There is an intuitive acceptance that the claim is 'generalisable'. That is, the claim is supportable until a counterexample is turned up!
- (ii) The cases studied could be categorised; consider odd numbers and even numbers as the categories of whole numbers. The claim appears to be supportable for even numbers, for odd numbers, and hence for whole numbers per se.
- (iii) The numbers themselves can be 'generalised' through the use of pronumerals and the claim examined in the light of the character of the 'number expression' involved.

Let n be any whole number.

Then $(n + n^2)$ is the number expression to be examined.

If n is odd, n^2 is also odd and thus $(n + n^2)$ even.

If n is even, n^2 is also even and thus so is $(n + n^2)$ even.

Hence for any and all n , $(n + n^2)$ is always even.

- (iv) The result can be reasoned as follows:

$(n + n^2) = n(n + 1)$ as a product. Now, n and $(n + 1)$ are consecutive whole numbers and hence one or other must be even. Thus, the product $n(n + 1)$ must always be even.

This analysis exhibits a high degree of deductive sophistication.

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About the presenters

Ken Milton is an honorary Research Associate in Mathematics Education at the University of Tasmania. In a teaching and research career spanning 'many years' he has published widely and made many presentations at MAT, MERGA and AAMT conferences. He is a Life Member of both MAT and MERGA, and Past President of both. His current interests concern the learning and teaching of algebra (the area in which he has completed his PhD 'in retirement') and mathematical proof. He is a long time, paid-up supporter of Essendon Football Club.

Howard Reeves is the Principal Education Officer (Mathematics) in Professional Learning Services Branch of the Department of Education, Tasmania, a position he has held since 1985. During this time he has been involved in a number of AAMT projects and activities and national Mathematics and numeracy initiatives. He is a Past President of the Australian Association of Mathematics Teachers, a Life Member of the Mathematical Association of Tasmania and 'long time loyal and financial supporter' of the Essendon Football Club.

The Bicentennial Conservatory in Adelaide

Carol Moule

The Bicentennial Conservatory was built as a lasting monument to South Australia's part in the bicentenary celebrations in 1988. The Mathematical Association of South Australia has produced a folder of activities explaining the construction of the conservatory, and the interesting mathematics required. Participants will make a model of the building and look at the method of construction and the mathematics involved. During the workshop a video showing the construction of the tropical rain forest will be shown.

The Bicentennial Tropical Conservatory in the Adelaide Botanic Gardens was our contribution to the celebration of the 1988 Bicentenary. It was to be an attempt to recreate a tropical forest — with the warmth, humidity and ecosystems of the tropics — in the basically drier, mediterranean climate of Adelaide.

The design is stunning, and, twelve years on, working very well. Admittedly we have had no hail storms like those in Sydney to test the glass, and no earthquakes or cyclones either, but we hope it is designed well enough to withstand such tests!

This workshop is to show you the conservatory, build a small model of it and consider the mathematics embedded in some of the various problems associated with the project.

The folder is to take home for use in classrooms: students can give it to a teacher in their schools and teachers might like to try the exercises on their own groups.

They are sold by the Mathematical Association of South Australia. A sub-committee of MASA, in conjunction with the SA Chapter of the Royal Institute of Architects, developed this set of activities which, in a simplistic manner, consider various aspects of the design and the construction.

For students, they provide opportunities to consider a wide variety of careers that use mathematics and they also illustrate some applications of mathematics.

For teachers, they provide fun activities but real applications from which the mathematics needs to be drawn out.

What are some of the requirements that you can suggest if we were about to plan such a building?

1. Adelaide is relatively dry and very hot in summer, and cold in winter — temperature and humidity controls?
2. light for plants – glass – heat transfer?
3. Humidity = water?

4. Aesthetics — design, practicalities?
5. Construction — ease of prefabrication?

This means a range of specialists:

1. Architect — design structure and make a small scale model to consider – pure geometry!
2. Site surveyor to check position, orientation etc. — peg out exactly — Activity 2.
Building is $100\text{ m} \times 45\text{ m} \times 27\text{ m}$ high. Estimate the volume?
3. Structural and civil engineers to test the structure under various loads — weight of building, additional loads due to movements, thermal loads, etc. Activity 6 shows some simplified calculations related to changing conditions. Need for computer simulations.
4. Quantity surveyors who calculate amounts of materials and costs associated with structure and construction.

Each side has 14 toughened glass panels shaped as a trapezium 24.8 m high, and 4.8 m at the bottom and 3.5 m wide at the top.

The glass is 6 mm thick.

182 tonnes of steel was used — had to withstand high humidity so coated with zinc and a polyurethane to marine standards.

5. Builders' schedule — the various stages have to proceed smoothly with each part ready to go as others are finished etc. This stage ensures that sufficient time is allowed for each stage and that the project is completed on time and within the budget.
6. The landscape gardeners now finish off the project with soil, water and plants. There are several thousand plants.

I hope that when you are next in Adelaide you will visit the tropical conservatory and perhaps view it with a more informed eye than most visitors!

About the presenter

Carol Moule is Head of Mathematics at Westminster School in Adelaide. She has taught mathematics in South Australia for many years, as well as being an active member of MASA and AAMT.

Working Mathematically: What Does It Look Like in the Classroom?

Thelma Perso

For too long Working Mathematically outcomes have been a by-product of mathematics lessons. In a world where it is becoming increasingly important for children to be able to process information and knowledge, working mathematically skills and processes have become fundamental to learning and conceptual understanding in mathematics. The mathematics content should be the vehicle for the processes — not the other way around. This workshop shows how to do this. It is presented in such a way that participants have a lot of fun and model a process which can be used in the classroom (so that kids can have fun too!) in order that Working Mathematically outcomes are achieved.

Background

Working Mathematically can be thought of by some as ‘something new’ in the teaching of mathematics. To some extent this is probably true in the traditional mathematics classroom where students are taught content, routines, procedures and methods and taught to practice these until they are fluent with them. Clearly, this is no longer appropriate. Children need to be taught to *think* in mathematics classes, not to simply inculcate routine procedures.

In the eighties this was flagged as essential through the emphasis on investigative processes and problem solving in both primary and secondary curriculums. For many teachers it was seen as something ‘extra’ and, as usual the changes did not generally filter through mathematics teaching but became something we left out if we ran out of time in which to ‘get through all the content’. I also think that we believed children would learn these skills almost inadvertently as a result of *knowing* the mathematics content and skills.

Working Mathematically and the mathematical modelling process

With the emphasis in the mathematics classroom now being on processes the mathematical modelling process provides a useful framework for teachers in understanding what this means for their classroom.

For too long we have concentrated on teaching students the ‘bits’ and the ‘tools’ for applying and solving mathematical problems but have paid little if any attention on teaching children how to use them. Someone once said that if we taught English like we teach mathematics children would spend all of their time practicing spelling, grammar, punctuation, and sentence structure without ever doing any creative writing. This is a very powerful analogy: we’ve spent most of the time teaching

children how to add, subtract, multiply, calculate, and evaluate but given them little opportunity to use these in a creative way.

Problem solving, which is the creative 'goal' of mathematics, has too often been used as something 'added on' to the mathematics lesson; problems are given to the academically able students who finish their work early, or they're given to children to do for homework at the end of an exercise. Rarely are they the *focus* of the mathematics lesson.

The mind-shift that teachers of mathematics need to make then is about teaching the repertoire of skills and the content *in order that* students have these to choose from when solving problems. We teach content and skills *so that* our students are empowered to solve problems.

The choice about what mathematical skills and knowledge to use is what empowers students. Too often in mathematics classrooms are children taught a specific skill or skills and then given problems which require that skill. This does not empower children, it disempowers them; they do not have the choice about which mathematical models and skills to use as this choice has been made for them in the context of the lesson or the textbook exercise!

The mathematical modelling process is a problem solving framework which will help explain what is being described here. It includes five steps:

1. The student clarifies the problem;
2. The student chooses an appropriate mathematical model (skill, method) with which to solve the problem and justifies the choice;
3. The student uses the model chosen;
4. The student interprets the solution obtained in light of the original problem in the given context; and
5. The student communicates the process (i.e. steps 1–4) for a given audience.

These steps can be summarised as *Clarify, Choose, Use, Interpret, and Communicate*. Other words can be used which may be more appropriate for the age of the students or their phase of schooling, such as *Read, Plan, Do, Check, and Share*

To *clarify* a problem children need to ask questions like:

- What do I know?
- What assumptions can I make about the context of the problem?
- What am I being asked to find out?
- What will I need to find out?

In order to answer these questions children can use various strategies like restating the problem in their own words, underlying key words, identifying any irrelevant information and so on.

On the basis of this clarification they will then need to make some *choices* about which mathematical skills, tools and knowledge they can use in order to solve the problem. They can ask question like:

- How do I find out the information I need with which to solve the problem?
- What mathematics will I need to do?
- Will I need more than one mathematical model?
- Which mathematical model/s shall I choose?
- How should I display my results — in a table or chart?
- How should I organise the mathematics I use — will diagrams and calculations be helpful?
- Which is the most efficient and appropriate mathematical model to use here?

After making the choice of the mathematics they will *use*, children then have to carry out the calculations using the model/s chosen. They will need to make decisions about how much of the actual calculation they will need to show on paper (and this is related to both the Choose and Communicate parts of the process) for the sake of the audience of their results.

Once the calculations have been carried out students need to *interpret* or check their results. They should ask questions like:

- Do my answers seem reasonable? Why?
- If my answer doesn't seem reasonable could it be that I didn't clarify the problem properly? Is there something I didn't take into account?
- If my answer doesn't seem reasonable could it be that I didn't choose an appropriate model to use? Could I have chosen a more appropriate model?
- Does the model I chose tell me what I want to know or will I have to choose another one?
- Did I use the model/s correctly or have I made some careless errors or mistakes?

The communication of the processes used is an extremely important part of mathematics. For too long we have focussed on answers so that teachers and students often believe their task is complete when the answer has been produced. Unfortunately this is insufficient. Employers want people who can solve problems and communicate not only the results but the processes used — including refinements and justification of choices made during the process — to obtain results. It is the responsibility of teachers to teach the skill of communicating both in verbal and written form, to students. Students need to ask themselves the questions:

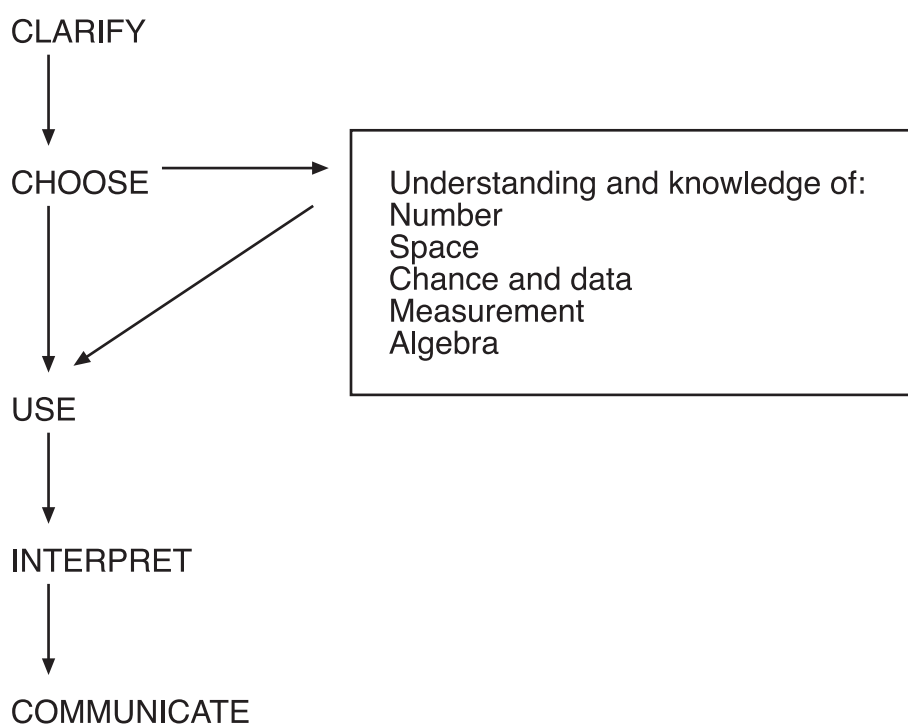
- What did I do?
- How did I do it?
- What results did I obtain?
- Did I have to redo anything? Why?

- What would I have done differently if I did it again?
- What assumptions did I make when clarifying the problem? Were these valid?

The responses to these questions need to be communicated in an appropriate format for the required audience.

In summary then, the Mathematical modelling process is part of *Working Mathematically*; a strand recognised in the *National Statement on Mathematics for Australian Schools* (1990) and more recently in the National Profile (1994).

The content still has an important place — indeed the processes cannot exist without it. It forms the repertoire from which the students choose what mathematics to use and how to use it. Diagrammatically:



Concluding remarks

How would you use this process in your classroom? First of all you need to make the mind-shift from focussing on content to focussing on mathematical processes; you still need to teach the content but only in so far that it becomes the essential resource from which children can choose in order to solve mathematical problems. Using the above model then will become a continual part of the teaching environment.

You may choose to teach the skills and knowledge as they are needed in order to solve various problems or you may teach the skills and knowledge for a period of time and then use problems intermittently but constantly so that children appreciate why they are learning the skills. Degree of reinforcement of this is a personal thing and is also related to timing and classroom organisation. The mind-shift made by the teacher will in itself affect the nature of the pedagogy or teaching style used.

About the presenter

Thelma has been a teacher of mathematics in a secondary school for over 20 years, eight of which as head of department. For the past three years she has held the position of Senior curriculum Officer Mathematics with the Education Department of Western Australia. She has a PhD in Mathematics Education and is the immediate past president of MAWA.

Aboriginal Numeracy

Thelma Perso

It is proposed that numeracy standards of Aboriginal children will only improve when teachers of aboriginal children take three things into account in their planning and teaching:

- the Aboriginal people, their culture and their transition into schools in the dominant culture;
- the Aboriginal children and the mathematical understandings they bring into the classroom;
- the explicit mathematics teaching required by all children in our schools.

Background

The release last year of the WALNA (Western Australian Literacy and Numeracy Assessment) data and the performance of Western Australian students on the national benchmark tests once again drew our attention to our failure to address the needs of Aboriginal children in the area of numeracy. As a result of the attention drawn to this fact I have commenced the writing of a book entitled *Improving Aboriginal Numeracy through Mathematics*. The book will be the end product of my attempt to bring together the many findings of Australia-wide research about the Aboriginal people and their mathematics (this also enhanced and supported by the input of many Aboriginal people themselves through discussion). I will share with you some of my writings, research and ‘musings’ to date.

Numeracy

What is ‘numeracy’? There are widespread disagreements as to a definition for this term. I prefer to work with a definition which is more about *functional numeracy*, which, I might add, is described in the Western Australian Curriculum Framework (*Curriculum Framework*, p. 219) as being the ‘disposition to use mathematics in situations outside of mathematics’. In my personal opinion the WALNA tests do not effectively test numeracy; they merely test children’s ability to *do* mathematics but not their disposition or attitude towards using mathematics. There may in fact be children who can effectively do mathematics but who are not numerate — that is, they have the skills to be numerate but are not confident to choose to use mathematics in contexts outside the mathematics classroom.

The implications for this and Aboriginal numeracy — and indeed for the numeracy of all children — is that generally tests that claim to measure numeracy are in fact invalid; they do not test what they claim to test.

Numeracy is also, in my opinion, a cultural construct. I may be numerate in an environment and culture with which I am familiar, but when placed in an unfamiliar environment I may not be numerate because I may not be confident to use the mathematics I know and apply it in an unfamiliar context. Aboriginal children are likely to be numerate in their environment — that is, they are confident to choose to use the mathematics they know in the environment they are familiar with.

Clearly, any test which measures numeracy is invalid if it does not test the mathematics you know and the confidence you have to use it *in an environment in which the mathematics you have learned to help you deal with that specific environment*. What we need to be aware of then, is that these tests test children's ability to be numerate in the environment of the dominant culture. We should not then, be surprised when Aboriginal children fail to meet benchmarks as set by the dominant culture — indeed, would we pass a test on numeracy as set by Aboriginal people — bearing in mind that the test would focus on mathematics skills and knowledge as needed to survive in a totally unfamiliar environment?

We should not fail to take into account, however, that the Aboriginal people themselves clearly want the mathematics and numeracy skills of the dominant culture for their children. In order to help Aboriginal children to become numerate *by the standards of the dominant culture* we must be inclusive in our teaching of mathematics. We should also remember that all children are on a continuum with respect to their numeracy — we are not dealing with two distinct groups: Aboriginal and non-Aboriginal. Some Aboriginal children may be more numerate than some non-Aboriginal children. This will depend largely upon the demands of the environment. To exemplify this point I point out that many Australian adults may fail the measurement section of the WALNA and national benchmark tests in numeracy because they still use imperial measurements. In their own environment however, these people *must* be numerate in measurement by the standards as set by their environment or the demands of their environment would have forced them to change to the decimal system.

The standards of benchmark tests then, are externally set as being a standard that is required or demanded by Australian society as a whole. Clearly the standard is an attempt to aggregate what is needed to be functionally numerate in all environments of the dominant culture. It would do well for us to remember this.

'Teaching' numeracy

You can't 'teach' numeracy. Numeracy results from two aspects:

1. A sound understanding of mathematics, and
2. A confidence to use mathematics when the environment demands it.

The implications for teachers are clear; teachers must teach mathematics in such a way that children understand it including how and when it can be used. They must also teach mathematics in an environment that fosters risk taking; in particular, taking away the 'fear' of being wrong, so common in classes where mathematics is

taught. It is this that so largely impacts on the confidence children have to use mathematics when it is appropriate to do so.

Inclusivity

Inclusivity means providing all groups of students, irrespective of educational setting, with access to a wide and empowering range of knowledge, skills and values. It means recognising and accommodating the different starting points, learning rates and previous experiences of individual students or groups of students (*Curriculum Framework*, p. 17).

This principle underpins the whole *Curriculum Framework*. The implication of this principle for Aboriginal numeracy — if we are serious about it — is that in order to assist Aboriginal children to become numerate in the dominant culture, we must take into account the previous experiences of students with mathematics and provide learning experiences that take different starting points and different learning rates into account.

Teaching mathematics to Aboriginal children

As previously mentioned, all children are on a continuum in their understandings of and abilities with mathematics. Part of the difficulties about writing a book on Aboriginal numeracy is that there are so many different Aboriginal cultures across Australia it is impossible to talk about ‘Aboriginal children’ as if I am identifying one group. It is almost as difficult as setting one test to test the numeracy of all Australian children!

In my attempts to identify the differing starting points that are possible for some Aboriginal children, I have read many research reports by authors who worked with specific cultural groups from different parts of Australia, and I have spoken to Aboriginal people from different parts of Australia. It is for this reason therefore, that all examples used and all suggestions made are couched in phrases such as, ‘some Aboriginal children *might...*’ or, ‘If you teach Aboriginal children you *may* have to be aware that...’. This is not an attempt at being apologetic, merely a recognition of the diversity of Aboriginal cultures and the enormous range in the extent to which each Aboriginal child may or may not have been exposed to the different environments and contexts of the non-Aboriginal people.

An example: understanding numbers; counting and whole number

It is impossible in the scope of this paper to give more than a brief example of inclusive teaching of mathematics for Aboriginal children. It should also be remembered that since all children are on a continuum in their understandings, the examples I use are just as applicable for non-Aboriginal children as for Aboriginal children. I will use the outcome of understanding numbers and the skills of counting with whole numbers as an example.

The understanding of numbers and number forms is crucial to being numerate. The reason many children fail to have numeracy skills is that despite being fluent with

mathematics computation — the ability to compute using mathematics procedures — they do not understand the numbers they are working with. It is because of this that they are unable to make choices about what mathematics to use and when.

Numerate people have an ability to recognise when numbers are being used as labels, as a solution for questions about ‘how many’ or simply to represent order, for example the ordering of book placement in a library or the representation of ‘overs’ in a cricket match. Failures to recognise these differences can cause problems for children learning mathematics in schools.

Teachers should help children to make connections between the different uses of numbers in society. For example, when a cricket commentator explains that there are ‘4.3 overs remaining’ in a cricket match, does that mean 4 and $\frac{3}{10}$ overs? If not, why not? If these sorts of discussions do not occur is it any wonder that children have difficulties with decimals? Many children are under the misconception that decimal numbers as used in mathematics classes are something entirely different than decimal numbers in the real world. It is these sorts of misconceptions which make children apprehensive and lacking in confidence about choosing to use mathematics outside the classroom.

For some Aboriginal children, particularly in remote communities, experiences with numbers used in an ordinal (ordering) sense, a cardinal (counting) sense or a labelling sense may be rare. To facilitate the learning of these, real life experiences in and around the school community may need to be ‘created’ so that there is an immediate and practical purpose in learning them.

Counting is one of the first encounters that children have with Number. In Western society and many other cultures, the ability to count by pre-school children is often used as an informal benchmark by parents; one often hears parents boasting about the fact that for example, ‘my daughter is only 3 and she can count up to 20’. It is used as a mark of intelligence or as a promise of future achievement. Unfortunately, it is often little more than a measure of a child’s ability to memorise a string of words. In Aboriginal cultures it is likely that this skill is not valued.

Numbers may be cardinal or ordinal. When numbers are used to tell ‘how many’ they are being used in a cardinal context. In Aboriginal society counting to find ‘how many’ may not have the same use. Indeed, in some Aboriginal cultures counting has little value. Some Aboriginal dialects often do not have words for numbers greater than three; this clearly indicates that the necessity for these is not demanded by the environment. After three it is just ‘big mobs’ or ‘lots’.

Many Aboriginal people use a deficit model when counting; when asked how many people were at a funeral for example, they may respond with, ‘There was a big mob but Auntie Betty and Auntie Dulcie weren’t there’. Clearly, it is more important to know *who* is there rather than how many are there. This is because for most Aboriginal cultural groups relationships and people are more important than quantities. There must be a purpose for numbers and counting for them to be important. For catering purposes an extra potato may be added to the pot for each person as a sort of one-to-one correspondence as extra people turn up but there may not be a notion of ‘preparing for 10 people’ for example.

Children from non-Aboriginal cultures are generally immersed in a ‘counting world’ before entering formal schooling. Parents believe they are helping their children to learn by teaching them to count (or at least, to say the counting words in order!) For example, a mother carrying her child up the stairs to bed may count the stairs as she goes; parents driving along with their children in a car may count the numbers of trees they pass or the number of cars on the road; a woman dishing up the dinner may count out two pieces of chicken for each adult and one for each child; and a child may be helped to count how many steps they take to walk along the footpath. These experiences are often missing for many Aboriginal children since this ability is not highly valued as either a skill or a necessity.

There are pitfalls here for teachers of both groups: does the child understand counting when they say the words and moreover, is a child less intelligent or less advanced if he/she cannot say or does not know the words? Many children can learn the words up to ten with ease. They may also be fluent with words for 21, 22, 23... 31, 32, 33... 41, 42, 43... and so on. Indeed, the patterns within the sequence make learning numbers relatively easy for many children. Unfortunately, the numbers between 10 and twenty do not ‘fit the pattern’ and can cause major obstacles for children; we don’t say ‘tenty five’ for example, we say ‘fifteen’. Particular attention needs to be placed on these numbers when helping children to say, read and write them.

Counting is clearly more than saying the words in the correct order. Once children are able to recite an accurate number sequence in words they then need to give them qualitative meaning. They need to coordinate their verbal counting with actions on objects: one word for each object touched. Children need to informally learn that they are ‘adding on one’ each time a new number is learned. Primarily the best way to do this is to teach children through the use of objects, using a one-to-one correspondence. Activities such as counting jelly beans, matches, stones or chairs could be used. Some children may skip numbers or count objects more than once. They can be helped by actually moving each object from one place to another as they count it or, if it is a drawing or picture, cross it off as it is counted.

Clearly, in order for this to be a useful learning strategy, children need to know that it is important to be able to count: this will need to be made explicit for many Aboriginal children who may not already have a sense of this importance from their home environment. Contexts should be used where this importance is clear through the purpose; for example, count six jelly beans each *so that you can eat them.*, or similarly, count twenty cards each *so that you can use them in your game*, or count these counters *so that you will know how many there are*, or count these pencils *so that I can find out whether you can count*.

Some children can count but cannot connect this activity with cardinality — that is, they don’t understand that the last number spoken tells them ‘how many’. When children learn this then they clearly understand the purpose of counting. Following this they need to learn that the last number refers to the whole group, not just to the last object touched or drawing crossed. This indicates that their counting knowledge has been connected to an understanding of groups.

Creating an environment that fosters a positive attitude

Since this is part of the requirement for Aboriginal numeracy it is clearly insufficient to just teach mathematics explicitly to the recognised different starting points brought to the classroom by children. Much work has been done in this area, particularly in the context of Aboriginal literacy. It is impossible to go into this in any detail here. Suffice it to say that we need to know about Aboriginal culture — aspects such as how questioning is used by Aboriginal parents and the wider family, learning styles and cognitive differences in Aboriginal children, and so on — so that Aboriginal children feel comfortable, accepted, able to take risks without fear and so on, in order to promote and foster a disposition that empowers them to choose to use Western mathematics in situation where this is required.

Concluding remarks

This has just been a ‘taste’ of the complexity of the attainment of numeracy for Aboriginal children. My work continues and hopefully the completed volume will provide teachers with some insight into what it really means to be inclusive in their teaching of mathematics for Aboriginal children in such a way that they may become numerate in the dominant culture if that is what they desire.

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About the presenter

Thelma has been a teacher of mathematics in a secondary school for over 20 years, eight of which as head of department. For the past three years she has held the position of Senior curriculum Officer Mathematics with the Education Department of Western Australia. She has a PhD in Mathematics Education and is the immediate past president of MAWA.

Developing Skills in Writing Proofs Throughout Secondary School

Diane Resek

Participants will work on problems that can be used to engage students in writing convincing arguments. These arguments evolve into proofs as students mature. The problems given will represent a wide range of subject areas: algebra, geometry, number theory, and discrete mathematics. A key to helping students develop their arguments is grading their work and giving feedback. Participants will look at student work and discuss grading rubrics.

We have all heard students complain: ‘I understand the math, but I just can’t write proofs.’ A few probing questions usually reveal that the student really does not understand the mathematics involved. She or he may be able to parrot back the key definitions, but does not know what inferences can be drawn or what relationship needs to be proved. What is really happening here?

First, students often believe that they understand mathematical concepts after they have read about them in the text or sat through a lecture on the material. They believe that, if they are unable to do a problem, whether a proof or another non-mechanical exercise, then there must be something wrong with the book or the lecture. If someone would just fill in a missing piece of explanation, they would be fine. They do not understand that what is missing must come from inside their own head.

Teaching exploration

Students need to learn what people who have become mathematicians instinctively know and probably cannot remember being taught. That is: in order to understand a new idea, you must play with it, pose examples, look for counter-examples, ask yourselves questions about it. In the words of *Everybody Counts*, a report on the future of mathematics education in the United States published by The National Research Council:

In reality, no one can *teach* mathematics. Effective teachers are those who can stimulate students to *learn* mathematics. Educational research offers compelling evidence that students learn mathematics well only when they *construct* their own mathematical understanding. To understand what they learn, they must enact for themselves verbs that permeate the mathematics curriculum: ‘examine,’ ‘represent,’ ‘transform,’ ‘solve,’ ‘apply,’ ‘prove,’ ‘communicate.’ This happens most readily when students work in groups, engage in discussion, make presentations, and in other ways take charge of their own learning. (National Research Council 1989, pp. 58–59).

The question is how do we, as teachers, help students to take an active role in their own learning. The key to stimulating students is to provide them with engaging mathematical problems, ones with answers they want to discover. Then a teacher needs to provide students with the right amount of support, not too much and not too little. Finally, the teacher must demand quality work, but cannot be too demanding. Just as coming to understand mathematics is an art learned over time, so is teaching students, or facilitating them to develop their skills in this area.

The right problems

For students to actively explore a problem or a situation, they must be curious about the answer. There are some students who will work on problems when given very general motivation such as ‘It will be on the test,’ ‘You’ll need it in math next year,’ ‘You’ll need it in college,’ or, more vaguely, ‘It’s important to know this.’ On the other hand, most students need more intrinsic motivation. They may not see themselves as going to college or as needing any mathematics in their future jobs. We need to reach these students as well as those who accept the idea that mathematics is a part of their future.

There are different ways that problems can appeal to students. One way is by putting the problem in a context of a real-life situation. Students will value a task if they believe it relates to their real lives, present or future. Examples of such problems are finding the incidence of false-positives in mandatory drug testing, solving a linear programming problem to maximize profit for a business, and developing a calculator program to generate a graphical display that moves on the screen. Although students recognize that they might not face these precise challenges in the future, the tasks have a feel of reality or relevance that maintains their interest.

Another reason students are motivated by a task is that the problem catches their imagination. An example of such a problem involves Edgar Allan Poe’s story *The Pit and the Pendulum*, in which a man escapes from the deadly swinging blade of a 30-foot pendulum. Students are asked whether the amount of time the prisoner needs for his escape in the story fits the reality of how pendulums work. Another problem asks students to figure out when someone should jump from a moving Ferris wheel to land in a moving tank of water. They are highly motivated to do the complex mathematics required to create a ‘splash’ instead of a ‘splat’.

A third way to motivate student is to give them situations where the mathematics itself is intriguing. One example is to ask students to investigate which numbers can be written as linear combinations of other numbers. But while the mathematics of this problem is an eventual ‘grabber’, students often need a context for such problems to get them started. That is, their intellectual curiosity often does not kick in until they begin working on the problem. It is not a lack of motivation that makes students hang back from the task. Rather, many students find it hard to understand a task when it is stated very abstractly. A context — even a far-fetched one — can make the situation concrete enough for students to begin thinking about the problem.

The right amount of support

Students will not learn to discover, explore and or 'get their hands dirty' with the math unless the problem is somewhat open. This means the problem cannot have a simple right answer that is accessible to students through a learned procedure. At the same time the problem cannot be too open or students will not know how to get started and will become bored and frustrated.

So, the first task for a teacher is to structure the problem so students can get started on it. The whole process will be easier if the students are working in small groups. In such a format, students have more ideas of ways to proceed when they are stuck. There is also more need for students to explain their ideas to each other and to clarify things. Sometimes the explanations will let students understand when they have taken a wrong turn and other explanations will eventually lead to a proof.

The teacher must be monitoring all of the groups, looking for several sources of trouble. One source is that some groups will at some point not know how to proceed. The teacher needs to judge when they might solve their own problem if given a few more minutes, and when it is time to give them a hint or to ask them a question that will move them in a productive direction.

The teacher also needs to be looking for groups that are taking a wrong turn. Sometimes it is important for students to make mistakes and work with erroneous assumptions. This is one way we deepen our mathematical understanding. We must play with the misconceptions to clarify our ideas. But there is limited time for classroom exploration and teachers need to redirect groups of students at times so they will have time to grapple with other important ideas.

To sharpen students' abilities at explaining or at constructing proofs, teachers must challenge their explanations. Students are often satisfied with vague understanding and a vague explanation. Teachers need to challenge those explanations and send groups back to the drawing board to do better. Students will give better explanations if they are expected to present them to their classmates. Teachers should try to create a classroom climate where students listen to peers politely but critically. This climate typically takes time to develop, sometimes several years.

Not all groups will finish solving and explaining a problem at the same time. Probably teachers need to bring the class together before the slowest group finishes, but there must be some extra extension problems up a teacher's sleeve for the fastest groups. So, during group work, teachers need to monitor the groups to see that all groups have appropriate tasks to work on.

The right demands

It is true that 'If you don't ask, you don't get' where student work is concerned. Most students will not extend themselves to do excellent work if they are not asked to. But, at the same time, students must feel it is possible to satisfy their teachers' demands before they will work on it. Probably the one thing holding many students back from becoming active mathematics investigators is lack of confidence.

Group work is often a good way to start students off to become independent investigators. It is easier to have the confidence as a member of a group to partake in a joint investigation. Encouraging students to work on homework with other students is another way to help them get confidence.

Finally, written feedback is a mechanism to encourage students at the same time one pushes them. It is certainly appropriate for a teacher to give students different levels of feedback on their work. A teacher can be more critical of the reasoning and exposition of more sophisticated students. She or he can also put encouraging remarks on the paper of a student who is doing better than average work in terms of that student's past performance. Such remarks would not be appropriate for a student whose previous work was stronger.

In commenting on student proofs, real slips in logic cannot be tolerated. But if a teacher comments on all ambiguities and instances of poor grammar in the writing of some students, it will only be discouraging. So, a delicate balance must be struck between praise and criticism.

Students can learn a great deal about what is expected by seeing or hearing other students' work. Having students read answers to problems or present them to the class can give other students models of how they could improve. Again a teacher needs to worry about setting too high a standard if the only work that is presented to the class is highly excellent. By asking for volunteers to present solutions, a teacher can have a good assortment of work presented. Such an assortment can give students more attainable goals and help them respect their own work.

Do not expect miracles

If there were a 'magic bullet' for learning to write sound explanations and proofs, it would be widely available. It does seem that just as in any form of communication, it takes most people years to become adept at writing good proofs. Therefore, teachers need to think of the process developing over several years. Taking a long view of the process means looking for growth in students' work and not expecting them to write 'ideal' proofs.

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About the presenter

Diane Resek is a professor of mathematics at San Francisco State University in the United States. She is one of the authors of a 'reform' secondary school program in that country, the Interactive Mathematics Program. One focus of this program is on student communication, which includes the writing of justifications or proofs. In addition, she has written a text with Dan Fendel, called *Foundations of Higher Mathematics: Exploration and Proof*, which is used in university courses for preparing

mathematics majors to write proofs. Her interests at the university include teacher preparation and the foundations of mathematics.

Problem Solving Approach to Teaching Mathematics

Katrina Sims

This interactive presentation involves teachers in using a Problem Solving Approach to teaching mathematics. The PBS approach to learning encourages students to investigate a real-life problem by identifying what they already know from data given, what else they need to know in order to solve the problem, and how to use this information to solve the problem. Students learn Polya's four-step process to problem solving, the varying strategies for problem solving and how to implement these strategies appropriately. Participants will take away with them a teaching process they can apply immediately in their own classrooms.

'In problem-based learning, the learner is confronted with an ill-structured problem that mirrors a real-world situation, thus, drawing the learner into a complex reality' (Journal for the Education of the Gifted, p. 366).

A problem-based learning (PBL) science curriculum for high ability learners in Kindergarten through to Year 8 was developed at the Center for Gifted Education at the College of William and Mary, Williamsburgh, Virginia USA. I have adapted this model for teaching mathematics in Years 5 and 6.

This interactive presentation involves teachers in using this Problem Solving Approach to teach mathematics. With this approach a teacher presents a real-life problem for students to solve, using one or more concepts and skills she/he may be about to teach. By teaching computational skills concurrently with concepts students understand why these skills are needed and learn them more efficiently.

This process allows the teacher to cater for a range of mathematical abilities within the class, and is an effective tool to assess what the students already know and what they need to learn.

Students learn Polya's four-step process to problem solving, the varying strategies for problem solving and how to implement these strategies appropriately. Students work on real-life problems, relating what they already know, to what they don't know.

During this presentation participants will discover how easily this approach caters for different students' styles of learning. They will also learn how this approach assists students record the processes they use to reach an answer, a difficult skill, especially for rapid thinking young mathematics students.

I have been developing my particular application of this process to teaching mathematics for a number of years with outstanding success.

Participants will take away with them a teaching process they can apply immediately in their own classrooms.

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About the presenter

Katrina Sims has been teaching at Kaleen Primary School for several years, most of them with the Year 5 and 6 students in the School's Gifted Program. Katrina has a special fascination with mathematics and enjoys working with her students on their mathematical journeys, discovering new perceptions along the way. In 1999 Katrina joined the Australian Maths Trust Challenge Problem Solving Committee, as one of the Primary School Teacher members. When not surfing the Internet for exciting activities for her students she may be seen developing her skills with the video camera, taking various shots of her not so patient cats.

Characteristics of a Good Mathematical Investigation

Beth Southwell

Investigations seem to be the current preferred mode of teaching in the primary school. Participants in this workshop will be asked to investigate various mathematical situations in space, measurement, chance and data, and number, and to reflect on their experience in doing so. From this reflection, it is anticipated that a set of characteristics of a good investigation will emerge as well as some strategies for implementing an investigational approach in the mathematics classroom. Special consideration will be given to language, reflection and assessment and the role of investigations in national development.

Let's investigate

1. Square tiles

With the square tiles provided, investigate the shapes that can be made with 2 tiles, 3 tiles, 4 tiles, ... by matching sides.

Investigate the number of matchsticks needed to form these shapes.

Investigate which shapes made of five squares can be folded to make an open box.

2. Circle tables

Complete a standard multiplication table.

Write down the table and underline the units digit of each product.

Factor	0	1	2	3	4	5	6	7	8	9	10
Products ($\times 7$)	<u>0</u>	<u>7</u>	<u>14</u>	<u>21</u>	<u>28</u>	<u>35</u>	<u>42</u>	<u>49</u>	<u>56</u>	<u>63</u>	<u>70</u>

Represent the units digits on a circle numbered from 1–10 by drawing lines in the order specified.

Discuss the patterns that form and make conjectures about all the tables for 1 through 10 and the reasons behind them.

3. Digital sums

(Adapted from Bonsangue, Gannon & Watson (2000).)

Complete a standard multiplication table.

Write down the digital sum of each product in the table, e.g.

Products: ($\times 7$)	7	14	21	28	35	42	49	56	63	70
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Digital Sums: 7 5 3 10 8 6 13 11 9 7
 1 4 2

Investigate the patterns in the digital sums.

Represent their digital sum patterns on a circle numbered 1–9 by drawing line segments between points in the order specified.

Select one pattern, recopy it onto a larger piece of paper, colour it and give it a name for display.

4. Who jumps?

The tallest person jumps the furthest. Investigate.

5. Fair go

In a game with 2 dice and 2 players, points are awarded as follows:

- if a player throws a total less than 7, s/he gets a point;
- if a total of 7 or more is thrown, the player gets a point.

Play the game several times and investigate a winning strategy.

6. Quadrilaterals

On 5×5 grid paper, draw as many distinctly different quadrilaterals, one in each 5×5 grid, by joining points.

7. Happy numbers

Think of a number. Square each of its digits and add the squares to get a second number. Square the digits of the second number and add the squares to get a third number. Continue in this way to get a sequence. The sequence starting with 7 is:

7, 49, 97, 130, 10, 1, ...

If a sequence reaches 1 then the original number is called *Happy*.

Investigate.

Now let us think about what we have done? Here are some questions to help reflection.

Questions to consider

1. What process did you follow?
2. Did you make and test conjectures?
3. Was it easy to pose problems or extend the situation?
4. What learning took place?

5. How did you feel about the activity?
6. How valuable are these types of activities for learning mathematics?
7. What then are the characteristics of a mathematical investigation?

Possible characteristics of a mathematical investigation

Investigations are sometimes called open-ended tasks and this title could be taken to imply that investigations cover a range of openness and levels. This suggests that investigations will vary in methodology according to the learners' interest and background. When introducing students to an unfamiliar piece of concrete material, the teacher might just empty the container onto the floor or the desk and let the children explore them through constructing patterns and pictures. To stretch a point, the children subconsciously ask themselves the question, 'What can I make of these?' and so have posed a problem that they then set out to solve. Having made one pattern, they pose a further question of the same type. On a more formal level, a situation such as number 7 above calls for a more formal response process. The natural process that individuals might follow is simply to attack the most obvious elements in the situation and they may even consider they have finished the task once they have made some kind of a discovery. The inherent benefit of such activities as mathematical investigations is in their capacity to be extended and their dynamic nature. While problem solving is only problem solving and has a fairly static feel about it, investigations are problem posing and problem solving and have a much more dynamic feel. The goal of a problem is given in the problem itself and once the solver has reached an answer (or more than one answer, depending on the problem), the problem is finished. In an investigation, however, the learner sets his or her own goal and the conclusion of the investigation is limited only by the creativity of the learner. The characteristics of a problem are threefold, viz.:

- there is a goal to be reached;
- an obstacle is blocking the path to the goal;
- the learner is motivated to reach the goal.

The characteristics of a mathematical situation include these same three but only after some preliminary exploration enables the learner to establish the goal for him or herself and followed by any extensions the learner may feel interested in pursuing. How formal the learner makes the investigation will depend on age and experience.

One of the fundamental processes used in a mathematical investigation is *conjecturing*. Mason, Burton and Stacey (1982) claim that conjecturing is an essential process in mathematics. It is certainly the process that links mathematics most closely to the scientific method of inquiry from which investigations no doubt derive. This is supported by the *Principles and Standards for School Mathematics* (NCTM, 2000) in that it states that 'a conjecture is a major pathway to discovery' (p. 57)

Whether investigations are superior to problem solving as a methodology for teaching mathematics has not been firmly established, despite the fact that many researchers have made claims in this regard. The writer (1998), for instance, reported

on a small scale study of eight teacher education students in which she sought to discover whether problem posing actually aided students in problem solving. While no definitive results could be claimed with such small numbers, there were outcomes that indicated possible advantages to a problem posing approach. These outcomes were related to the number of solution strategies the subjects developed, the capacity of the problem attempted to employ visualisation and the subjects' preference for problems that have a practical basis.

This preference for work that is situated in real life was a result of a study in England also. Boaler (1998) reported a three year case study of two similar schools in which she compared the outcomes of two different styles of teaching. Mathematics in one school was taught in a traditional textbook oriented mode, while the other followed a more open-ended project approach. At the end of the three year study, Boaler concluded that the traditional approach led to what she calls procedural knowledge that the students could not transfer into other situations, whereas the open-ended approach developed in students much greater confidence and the ability to apply what they learnt to other situations.

Klein (2000, p. 136) writes that, 'Mathematical reasoning where learners explore, investigate and communicate abstract mathematical ideas and events, can be seen as a tool that builds a firm foundation of generalisations, patterns and connections' (p. 136). Steen, (1990, p. 338, cited in Reys, Suydam, Lindquist & Smith) also stresses the value of patterns and goes so far as to claim that to 'grow mathematically, children must be exposed to a rich variety of patterns appropriate to their own lives'.

Brown (1984) contextualises his discussion about problem solving and problem posing in a moral dilemma. His conclusion would support that of Boaler to a certain extent in that in problem solving, there is a clear result which can be obtained by following certain rules. Problem posing, on the other hand, looks for other solutions even when one might seem obvious and dominant. This is an argument to support the process of a mathematical investigation being a valuable methodology.

Reflection in mathematical investigations

One of the benefits of a mathematical investigational approach to teaching mathematics is its capability of encouraging students to think and take responsibility for their own learning. It causes students to reflect on what they are doing and learning. Reflection is seen by Boud, Keogh and Walker (1985) as the key to learning. They make the point that the act of reflecting is so common that it is often overlooked in education. For learning, however, the reflecting must be with intent — not just musing or day-dreaming — but with purpose directed towards a goal. Boud, et al. also say that reflection is a complex process which engages both cognition and affect and that negative feelings can cause barriers to learning whereas positive feelings can enhance learning. Accordingly, in their model of reflection (p. 36), there are three stages. The first is *Returning to experience*, the second *Attending to feelings* and the third is *Re-evaluating experience*. *Returning to the experience* is simply recalling the most important aspects of an experience, going over in the mind the events that are relevant. *Attending to feelings* calls for two approaches. One is focussing on positive

feelings by recounting good feelings and the other is expressing negative feelings. *Re-evaluating the experience* has four elements: association, integration, validation and appropriation. This third stage is the most important in the reflective process. Association involves the connecting of ideas and events, integration involves the synthesis of the associations into a whole body of knowledge, validation involves testing the new ideas for internal consistency and appropriation involves accepting this new learning as our own.

This idea of reflection in learning is not a new one. Dewey (1916) wrote about reflection as the process of making connections between parts of experiences. Schank and Cleary (1995) also claim that learning takes place through reflection. They give great emphasis to reflection as the process through which learners can promote and accomplish their own learning.

Resources in mathematical investigations

As Cockcroft (1982) said, the best situations for investigation arise in the classroom itself. These are the ones that children will find more interesting and profitable. This puts the onus on the teacher to be aware of and grasp the opportunities as they arise to engage students in meaningful investigations. This has several implications for the classroom. Being tied to a rigid timetable will not enable a teacher to ‘grasp the moment’ and develop it into an investigation. Then, too, there are difficulties that arise when team teaching or with parallel class timetabling arrangements.

Other sources of investigations exist. One is the development of investigations from problems. By gradually opening up the original problem, a teacher can generate investigations at different levels of openness. An example is as follows:

<i>Problem</i>	If every one of the 17 people in the room shakes hands with every one else, how many handshakes would take place?
<i>Slightly more open</i>	If everyone in the room shakes hands with everyone else, investigate the number of handshakes that would take place.
<i>More open</i>	Investigate what would happen if everyone in the room shook hands with everyone else.

A third source of investigations is to make them up or get them from books. Careful selection is then necessary to ensure as close a match as possible to the students’ interests.

Implementation

The process of investigation is natural for very young children but somehow as children proceed through school they become less and less willing to risk their peers’ or their teachers’ displeasure. Therefore, it is not easy for them to move into the investigational approach without being gradually led to it. A wise teacher will provide opportunity for students to ask questions (K–6 Mathematics Outcomes and

Indicators, 1998) on a range of mathematical ideas. Pictures or diagrams may well be stimuli for students in posing problems. When students are comfortable about posing problems, the situations might be slightly formalised before moving on to the more formal investigations. This, of, course, will need to be varied according to the level and ability of the class concerned.

Assessment of mathematical investigations

When investigations are so open, how can they be assessed? Clarke (1992) has provided a possible strategy for assessment. It seems to me that investigations cannot be assessed in the same way as an arithmetic exercise. An alternative strategy is required. One such strategy is the four dimensional approach suggested by Clarke (1992). These four dimensions are mathematics, strategies, structure and personal (p. 39). Particularly where group work is used, we must take into account the interactions within the group as well as the mathematical content explored, the strategies used and personal qualities such as perseverance.

Another strategy provided by Conway (1999) suggests that the qualities that need to be assessed are fluency, flexibility and originality. Applying these processes to the Clarke model would give more direction to the process dimension and therefore strengthen the total assessment.

A further strategy that could well be used is self-report and self-assessment. Keeping a journal of the investigation is an ideal way of allowing the student to follow and record the investigational process. The teacher can then assess the depth of thought revealed in this way and subsequently help students who are struggling. A good way to assess the language used in an investigation is to use the Newman language kit or carefully examine work samples. The latter can be very helpful when they are thoughtfully examined, ideas recorded and communicated. Annotations can be used to identify areas of achievement and possible misconceptions. There are several other useful strategies included in the Board of Studies Support Document (1994).

Contribution of investigations to national development

There are many benefits in teaching mathematics through investigations. The following is a summary of some of these benefits.

1. It is more related to every day life and therefore more realistic for students.
2. Its dynamic nature makes it more motivating.
3. It is geared towards the learner's ability and interest.
4. It lends itself well to group work.
5. It allows for the learner's creativity.
6. It integrates diverse areas of knowledge and different processes.
7. It puts the emphasis on the learner's responsibility in learning.
8. The learner feels greater ownership of the process and knowledge.

National development is achieved through the effective decision-making of thoughtful citizens who can analyse a situation, thoroughly explore its elements and, with great sensitivity and understanding, determine the path that will lead to the greatest good. In an approach to teaching mathematics through investigations, students are enabled to practise these skills and understandings. They are encouraged to take control of their own learning, to take initiatives and, where necessary, risks, and to make responsible decisions based on a thorough examination of the facts available. They are challenged to persist in the face of difficulties and to maintain a high level of interest and motivation. These are all qualities or characteristics that will contribute to national development.

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About the presenter

Beth Southwell has taught mathematics and mathematics education at tertiary institutions for over twenty years and has research interests in problem solving, concept development and a broad range of curriculum areas. She has also been a consultant on mathematics education both in Australia and overseas including the Lao People's Democratic Republic. She is currently the Primary Publications Manager for the Mathematical Association of NSW and regularly presents papers at a range of professional conferences. So that she does not get lazy, she does enjoy an opera every now and again and has a fine collection of photographs that will take the next thirty years to organise.

Not Another Worksheet!

— Activities for Bright Maths Students

Jenny Tayler

Too often, talented mathematics students find that their reward for being quick on the uptake is to do more of the same. It is no wonder that many of our brightest students keep their light hidden firmly under a bushel. Surely these are the people we should be nurturing as the future shapers of Australia! This workshop highlights some ways in which the interest and talent of students who are gifted in mathematics may be engaged and stimulated. Bright young children's thirst for knowledge needs to be satisfied by meaningful experiences that challenge and build upon their current mathematical understanding. Genuine extension in mathematics is not about handing out next year's textbook.

Identifying gifted mathematicians

A working definition of giftedness:

... giftedness is *potential or demonstrated achievement* in any one of the following areas, singly or in combination:

- general intellectual ability
- specific academic aptitude
- creative or productive thinking
- leadership ability
- visual or performing arts
- psychomotor ability.

The United States Office of Education (Marland, 1972)

In the same publication, Marland defines gifted children as 'those who, by virtue of outstanding abilities, are *capable of exceptional performance*, and ... thus require differential programs to realise their potentials and contributions.'

This definition is generally accepted among educators, and is significant in its emphasis on *potential* as well as *demonstrated* abilities, and in its clear description of the *multifaceted* nature of giftedness.

It seems apparent that identifying gifted mathematics students, and subsequently catering for their special needs, is one of the many challenges of teaching mathematics.

So ...

- Who are the gifted mathematicians?
- Are there students who are gifted at mathematics who are not displaying their talent? If so, why?
- Are the gifted students those whose parents assure us they are gifted?
- Are they the students who always come top of the class in traditional tests?
- Is there a fail-safe diagnosis?
- Are they always the enthusiastic, diligent, self-motivated students?

Some subjective observations that may help in identifying students with special abilities in mathematics include:

- An unusually keen awareness of, and intense curiosity about, numeric information.
- An unusual quickness in learning, understanding and applying mathematical ideas.
- A high ability to think and work abstractly, and the ability to see mathematical patterns and relationships.
- An unusual ability to work with problems in flexible, creative ways rather than stereotypical methods.
- An unusual ability to transfer learned skills to new, unfamiliar situations.

Adapted from: Miller, R. (1993). 'Discovering Mathematical Talent'. *Gifted*, 80, December.

Further subjective advice can be obtained through annotated class lists, while test results and knowledge of previous performance can support an objective analysis.

Some classroom strategies

Strategies that are designed to meet the needs of *academically gifted* students in a mathematics classroom should be considered in the context of meeting the academic needs of *all* students in the class. However, gifted students may be unchallenged by activities designed for the so-called 'top' of the class. Negotiation with individual students can provide appropriate direction in 'twisting' activities in such a way that the bright students are genuinely challenged. The following strategies allow for engagement with mainstream students, but also can be individually modified to address the needs of the gifted mathematicians in the class.

- Graded questions
- Variety of learning style environments, such as auditory, visual, cooperative, individual, competitive
- A range of methods of inquiry moving from concrete to abstract
- Genuine problem-solving

- Open-ended tasks
- ‘Good’ questions
- Graded investigations or projects involving extended work
- Alternative assessment procedures that pick up on the preferred learning styles of students
- Developing mathematics task centres
- Computer-aided learning
- Appropriate video material
- Individual research projects
- Individual extension modules
- Short-term withdrawal programs for particular instruction
- Co-curricular interest groups

Some activities to try

Happy and Sad numbers

Pick any number between 1 and 100. Square each of the digits and add the result. Continue this process until ‘something’ happens.

What does happen (if anything)? How can you explain the result(s)?

Omar’s rope

Omar the rope-maker wanted to make a rope long enough to stretch around the earth (distance at the equator = 40 000 km). On completing the task, he found he had actually made the rope 12 m too long. Rather than cut off the extra, he joined the two ends together and summoned all of his friends and relatives to help him hold it at an equal distance above the ground, all the way around the earth.

Which of these could pass under the rope?

A. ant B. snake C. Omar D. elephant E. blue whale

Word holes

Think of a number (any whole, positive number) and write it down in words. Count the number of letters used, and write this number down in words. Continue until ‘something’ happens. What does happen? Why? Try another language!

Number holes

Think of a three-digit number with non-identical digits. Make (a) the biggest and (b) the smallest number from these three digits. Subtract the smallest from the largest.

Continue until 'something' happens. What does happen? Why? Is any generalisation possible?

Vacant seats

In a row of four seats, if two are occupied like this: $_ x _ x$, then the next person must sit next to someone. If there are five seats, and two are occupied like this: $_ x _ _ x$, then the next person must sit next to someone. Continue this pattern to find the *least* number of seats, s , that must be occupied in a row of n seats in such a way that the next person must sit next to someone. Express your result as a rule. Investigate the mathematical definition of your rule.

Variation on Pascal's Triangle

The positive, odd integers can be arranged in a triangular array as shown:

			1			
			3	5		
		7	9	11		
	13	15	17	19		
	21	23	25	27	29	
31	33	35	37	39	41	

By extending the triangle for a few more rows, investigate the patterns for:

- the sum of each row;
- the number of terms in each row;
- the first term of each row;
- the last term of each row;
- the middle term of each odd row.

In each case, write a general expression in terms of n , the number of rows.

Variation on Fibonacci

The familiar Fibonacci series of numbers, i.e. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89..., can be generalised algebraically by the series $x, y, (x+y), (x+2y), (2x+3y), (3x+5y) \dots$

Extend this series to a total of 12 terms. Find:

1. the relationship between the sum of the first 5 terms and the 7th term;
2. the relationship between the sum of the first 6 terms and the 8th term;
3. a generalisation for this relationship.

By considering three consecutive terms of a *numerical* Fibonacci series, find the relationship between the square of the middle term and the product of the outer two.

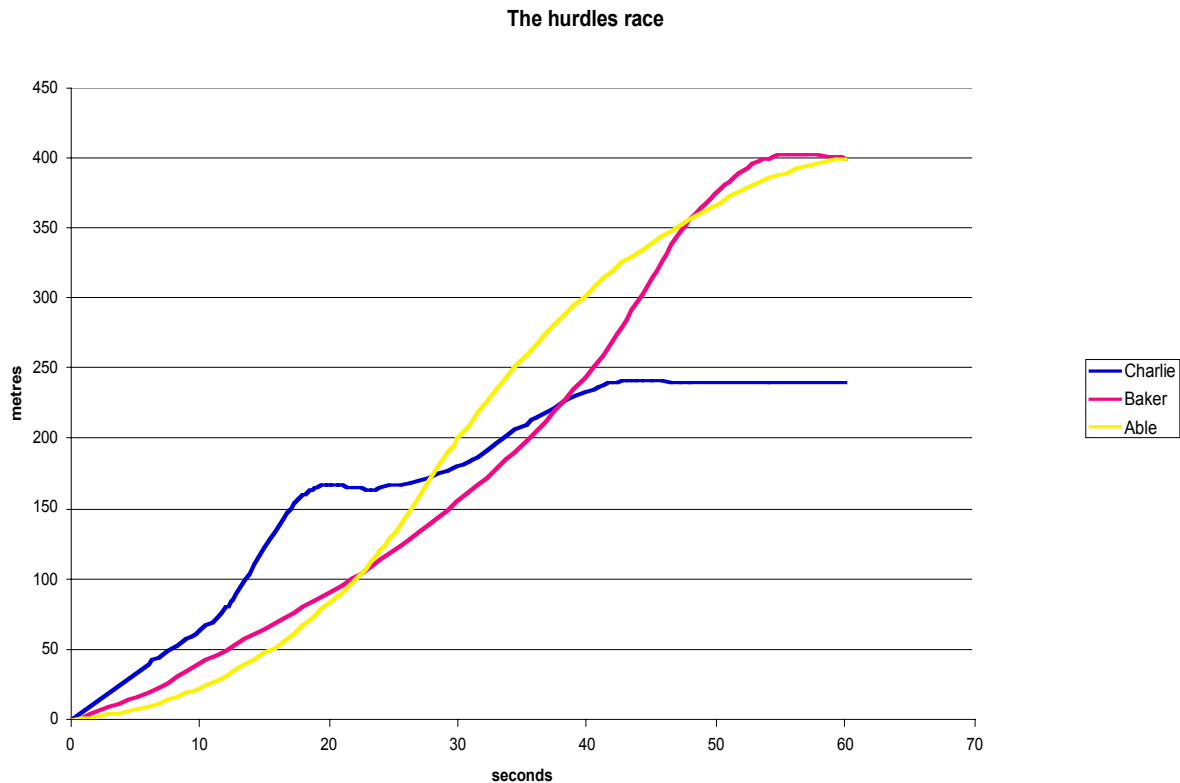
Express this relationship in reference to three consecutive terms of the *algebraic* series given above.

Some 'good' questions

- The average age of five people living together in a particular house is 33. Describe the occupants of the house.
- While watching television, I notice that the hands of the clock are making an acute angle. What program am I watching?
- Name some things that are north of you right now.
- I have 75c in my pocket. What combinations of coins might I have?
- I have 60 m of edging to go around a rectangular garden. What are the dimensions (or what is the area) of the garden?
- The area of a triangle is 24 sq. cm. What are the possible dimensions of the triangle?
- Write as many equations as you can for which the solution is 6.

Some open-ended tasks

- Which is the better fit: a square peg in a round hole, or a round peg in a square hole?
- I have two clocks in my house. One loses a minute each day, and the other one doesn't work at all. Which is the better clock?
- My friend's flat has five rooms, with a total floor area of 60 sq. m. Draw a plan of the flat, showing the dimensions and purpose of each room.
- Phil wanted to give his girlfriend a birthday present, but didn't know what to buy. However, in a magazine he read the results of a survey of girls of the same age as his girlfriend, which showed that 50% of them liked chocolates as a present, and 50% liked a 'Popstars' CD. Phil is pleased with this, since it means that he can buy either chocolates or a 'Popstars' CD and be 100% sure of a happy girlfriend.
Explain why Phil may or may not be right. Try to include diagrams in your explanation.
- The graph below shows a 400 m hurdles race with three competitors, Able, Baker and Charlie. Imagine you are the race commentator, and describe what is happening as carefully as you can.



An example of a graded investigation

Consider a typical, two-dimensional, 3×3 noughts and crosses grid. A player wins when three of the same symbol (O or X) line up vertically, horizontally or diagonally.

Using words and/or diagrams, explain your answers to the following as clearly as possible.

- How many winning lines are there on the typical grid described above?
- The game can also be played on a 2D, 4×4 grid. How many winning lines are on this grid?
- How many winning lines on a 2×2 grid?
- Summarise your results so far. Can you extend your findings to a 5×5 grid? How about 6×6 ?
- How many winning lines would you expect on a 20×20 grid?
- Can you write a general relationship for the number of winning lines on an
- $n \times n$ two-dimensional grid?
- 'Connect 4' is played on a rectangular, 6×4 grid, where a win consists of four like symbols lined up either vertically, horizontally or diagonally. How many winning lines are there in a game of 'Connect 4'?

- A three dimensional game of noughts and crosses is played on a $3 \times 3 \times 3$ cuboid. How many winning lines are there here?
- How about further cuboids: $4 \times 4 \times 4$, and $2 \times 2 \times 2$?
- Summarise your results for the 3D versions of the game so far. Can you extend to the next ones: $5 \times 5 \times 5$ and $6 \times 6 \times 6$?
- How many winning lines would you expect on a $20 \times 20 \times 20$ cuboid?
- Write a general expression for the result for an $n \times n \times n$ cuboid.

Some mathematical ‘jokes’

- ‘It is a well-known fact that if any New Zealander moves to Australia to live, the average IQ of each country will rise.’

Who is talking? Why is it so?

- ‘Did you know that 40% of road accidents are caused by drunken drivers? Hang on, if that is the case, then 60% are caused by people who are sober! Why don’t they get off the roads and leave the driving to us drunks?’
- Attributed to Dave Allen, on the eponymous TV show:

A paranoid air-traveller asked the airline staff the chances of a passenger bringing a bomb on board the aircraft. ‘About one in a million,’ was the response. ‘How about two passengers?’ further asked the traveller. ‘Much less likely — about one in a billion!’ replied the crew-member. So the traveller took a bomb aboard, just to be safe.

Try to unravel the logic (or lack of it) in these stories. Use as much mathematics as possible in your answer.

Resources for further activities

Journals

The Australian Mathematics Teacher. AAMT.

Australian Primary Mathematics Classroom. AAMT.

Talented. Ed. Stan Bailey, UNE.

GIFTED. NSW Association for Gifted and Talented Children Inc.

Gifted Child Quarterly. National Association for Gifted Children, Washington D.C.

Books etc.

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Additional thoughts

Resources from the Australian Mathematics Trust — Olympiad solutions, Toolchest, Mathematical Challenge activities.

Kanevsky resources — units on Maths, Science, Environment, Computers etc. from UNSW.

Mathematics competitions — AMC, Canadian and others.

About the presenter

Jenny Tayler is currently the Principal Education Officer (Gifted Education) for the Tasmanian Department of Education, a position she has held since making a long-anticipated lifestyle move in late 1999. Although the focus of her work is with gifted students generally, her extensive background in mathematics education as a secondary teacher, consultant and curriculum writer has found her active in the mathematics scene in Tasmania virtually before her feet touched the ground. She relaxes by getting her hands dirty in the garden and gazing at her recently acquired beautiful view.

Listening to Children's Learning — Using Conversation to Shape Children's Understanding of Mathematics

John Truran

The author has had twenty-five years' experience of teaching secondary school mathematics on a one-one basis. Such a teaching environment makes it possible to identify beliefs and understandings which often remain hidden within a large classroom, and also to provide situations which stimulate intellectual growth.

This paper reflects on conversations as a pedagogical approach, presents examples of different types of conversations, and proposes some criteria for good conversations. It then provides examples of conversations for participants to analyse and use as a basis for assessing their value within classroom practice.

This paper has three main parts:

- background remarks about the place of conversations in education and nature of authority in mathematics education;
- examples of mathematical conversations which will be used for analysis in the Conference session;
- observations on the place of conversations in classroom teaching.

While my early teaching experiences were as a secondary classroom teacher, but I have now spent many years as a freelance teacher of mathematics, some of which as a one-one teacher of secondary students. Most of this has been concerned with providing children with a deeper background and understanding of mathematics.

I have described elsewhere some of the strengths and weaknesses of one-one teaching (Truran, 1983). Here I want to use my experiences to discuss the special benefits of conversation as a teaching method. This is, after all, a way in which adults do much of their learning, and there is some evidence that this is a principle way for practising teachers to learn about teaching (Swinson, 1993).

Conversation has been extensively used in mathematics education research, where it is usually called 'clinical interviewing' and so there is a wide range of carefully thought out questions which are fairly readily available to practising teachers and which can help them to start conversations off constructively. There is some need for caution. Clinical interviews almost always try to find out what a child understands without doing any overt teaching, though merely asking a child to focus on a certain idea is almost certain to generate some element of learning. Such interviews focus on listening, rather than telling, and this is probably not a bad thing. The old adage 'start from where the child is' is as true as ever, though it is now dressed up in the

fancier language of Constructivism. So the listening constraints of clinical interviewing remain a good place to start conversational teaching, and this is what we shall do.

Conversations and authority

But first we need to address the issue of authority. Conversations in schools inevitably raise such issues, even in contemporary schools where relationships between teachers and most students are pleasantly relaxed. The ultimate authority in conversations about mathematics must always be the mathematics, though it may take time for children to learn this. Responses of peers, flickering eyelids of teachers, expressions of personal opinion — none of these is ultimately relevant to the learning of mathematics, though all may be important at various stages in the learning process.

In recent years much greater emphasis has been placed on openly encouraging children to construct their own mathematics. This pedagogically sound emphasis has been interpreted by some to mean that whatever mathematics children construct is good, regardless of how eccentric it may be. They have overlooked the fact that children often construct what Ritson (1998) has called ‘transitional conceptions’ on their path to obtaining secure understanding. Their view seems to have developed from one held by some Constructivists that there is no place for traditional teaching because, since knowledge can be constructed only by the learner, there is no knowledge to be transmitted. Such a Post-modernist position eschews all forms of traditional authority, and has been seen by Hargreaves & Fullan (1992, p. 5) as

... the dark side of the postmodern world: a world from which community and authority have disappeared. It is a world where the authority of voice has supplanted the voice of authority to an excessive degree.

It is important to make this point at the beginning. We live in a world where teachers will berate examiners for not accepting incorrect definitions which are in commonly used textbooks, and where those involved with comprehensive testing programs do not seem concerned (or are not adequately trained to appreciate) that some of their questions are mathematically incorrect. The failure of some of those involved in teaching mathematics to uphold a position that mathematics does have some external authority and is subject to reasonably good internal control mechanisms has done mathematics education little good in the standing of mathematicians, politicians or parents. So the conversations which I shall discuss here will be ones which are directed to the learning of mathematics as constructed by society as a whole but which are tolerant of the slips and false starts which all learners experience.

Some examples of conversations

In all of these examples questions predominate. But they are not the standard questions found in many classrooms. They are not questions for which the questioner knows the answer. As Ainley (1987, p. 25) has pointed out:

Asking questions to which you already know the answer is a very odd linguistic activity, almost entirely restricted to classrooms, or at least to teaching situations. In other circumstances it would rightly be regarded as bizarre, except as a conversational gambit (where it is not apparent to the person you are talking to that you already know the answer). And yet this activity is what is generally meant by 'questioning' children.

In the first two cases the questioner is seeking to find out what a student is thinking. Again, as Ainley has noted (p. 26)

[b]y asking questions you indicate what is of interest to you. When a teacher asks a question, she is drawing attention to those aspects of a situation which are important. ... Thinking time is essential if this type of questioning is to be effective

Do not many day-to-day adult conversationalists focus on what is important to them? And how often do we allow children time to think? So the third case illustrates children thinking together without adult intervention. Here some of their questions are ways of proposing suggestions or perhaps 'thinking aloud'.

A clinical interview

In the following interview an able twelve-year old boy was asked about the number of heads obtained when tossing 12 coins simultaneously. He had to answer by ticking one box out of several which specified various combinations of heads and tails and also 'all have the same chance'. Green (1983, pp. 549-550) set out to elucidate the thinking which underpinned the boy's answer. (In the transcripts used here 'I' will refer to the teacher/interviewer and other letters of the alphabet to the children in the conversations

- I What is the answer?
 D All have the same chance.
 I What does that mean?
 D Can vary a lot.
 I Will any [faces] occur more often than others?
 D Yes, 5-7, 6-6, 7-5 more than 2-10.
 I Which one most?
 D Those three about the same.
 I And 8-4?
 D About the same.
 I And 9-3?
 D A bit less.
 I And 2-10?
 D Very unlikely.
 I Which answer would you pick then?
 D All have the same chance.

I am still not absolutely sure what the boy meant by his answer. But I am sure that his understanding of binomial distribution probabilities is much better than I would have deduced by considering his written answer alone. This is because of Green's

excellent questioning which probed deeply, but without hinting what a ‘correct’ answer might be, and left the boy free to stay with his initial response.

A didactic conversation

The following is a reconstruction of a common conversation which I have with students. It is slightly abbreviated, but the gist of the approach is clear. It is, of course, easily usable in a whole-class situation as well.

- C $4 \div 0$ is nothing.
 I I see, Let’s consider this story. [Writing to record the conversation as he talks] Four robbers steal \$12. They share the proceeds equally. How much do they get each?
 C \$3.
 I How did your work that out?
 C $12 \div 4$.
 I Now. Δ robbers steal \diamond dollars. How much each?
 C $\diamond \div \Delta$.
 I Four robbers steal \$0. They get sprung. How much each?
 C $0 \div 4$.
 I And how much in reality?
 C 0.
 I So $0 \div 4 = 0$. Now [Watching child’s face carefully]. Zero robbers steal \$4. How much each?
 C [Silence]
 I Why did you make a strange face?
 C Because no robbers can’t steal \$4.
 I What did you think of my question?
 C It’s a bit silly.
 I What would you get if you used the rule?
 C $4 \div 0$.
 I So to ‘ $4 \div 0$ ’ is the answer to a silly question. In mathematics we use the term ‘undefined’ to represent situations like this.

This is a different type of situation. It is certainly didactic, and the teacher is certainly in control. But it is not Socratic teaching where a student has no options available and only the ‘correct’ reply is possible. Students do give other answers, particularly arising from confusion between ‘nothing’ and ‘zero’, so the conversation may need to be extended. But the defining authority of the teacher is only used *after* the student has demonstrated an understanding of the correct mathematics underlying the point which caused confusion. The teacher shows where the student’s new understanding fits into the language and usage of the mathematical world. It is the student who is doing most of the thinking.

Conversations like this which often rest on helping a child to see a contradiction in his or her thinking do not always work as the teacher would like. For example, when comparing ratios which form probabilities ‘children as young as 5 years possess a repertoire of strategies to select from in reaction to the type of ratio pairs presented to them’ (Way, 1997, p. 574). These strategies seem to be used idiosyncratically and subconsciously, making focused pedagogic conversation almost impossible,

although it may still be useful as a clinical tool. Such difficulties often arise in probability conversations because probability is a topic where counter-examples and contradictions are almost impossible to generate. In ordinary life there are times when conversants fail to gain purchase on each other's ideas and hopefully simply break off the conversation and agree to differ. Teachers, of course, remain responsible for teaching, but if the conversation is proving fruitless, then this is a clear sign that alternative approaches are needed. I do not go as far as some and claim that '[i]f [mathematics] is a logical subject then we should not, in theory, need to tell anything. It can all be deduced, given the right questions' (Smith, 1986). But I would suggest that there are often more constructive approaches than mere 'telling'.

A small group conversation

Reports about conversations between small groups are not easy to find, because they are technically difficult to record. The English journal *Mathematics Teaching* is the best easily available source of which I know. Here is an example of a conversation between four children who were working from a textbook on a question about costing something, a task which required several calculations (Jaworski & Hall, 1997, pp. 35–36, layout altered).

- E If we need the cost of the tiles, do we need the area?
 F What do we do with the area?
 G What if we find the area of a tile?
 F What about the bits at the edges? Do they [the tiles] fit?
 H [Having done a calculation] There's 12 down here and 6 across.
 E So we need 60 tiles.
 H, F & G 72!

Personally, I find this conversation a little threatening. Not, I hope, because I do not see any teacher in 'control', but because I wonder what has been learned by each of the individuals, How many are passive followers? What calculation did H do? How much was H's thinking influenced by the earlier comments? I also wish there were more 'cut and thrust' and I can see several ways in which it might be appropriate for someone (not necessarily the teacher) to join this conversation with some more challenging questions. All the same, considering it is only seven lines long, it does seem to be representative of purposeful interactions between students.

Some principles of good mathematical conversation

What principles of good mathematical conversation can we deduce from these examples and our own experiences? I submit the following list with no claims to its completeness.

- All participants treat each other with mutual respect.
- All participants are actively involved in making meaning of each others' contributions.
- Time is necessary for ensuring that all participants have understood what each one of them has been saying.

- Diversity of opinion is to be encouraged.
- Logical and mathematical consistency are the principal criteria for reaching agreement.
- A more experienced and insightful participant may sometimes be of special use for presenting (but not enforcing) important views which have not been considered by the others.
- The process should often be as intellectually demanding and challenging for teachers as it is for children.

Some unfinished conversations

Space permits only a few, all deliberately unfinished, examples to be used in the Conference session. The first two address common issues in algebra learning.

- I [Writes down] $y = 3x + 4$
If I gave this to you, what would you do with it?
- J Solve it.
- I Could you show me how?
- I [Writes down] $y = 3x^2$
What is the value of this expression when $x = 2$
- K 36
- I Can you tell me why it would not be 12?

The next example about understanding of chance comes from Piaget, the man who did so much to make the clinical interview a well-accepted research tool (Piaget & Inhelder, 1951/1975, p. 115).

- L By chance I did something wrong.
- I Why by chance?
- L Because it does not happen often.
Ah, if that happens often, then it is not chance.
- I Give me another example.
- L By chance, I am sick.

The final example comes from a discussion about children's control over the tossing of coins (Truran, 1985, p. 71).

- I Have you ever tried 'pleasing' a coin?
- M No. I tried pleasing a dice.
- I Did that work?
- M Sometimes it did and sometimes it didn't.

Using conversations within classroom teaching

Is a conversational approach to teaching compatible with the demands of classroom management? I am too far away from the standard classroom to make a definitive statement for today. But in the classrooms I visit I see many interactions between teacher and student which are long enough for working through the sort of conversations I have illustrated here. In some classrooms I see more than enough time available for students to interact with each other. But many of the conversations

I see are largely one-way ‘telling’ — manifestations of what some have called ‘teacher lust’.

I am only too well aware of how much university teaching is mere ‘telling’, and of how ineffective so much of it is. I am also aware of how often I fail to make my own teaching something more than ‘telling’, and of how ineffective this is too. Finally, I am aware of how much that teachers present to students of all ages is simply not learned by them, no matter how clearly and carefully it is presented.

Reading a journal like *Mathematics Teaching* will make it clear that many classrooms do work effectively on a basis of extensive conversations. At the 2000 MERGA Conference we watched a tape of Deborah Ball skilfully leading a class through a difficult discussion in a way which used most of the principles listed above. The transcript is not available, but Ball (2000, p. 7) pointed out to us that

[b]eing able to see and hear from someone else’s perspective, to make sense of a student’s apparent error or appreciate a student’s unconventionally expressed insight requires this special capacity to unpack one’s own highly compressed understandings [which] are the hallmark of expert knowledge.

This paper is arguing that it is worthwhile making the effort to do such unpacking. Listening to children and talking *with* them are privileged and challenging tasks which may be utilised to ensure sound learning.

Conclusion

The English roots for ‘teaching’ and ‘learning’ are different. But the Russian word ‘obuchennyi’ covers both learning and teaching and sees these activities as being ‘deeply inter-related in complex ways’ (Adler, 1998). English needs a word like this to encapsulate what it is we do when we use conversation as a teaching aid.

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About the presenter

John Truran has worked as a secondary mathematics teacher, a tertiary statistics teacher, a tertiary lecturer in mathematics education, and as a private tutor. His research has investigated the teaching and learning of probability and statistics, and aspects of the history of science. He has written textbooks, has been on the editorial board of *The Australian Mathematics Teacher* and is currently Associate Editor of the *Statistical Education Research Newsletter*. He is deeply involved in the international stochastics learning research community, and is particularly concerned about linking research with practice in mathematics education.

Data For Primary Classrooms: Making Shape Out Of Chaos

Kath Truran

This hands-on Workshop for teachers of Years 3–7 inclusive, focuses on classroom activities for the exploration and development of Data in the Primary classroom.

The aim of this workshop is to investigate strategies for collecting and recording and interpreting data for use with primary students. In it we will ask interesting questions appropriate to the collection of data with the emphasis going beyond simply collecting and displaying data, to questioning the implications of the data being investigated.

The basis of the workshop will consist of engaging in challenging classroom activities and an open discussion about the way the data activities undertaken fit within the primary curriculum. Some of the graphs that we will create are not often used in primary classrooms. We will begin by investigating, in small groups the many graphs and tables that can come from one simple activity.

A packet of M&Ms can provide the basis for a series of graphing activities, many of which may be new to middle or upper primary mathematics students. The obvious, simple object and picture graphs, bar graphs; the easiest ever pie chart without the difficulty of calculating area; a stacked bar graph using percentages of colours from a single, fun size packet will be created. Then data from each group will be pooled and compared. Which is the best graph and why, how much information can be found in each graph will provide the focus of discussion.

Stem and Leaf plots will be created using two non-threatening personal measures (hand length & height from heel to shoulder) then scatter plots comparing these two measures will be constructed. Again we will consider what these graphs tell us? Are they different? How easily can we access the data?

‘Skeletons’ made from coloured strips of paper will provide a very different representation of information about the height of everyone in the group.

It is generally easy to interpret the data that we have collected and represented. It is not always easy to interpret data collected and represented by someone else. We will look at some graphs that have vital information missing; and will try to ‘write the story of the data’. In other words we will start at the finished product and try to find the initial question that the graph answers.

These varied activities are intended to show that graphing is much more than creating a simple bar graph; that many graphs and charts based on the same information can be created and compared; that stem and leaf plots and scatter plots

can be demystified and presented to an upper primary class and that graphs represent data that can be investigated and interpretations made about the data they contain.

Linking the paper to the workshop

This is a background paper for those wishing to take part in the ‘hands on’ data workshop being offered at this conference. The paper:

- provides a brief survey of the teaching of data;
- provides a framework for teaching data;
- considers the place of data within the mathematics, and wider primary curriculum.

Probability and statistics are highly visible topics in the primary mathematics program. The shift of attention to these areas has been dramatic and reflects the importance which probability and statistics have in present day society. While providing a context for promoting critical thinking, developing number sense, and applying computation, statistics also provides links to other curriculum areas particularly Science and SOCE.

Because Data are all around us children are bombarded with real life data that can often provide a starting point for a valid investigation. There must be a reason for collecting data and much of the study of data in a classroom comes from the children’s own observations and questions.

Statistics is essentially a practical subject and its study should be based on their collection of data, wherever possible by the students themselves. It should consider the kinds of data which it is appropriate to collect, the reasons for collecting the data and the problems of doing so, the ways in which the data may legitimately be manipulated and the kinds of inferences which may be drawn.(Cockcroft, 1982)

Although not from a recent document, this statement provides a clear strategy for approaching any activity involving data, and almost certainly provides the basis of the later curriculum documents that came from it.

It is very difficult to handle data *to a purpose* unless you know why the data were collected in the first place. There must be an initial question which provides the impetus to collect the data; by using an appropriate question as the starting point, the rest of the process is seen as logical and purposeful.

However, unfortunately the collection of data specified by the teacher is often seen as the starting point, students are encouraged to record, process, and represent the data about which they may have little interest, with often attractive collection of graphs around the classroom being presented as the most important part of the exercise.

Frequently there is a misunderstanding that Data at the primary school level is best (and can only be) represented and classified by graphing. This is not the case. The writers of the *National Statement on Mathematics for Australian Schools* tell us that:

Data provides us with a powerful means of forming opinions and reaching conclusions quite different from those we would reach if we relied on for example 'authority' or 'hearsay' (Australian Education Council, 1991, p. 163).

Not only are these processes relevant, interesting and important for daily living they provide a form of real problem solving, a variety of techniques for collecting, presenting and making sense of data are simple and accessible to primary students. These same techniques can be applied directly to the daily world (Van de Walle, p. 418)

A suggested framework for data lessons

Statisticians work from a structured framework like the one defined by Wild & Pfannkuch (1999) which they have called the PPDAC cycle. Professional statisticians have the questionable advantage of someone else (usually a client) asking the question, thereby defining the starting point of the study. The steps in the PPDAC framework for them however, follow the same cycle as the modified version that would be used by primary students in order to come to a result:

- Problem — defining the problem, pose the question
- Plan the study — this includes defining a timescale
- Data collection
- Analyse the data
- Conclusion — interpret the results, reach a conclusion.

Defining the problem — posing the questions

This is the crucial element in any data-handling activity which gives it purpose and direction. The kinds of questions that might form the basis of investigations and give pupils the opportunity of handling data purposefully; such questions might arise from discussions or organisational problems like the following:

- How much shelf space is needed for lunch boxes/reading books/paints?
- How quickly could we get everyone out of this room if there was a fire?
- Do kids watch more TV than adults?
- Which kind of balls bounce highest?

It would be difficult to answer any of these questions without collecting data of some sort. Of course, we already have quite a lot of information stored in our memories as the result of our own experiences. As adults we may 'know' that tennis balls bounce higher than squash balls, we have past experiences to draw on.

Children, especially young children do not have this store of experience to draw on. Where do we find appropriate questions that can form the basis of classroom investigations?

- Questions already posed in other areas of the curriculum

- Usual mathematics activities
- Cross-curricular themes
- Children’s every day experience.

Sometime questions come from references and describe tasks that have already been done. There is an interesting study detailed in Mathematics — work samples (Australian Education Council, 1994, p. 128).

The task was that students should conduct a detailed analysis of their school to explain how their school used 927 kL of water last year and to prepare a report for the principal, School Council and Schools and Properties department.

The outcomes and summary content were given and examples of the children’s work shown, it was clear from the details that the class members understood what was required and knew what the task involved.

This study was read about by a class teacher at an Adelaide school and begun by the teacher’s students. However the task proved too complex for the children and it was never finished. The timescale and complexities of the task had probably not been considered. Sometimes reading about an activity done by another class fails to mention some of the difficulties that may be encountered.

Plan the study

It is often suggested that younger students begin with data that are close to them such as their own family or local data before they explore other data. Unfortunately this suggestion has led to a proliferation of boring and identical collections made by year after year of primary students. It has been my observation during school visits in recent years that this idea has not been thought about, and that the same data questions are provided year after year; graphs based on the following questions are found on the walls of many primary classroom walls:

- the colour hair/ eyes of members of the class
- the number of people in our families
- favourite canteen food
- colours of cars parked in the car park
- counting the kinds of traffic that pass the school in a given period.

Some topics can be too relevant to the point of intruding on individuals or groups of children in the class and causing embarrassment. A class graph showing height or weight of class members, my family, holidays and Christmas presents are some that spring to mind. Children can be sensitive about their height or weight, The cost of holidays may be beyond some families and when family composition is different from that of other children in the class this can lead to embarrassment also. Clearly, sensitivity is needed when choosing or supporting pupils in choosing and area of investigation

An alternative context for considering data-handling questions is the variety of everyday situations in which children and adults can find themselves. These are by no means formal investigations, but simply consist of the many brief encounters with data which crop up every day in conversation, on TV, in magazines. Not many primary children read newspapers regularly, but showing them the occasional interesting cutting can engage their interest and move them towards taking an interest in the wider world. Television provides a vast source of data which most children are exposed to every day. Much media advertising, particularly during the months of November and December, is aimed at the under-twelve market, and fairly sophisticated consumer education is needed to try to ensure that children are not coerced or misled.

Collecting data

For many investigations involving the collection of data there will be little argument about what needs to be measured. The question to be asked will make clear the variable to be measured. The questions that children ask are often ill-defined and have more than one possible solution. The question, 'How many buses will we need to take us to sports day?' for example, depends on many factors.

- Who and what need to be transported?
- How much space will be required?
- What is the capacity of available vehicles?

If the children have proposed the study they are more likely to find reasonable solutions. However, selecting the key factor or factors is not always easy. For example one school that I visited had carried out the activity involving counting the traffic that passed the school with a view to investigating the safety of the school crossing. Because there is no way to measure 'safety' the recording of traffic data that they made had little or no reference to the reason for collecting it. A simple graph is not going to answer a complex question even if the children have proposed the question themselves and are motivated to answer it.

One of the easiest ways to collect data in the primary school is to use a simple survey to get one piece of information from each student in the class. The resulting information will have manageable numbers, and everyone will be interested. A wider set of information, that may not be able to be provided by class members can be found by using a questionnaire.

Before you or your pupils collect and record data some organisation and planning are needed. Writing a questionnaire is a difficult task and while administering a badly planned questionnaire can sometimes be a valuable learning experience, pre often it is frustrating for all concerned and can cause disruption around the school. In fairness to teaching colleagues and pupils the conducting of a survey should be expected to give considerable thought to the purpose and organisation of any questionnaire they prepare.

Use of a pilot study can enable students to correct many of the more obvious problems and to anticipate what may be involved in processing their results.

Collecting data from a large population is too costly or time consuming to undertake with a whole class. Choosing a representative sample of the population allows us to investigate a range of characteristics of a whole population by looking only at a selection of its members.

Analyse the data

One way to analyse data is to do a calculation, such as finding the mean, median or mode. But perhaps the most useful tools for analysing data are graphs. Graphs are powerful ways of analysing and interpreting data and should be seen by students primarily as useful rather than decorative!

Once data has been gathered what are you going to do with it? Children with little experience of the various ways of depicting data will not even be aware of the many options available to them. Sometimes you can suggest a way of displaying data and have children learn to construct that kind of graph or chart. Once they have done this then they can discuss its value.

- Did this graph/ chart tell about our data in a clear way?
- Is there a better way to tell about our data?
- Compared to other ways of displaying data how is this way better?

Technology can be used to create very accurate graphs. The use of the computer or graphing calculator have provided us with many tools for constructing simple representations; it is possible to create several different pictures of the same data with very little effort. The discussion can then focus on the information that each format provides and comparisons can be made about how a pie chart for example shows the same information differently than does a picture graph.

Interpreting the results

In the Conference presentation we will carry out a series of experiments, the results of which will be shown as data. This will not be the end of the exercise because it is at this point that the most important aspect of the exercise will take place. We will interpret the results that appear in our data. We will not be satisfied with the physical appearance of the data; but will be investigating whether the data gives us an answer to the Problem that we began with. (This is why the PPDAC framework is a cycle). If it does, we will ask, 'Is this the best answer that we can find?'. If the answer is 'no' then we have to begin again!

If there is more than one representation of the data, we will ask:

- Which is this the best graph?
- Which graph tells us most?
- How easily can we access the data?
- Could we have done this more effectively?

Answering these questions is what Data is really about and the power of statistics becomes clear.

It is important for children to see that one set of data can be represented in many different ways to allow them to compare which data best represents the information that they have collected. The teacher who has many sets of representations at his / her disposal will be able to help the students develop the most important skill in graphing, that of interpreting the data and the way it has been presented.

Beyond mathematics

As was stated earlier, Data provides links to other curriculum areas and those that most readily spring to mind are Science and SOCE.

Science topics like:

- Is it getting warmer/wetter/winder compared with 20 years ago? 50 years ago?...
- records of weather patterns over a month

encourage collecting of data and its representation.

SOCE questions like:

- How we lived 50 years ago, with the comparisons of prices, food and clothing would lend themselves to representation by data as well as text
- What percentage of a population from some country's (Ethiopia, Papua New Guinea, Haiti ...) rural areas have access to safe water?

can all be answered by using data in some form.

Many children's story books also lend themselves to use as the basis for a data activity. Participants will be shown how one year 6 & 7 class used Pamela Allen's *Mr Archimedes' Bath* as a basis for a graphing activity. There are many other texts for which graphing is a logical extension. The *Very Busy Spider* by Eric Carle; *Phoebe and the Hot Water Bottles*, Terry Furchgott and Linda Dawson can also provide potential material.,

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About the presenter

Kath Truran has worked as a primary and a secondary mathematics teacher, and a tertiary lecturer in mathematics education. Her research work has investigated the understanding of probability by young children. She is involved in the international stochastics research community, and is concerned with the quality of classroom teacher preparation especially in mathematics.

Shaping Primary Teachers' Beliefs About Mathematics

Kath Truran

Considerable research has been done to investigate student teachers' beliefs and attitudes about mathematics and mathematics teaching and learning. This research has told us a great deal about the attitudes of students who enter pre-service courses. However, the question we need to ask may be less about the influences of attitudes and more about students' knowledge of mathematics. Implied in this is a concern about the quality of teacher preparation.

In this session I wish to discuss with classroom teachers their views about the training they received as students in preparation for mathematics teaching.

Introduction

Important curriculum and planning documents, for example the *National Statement on Mathematics for Australian Schools*, stress that:

...students should develop confidence and positive attitudes towards mathematics and ... should gain pleasure from doing mathematics and seeing its relevance (Australian Education Council, 1991).

This statement focuses on attitude, while the authors of the National Council of Teachers of Mathematics (1989) *Curriculum and Evaluation Standards* document focus on knowledge:

...teachers should have a range of knowledge about mathematical concepts and procedures and that teachers need this knowledge in order to decide how best to help their students learn mathematics.

These are appropriate aims for pre-service teaching courses. However, frequently we are meeting students for whom confidence in and knowledge of mathematics is a major issue.

In this presentation participants will discuss some of the issues that influence good teaching in primary schools in order to further inform the thinking of the presenter. Classroom teachers have definite views about the training they received at Teachers College or University. There are many indications that content is a major concern, but there are others like:

- cognition
- young children's ways of learning
- how older children learn
- the influence of diversity

- forces of change.

These should be part of an effective pre-service teaching course. Because this paper is a guide for my further work in this area I want to talk discuss issues like:

- What was your background before you began teaching?
- How well prepared were you to teach mathematics?
- How effective was the training for teaching mathematics that you received?
- What could have been added to, subtracted from the course to have made your course more appropriate for your needs?
- What have you done since your graduation to extend your knowledge of mathematics teaching?

Previous research

There has been considerable concern about the standard of mathematics knowledge and teaching in primary schools for some time. As long ago as 1982 evidence produced by the committee which compiled the Cockcroft report in the United Kingdom, made the claim in paragraph 679 that:

It is not surprising that some students start their training with fears about the teaching of mathematics and that training institutions should have difficulty in giving to some of these students the positive attitude of the subject and the confidence which are necessary if these students are to be able to teach mathematics well. It must therefore be a major task of those who train these students to establish positive attitudes to mathematics.

This report influenced research into students' attitudes to mathematics and its teaching. Australian studies reported a similar problem. Sullivan (1987) reported that about half of a sample of beginning teachers had negative attitudes to mathematics when entering teachers' college; and Watson (1987) found that about 40% of a cohort of education students felt uneasy and less than confident about mathematics.

The DEET report into the teaching of Mathematics and Science of 1989 observed that:

It was consistently reported to the panel that students entering primary programs do so with feelings of fear and anxiety, and with negative attitudes to mathematics. Teacher education programs will need to give special attention in courses to turn these negatives to positives. (DEET, 1989, p. 66).

As a result of this report funding was made available for the creating of specific mathematics projects as part of existing primary pre-service teaching courses. One developed at the University of New England will be discussed in more detail later in this paper.

The constructivist view that knowledge is constructed actively by learners as a result of their experiences has been interpreted by some to mean that there is no need for a teacher to be an expert in the subject matter. On the other hand (Shuck, p. 110) goes on to argue that there is a need for three forms of content knowledge essential for teachers:

- subject matter knowledge
- curricular knowledge
- pedagogical content knowledge.

Yet many recent researchers are more concerned with beliefs and attitudes. A study of beliefs and attitudes to mathematics of pre-service teaching students. One by Shuck (1999, p. 109–123) investigated:

- Beliefs and attitudes about mathematics and mathematics education prospective primary school teachers bring to their tertiary education.
- How these beliefs and attitudes affect the learning of mathematics and mathematics education in teacher education courses.
- How these beliefs and attitudes affect their ideas on good practice in the teaching of mathematics in the primary school.

Schuck's students valued similar approaches to those espoused by University of South Australia students, to be discussed later, particularly that maths should be fun and that mathematics should be taught in a way that encouraged the school student's active participation in learning. For example, use of practical activities, interesting and easy challenges, games and puzzles and group work. This approach to teaching mathematics was believed to lead to enjoyable lessons because the experience of having fun was believed to be what lead to learning, rather than the content of the practical activities themselves.

Shuck describes the belief that teaching is dependent on these things is like,

offering students a brightly wrapped but empty gift box. While the offering appears to be an enticing and attractive gift, when it is unwrapped and examined further it entirely lacks content. (p. 120)

In contrast the negative responses of the Cockcroft and DEET reports, and much of what was found by Schuck, a more recent study by Askew et al. (1997) sought to identify 'effective teachers' and to explore the knowledge, beliefs and practices of these 'effective teachers of numeracy'. Numeracy was defined by the authors of the report as 'the ability to process, communicate and interpret numerical information in a variety of contexts'.

This report considered that the teachers who were identified as 'effective' were those who thought about identified learning outcomes. The factors that were considered to have most influence on effective numeracy teaching were:

- The nature of knowledge of mathematics that teachers have.
- The kind of teaching orientation that effective teachers have — the highly successful teachers demonstrated a range of classroom organisational styles including whole class teaching, individual and group work.
- The knowledge of pupils learning and using this knowledge to inform and develop their teaching
- Teachers' continuing professional development.

By considering 'effective teaching strategies' the Askew study has been able to define effectiveness in teaching numeracy and to suggest specific teaching strategies engaged in by teachers trying to improve their teaching. It has also informed the planning of the mathematics education subjects at the University of South Australia.

A very recent publication which analyses the mathematical understanding of Chinese and American teachers (Ma, 1999). Ma's claims are that Chinese teachers involved in the study were far more likely to have developed a profound understanding of fundamental mathematics. They may have studied far less mathematics at school 'but what they know they know more profoundly, more flexibly, more adaptively'. Another attribute of Chinese teachers is that, like Askew's effective teachers, they show willingness to continue to learn. For American we can read Australian, as the education and working lives of American and Australian teachers are similar.

University of South Australia experience

A group of second year teaching education students at USA was asked to list what they saw as the characteristics of a 'good mathematics teacher'. The resulting lists produced these and similar statements about the qualities of a maths teacher. Their view of a maths teacher is someone who fits one or all of the following categories:

- Sugar and spice and all things nice
- Understanding, kind, encouraging
- Uses materials
- Lets the children work together
- Does not correct errors
- Makes maths fun
- Is enthusiastic.

At no point were there any indications by these students that the mathematics teacher should have a strong background in, or be confident about, mathematics.

These same students evaluated some mathematics lessons which they taught during a practicum. Few students mentioned observed mathematics confidence or skills on the part of the children, but did see as important that 'the children enjoyed the lesson'.

Like Schuck's (1999) students many of the University of South Australia students did not believe that good teaching was dependent on being knowledgeable about good mathematics; on the contrary, these student teachers believed that a strong knowledge of subject matter might cause them to lose their empathy with struggling school students. This acceptance of poor content knowledge is a serious barrier to change and contrasts strongly with the Chinese teachers investigated by Ma (1999)

University of South Australia dilemma

I had taught the second year pre-service teaching students described above in 1999 in their first year at University of South Australia. There was a group of mature aged students who were particularly vocal about mathematics learning and teaching; a few had not done mathematics past primary school, and were 'terrified of teaching maths'.

Others in this group regarded having to work through a semester of mathematics education a waste of time and at the end of the semester the formal evaluation of the subject compared the maths unit unfavourably with the Science unit (which had been done concurrently), because Science was 'fun', 'especially the Snail races'. It was interesting to observe the dynamics of this particular group of mature aged students and the influence they had on the attitudes of the rest of the cohort. This group was also angry and terrified about the prospect of having to do the PMPs in second year.

These students also claimed to prefer teaching junior primary students and arranged their teaching practice so that they got their experience with older students 'over and done with' in the shortest practicum; this was done in the belief that teaching mathematics to young children is easier than teaching it to older ones. Particularly cited were the difficulties of understanding and teaching operations like long division and operations on fractions.

At this institution the third year practicum is currently the longest and the most important from the point of view of future employment. Some third year students believe that if they tell me the year level and mathematics topic they have been asked to teach during this practicum I can tell them what to teach. They appear to see mathematics as a fixed and sequential body of knowledge and that the role of a teacher is to 'slot' the class into the appropriate place. My response that I cannot tell them 'what to teach' is sometimes seen as deliberate obstruction!

An alternative plan

The two members of the mathematics department of the University of South Australia are concerned about student attitudes and the mathematics knowledge exhibited by some students in order to overcome some of the difficulties that students have with their mathematics knowledge and attitude we are working to provide pre-service teaching students with mathematics skills and confidence which will make them better teachers of mathematics at junior primary and primary levels. This is a two phase approach targeting first and second year students which began in Semester 1, 2000.

First year cohort

I teach the first year subject and replanned the first year subject because of the perceived need to develop students' understanding of the pedagogy of primary mathematics. Lectures are off-set with the workshops so that the lecture for a topic is presented the week before that topic is investigated in a workshop. Workshops are designed to engage students in participatory, relevant and enjoyable mathematics

activities. Weekly workshops include a presentation by the students, working in pairs, present a mathematics activity to their peers. Peers engage in the activity and also allocate marks to the presenters.

Students are also required to present a summary of a mathematics journal article related to the activity topic: students are provided with a list of appropriate mathematics teachers journals from which they are expected to access their material. This strategy is designed to encourage students to access teachers' journals, to realise how much information is to be found in them, and to focus on recent research as well as practice described in the article.

At the end of each lecture a question relating to the ideas presented is given to the students for study. This question becomes one of a set of possible questions that will be asked in the examination at the end of the subject. Because examinations are viewed negatively by students. I worked with a Student Support lecturer to create a web site that students were encouraged to access. It presented strategies for answering each weekly question. We used headings like:

- To answer this question ask yourself the following...
- What is the lecturer looking for in your answer to this question?
- Let's look at what you are being asked to do in the question.
- Finally, is there a general statement you can make about this topic?

An on-line discussion was also maintained so that there was a different kind of interaction; from the usual face to face interaction between students and lecturers.

The response by students to this was uniformly supportive; they saw it not just as a system to help them pass an exam but a way in which their ideas about teaching the mathematics topics discussed in the subject could be further investigated and discussed. Students formed friendship groups to discuss the ideas behind the questions and to share strategies and resources for answering the questions. This was more than supportive preparation for an exam but an awareness was developing that there were important pedagogical issues contained in the readings and references that they had been directed to during the semester.

Second year cohort

When one member of the University of South Australia staff was course co-ordinator at the University of New England, Personal Maths Profiles (PMPs) were developed. These consist of a series of 4 assessment tasks; two in number, and one in each of measurement and space & graphs. Each task has 5 sections with 4 questions in each section; three of the questions are considered to be 'routine' and one is problem solving. A mark of 75% is required for mastery and the tasks have to be worked at until mastery is achieved. The PMPs are administered during one semester in the student's second year of the course.

Students are given help to prepare for each task and have access to examples of the kinds of questions that are on the papers. They often choose to work in study groups to support each other as they prepare for each test. Most students take these tasks

very seriously, while others regarded the process as a bit of a joke until they had to repeat a task 3 or 4 times to gain mastery.

In 1991 a similar program of using a four part criterion referenced test was used as part of a course for Early Childhood teachers at the De Lissa Institute at University of South Australia. Some of these students admitted that they had not tried to 'pass' the tests until the third test had come up and the students realised that they had to learn something. (J. Truran, 1993)

We are attempting to enable students to develop their own mathematical understanding, and to extend their own knowledge of mathematics and the way that it can be taught, and to help them to realise that there is more than one way to present an idea to a class or to work through an example of a topic like division.

Some conclusions and questions

Now what should we do next? We are trying to provide an environment where a knowledge of mathematical understanding is the first criteria for our subjects. We are also working to encourage student's continuing professional development by making them aware of the teacher's journals that exist and the materials that can be found in such journals. Would you have appreciated this during your training?

Investigation of recent research and a supported assessment process are designed to encourage students to reflect on the kind of teaching orientation that effective teachers have, and the range of classroom organisational styles employed by effective teachers. The PMPs administered in second year are designed to develop further students' confidence in mathematics and to provide experiences for working through the mathematics that they will need to know and understand before they begin to teach.

Listed at the beginning of this paper are the issues that it would be profitable to discuss today, after having heard about some of the difficulties experienced by students at our university and some of the techniques that are being used to try to support them

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About the presenter

Kath Truran has worked as a primary and a secondary mathematics teacher, and a tertiary lecturer in mathematics education. Her research work has investigated the understanding of probability by young children. She is involved in the international stochastics research community, and is concerned with the quality of classroom teacher preparation especially in mathematics.

Chaos and Disorder

Michael Wheal

Graphic calculators are ideal for providing a quick transition between numeric, graphical and symbolic representations of mathematical models. Tying together several big ideas is what many teachers dream about but few do because of the time, effort and knowledge required by our students if the exercise is to be successful. The HP38G graphics calculator can be used to suggest some of the iterative properties of the equation $f(x) = rx(1 - x)$.

Introduction

These investigations should be introduced graphically and numerically. It is then feasible to verify some conjectures and even make further discoveries using the algebra of quadratics and other polynomials.

It is standard practice to use the past and present to predict the future. Mathematically this can lead to sequences of iterated values and under some circumstances these sequences are quite sensitive to the initial conditions. Recognition of this sensitivity by Lorenz in 1961 was one of the seeds for the growth of chaos theory.

The logistic difference equation $y = rx(1 - x)$ is one of the standard vehicles for introducing the Mathematical ideas of chaos theory. The values of this function can be considered to represent a population as a fraction of some limit. The domain is chosen to be $0 \leq x \leq 1$ whilst r is chosen to such that $0 \leq r \leq 4$. This will ensure values generated lie between 0 and 1.

The parameter r and the variables x and y can be considered respectively to be the growth factor, the present population and the next value of that population. As r takes larger values the population has the potential to grow by greater amounts, but with a limit on resources available for growth, if the population is too great, the amount of growth will be restricted and even negative. The term $1 - x$ serves to limit population growth.

It is the sequences of iterates $x_{n+1} = rx_n(1 - x_n)$ which exhibit either orderly or chaotic behaviour.

In an orderly sequence there is some pattern in the numbers which are generated: they either converge to a single value or produce a cycle, a sub-sequence which is repeated endlessly.

In a chaotic sequence the numbers have no periodic properties.

Convergence to fixed points

Press ON

Press ON, SK1 and SK6 simultaneously

Press SK6 (OK) to clear the memory.

Instructions

In the HOME screen press 2.8, SK1 (STOre), A...Z and * and finally ENTER to store 2.8 as the value of R

Press LIB

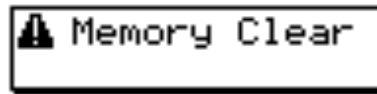
Highlight Sequence

Press ENTER

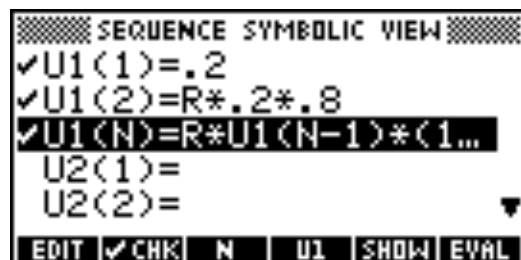
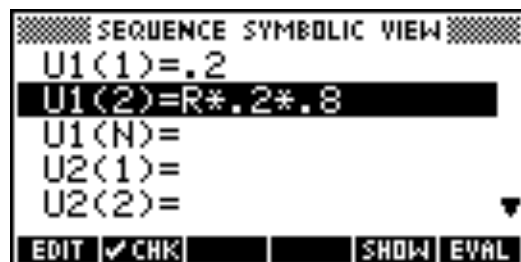
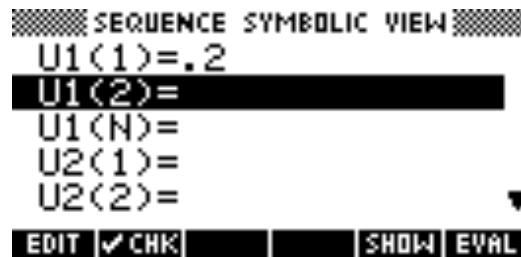
Enter a number between 0 and 1 for $U_1(1)$

For your chosen value of r between 0 and 4 enter $U_2(2)$ as $(r) \cdot (U_1(1)) \cdot (1 - U_1(1))$

With the same value of r enter $U_3(N)$ as $(r) \cdot U_1(N-1) \cdot (1 - U_1(N-1))$

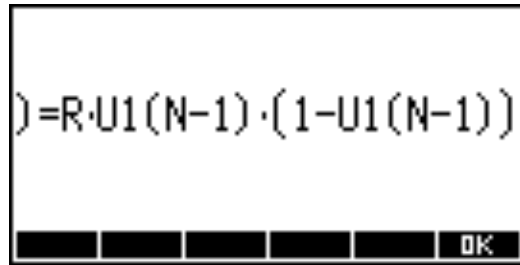


OK



Press SK5 (SHOW) to check the formula you have input.

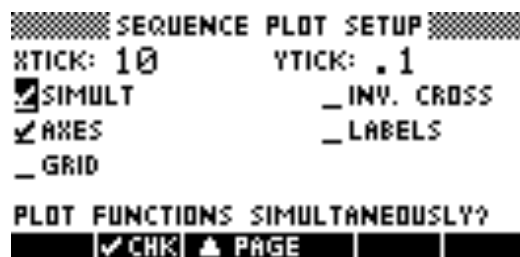
Press SK6 (OK) and then SK1 (EDIT) if corrections are needed.



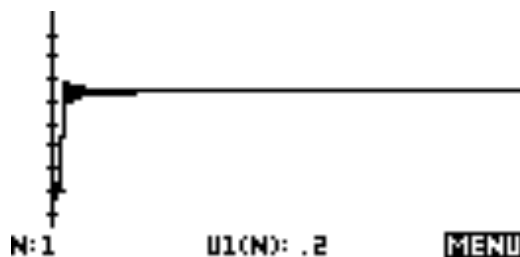
Press blue (shift) PLOT and enter the parameters shown.



Press SK4 (PAGE) to go to the next set up page.



Press PLOT to see the sequence graphically.



Press NUM to see a table of values of the terms of the sequence.

N	U1
1	.2
2	.448
3	.6924288
4	.5963192
5	.6740233
6	.6152045

1
ZOOM BIG DEFN

Scroll down to see other terms or with the highlight somewhere in the N column type in a term number, e.g. 50 and then press ENTER.

N	U1
50	.6428557
51	.6428583
52	.6428563
53	.6428579
54	.6428566
55	.6428576

50
ZOOM BIG DEFN

The sequence is settling down or converging to a value of approximately 0.6428571

This means that

$$2.8 \times 0.6428571 \times (1 - 0.6428571) = 0.6428571$$

With $r = 2.8$, iterates of the logistic difference function converge to a constant value.

N	U1		
200	.6428571		
201	.6428571		
202	.6428571		
203	.6428571		
204	.6428571		
205	.6428571		
<hr/>			
200			
ZOOM		BIG	DEFN

Figures 1 and 2 show $y = 2.8x(1 - x)$ and $y = x$ with iterations beginning at 0.2 and 0.95 but with both converging to a value near 0.65

The illustration of the iterations begins with an initial value x_0 on the x axis.

The vertical line segment meets the curve at (x_0, x_1) where $x_1 = 2.8x_0(1 - x_0)$

The horizontal line segment meets the line $y = x$ at the point (x_1, x_1)

The vertical line segment meets the curve at (x_1, x_2) where $x_2 = 2.8x_1(1 - x_1)$

In this way the cobweb diagram weaves from (x_{n-1}, x_n) on the curve to (x_n, x_n) on the straight line and then to (x_n, x_{n+1}) on the curve.

While the graph cannot provide great precision, the convergence appears to be to a value near that shown in the numerical view from the sequence applet.

The iterates of $x_{n+1} = 2.8x_n(1 - x_n)$ with $x_0 = 0.2$.

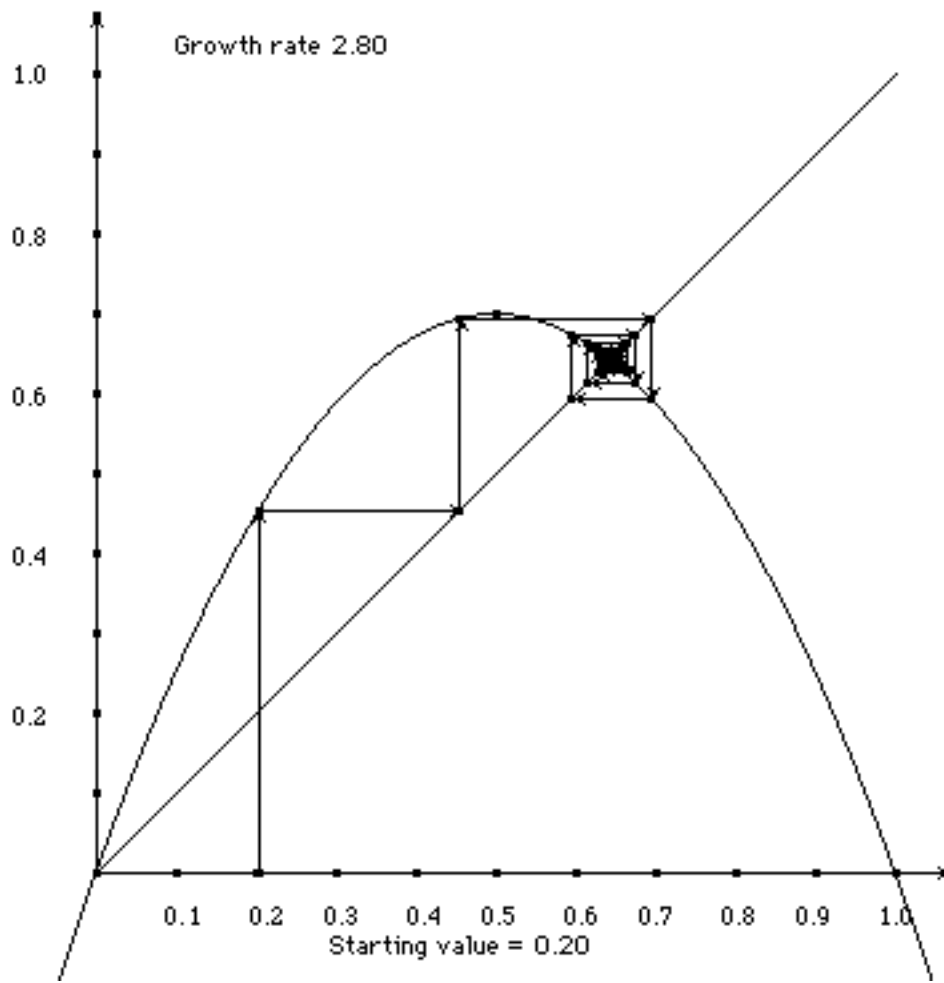


Figure 1

The iterates of $x_{n+1} = 2.8x_n(1 - x_n)$ with $x_0 = 0.95$.

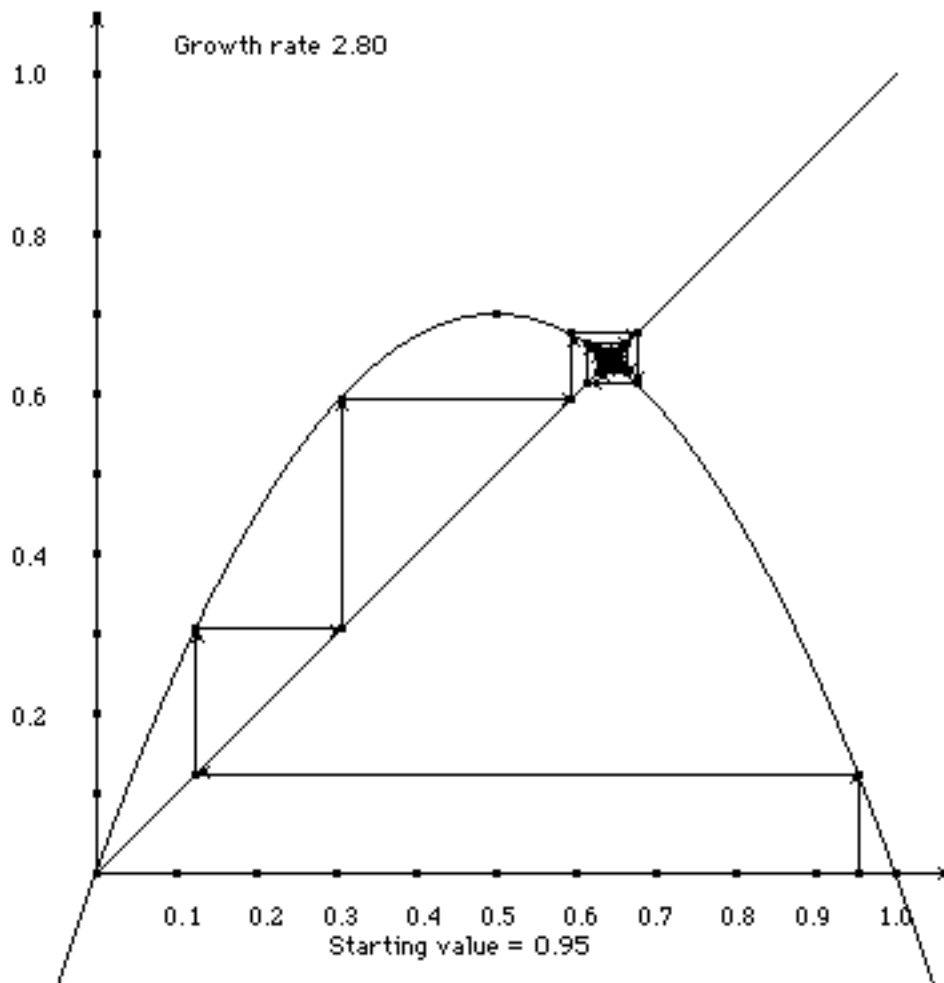


Figure 2

Activities

1. Solve $x = 2.8x(1 - x)$ by hand and find the exact values to which the population can converge.
2. Change the value of 2.8 in U1 successively to 2.9, 3.1, 3.3, 3.5, 3.7 and 3.9 and describe what happens in each case. You will need to change the value of R in the HOME screen and then EDIT the expressions in the SYMBolic screen by pressing SK1 (EDIT) and SK6 (OK): there is no need to do anything else.
3. Solve $x = rx(1 - x)$ by hand for $r = 2.8, 2.9, 3.1, 3.3, 3.5, 3.7,$ and 3.9 and find the exact values to which the population can converge.
4. What is inconsistent in your answers to questions 2 and 3?

The iterates of $x_{n+1} = 3.2x_n(1-x_n)$ with $x_0 = 0.4$

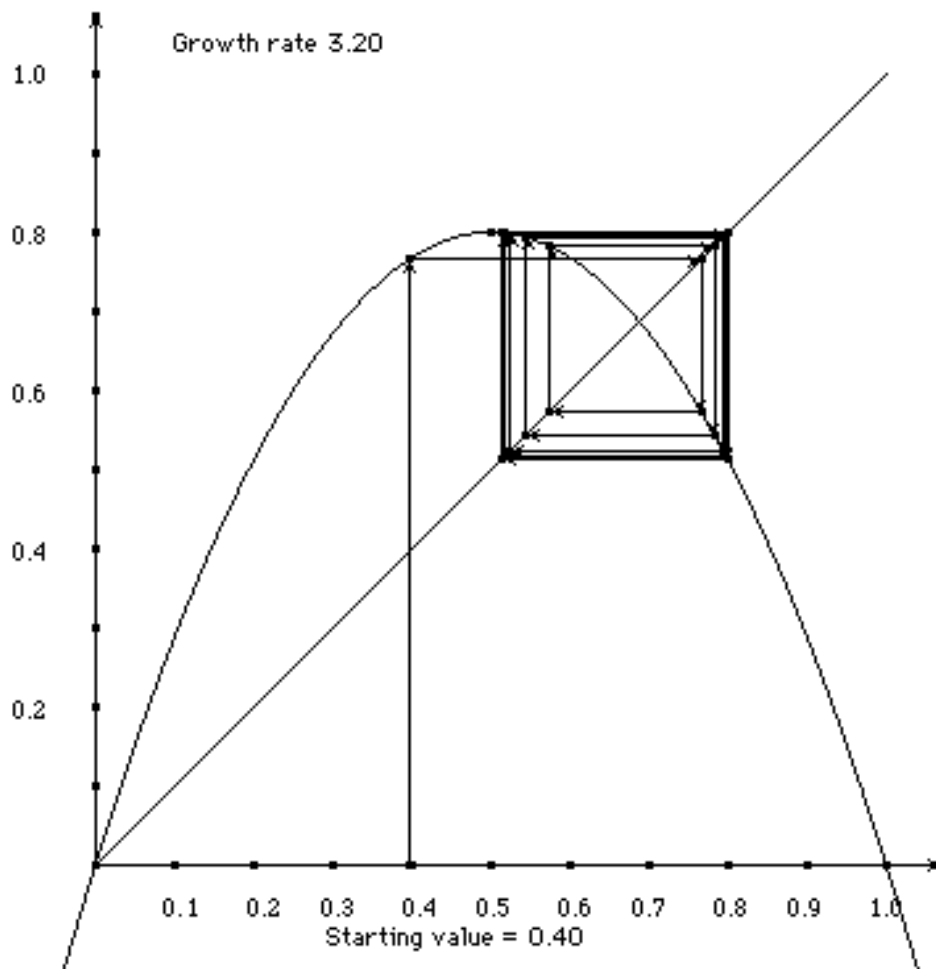


Figure 3. Convergence to a cycle of period two.

The iterates of $x_{n+1} = 3.47x_n(1-x_n)$ with $x_0 = 0.34$

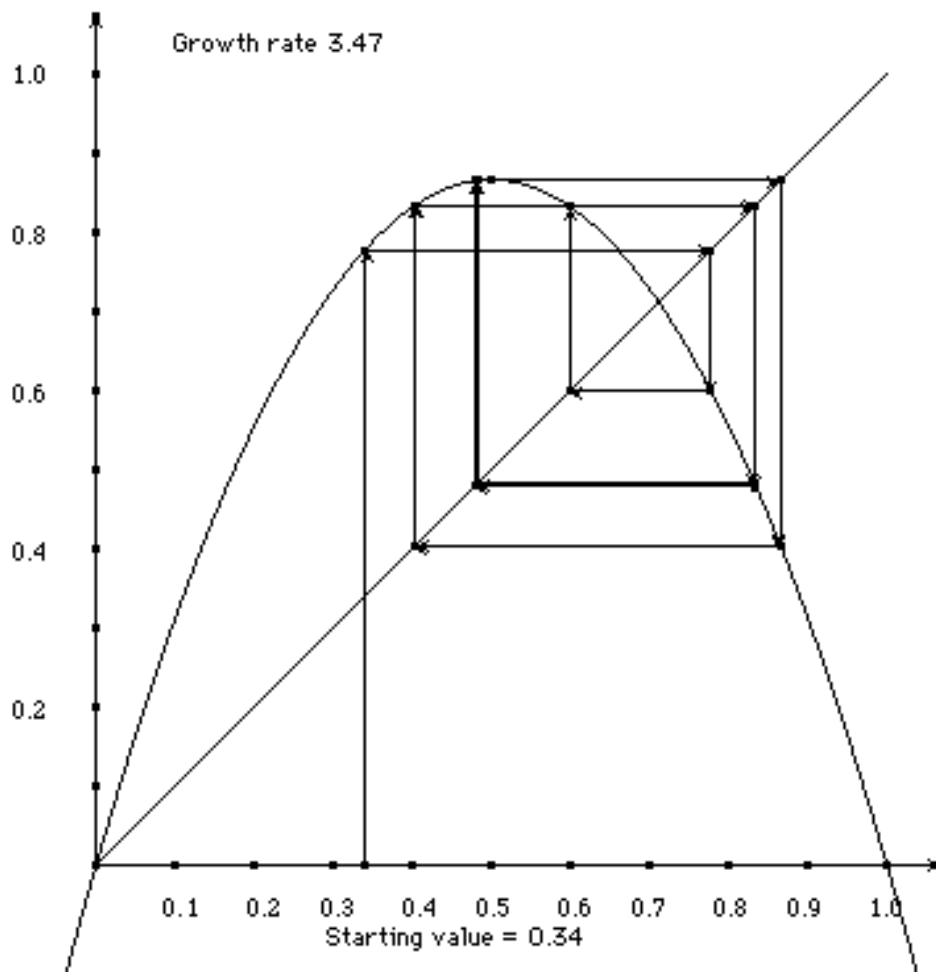


Figure 4. Convergence to a cycle of period four.

The iterates of $x_{n+1} = 3.69x_n(1 - x_n)$ with $x_0 = 0.32$

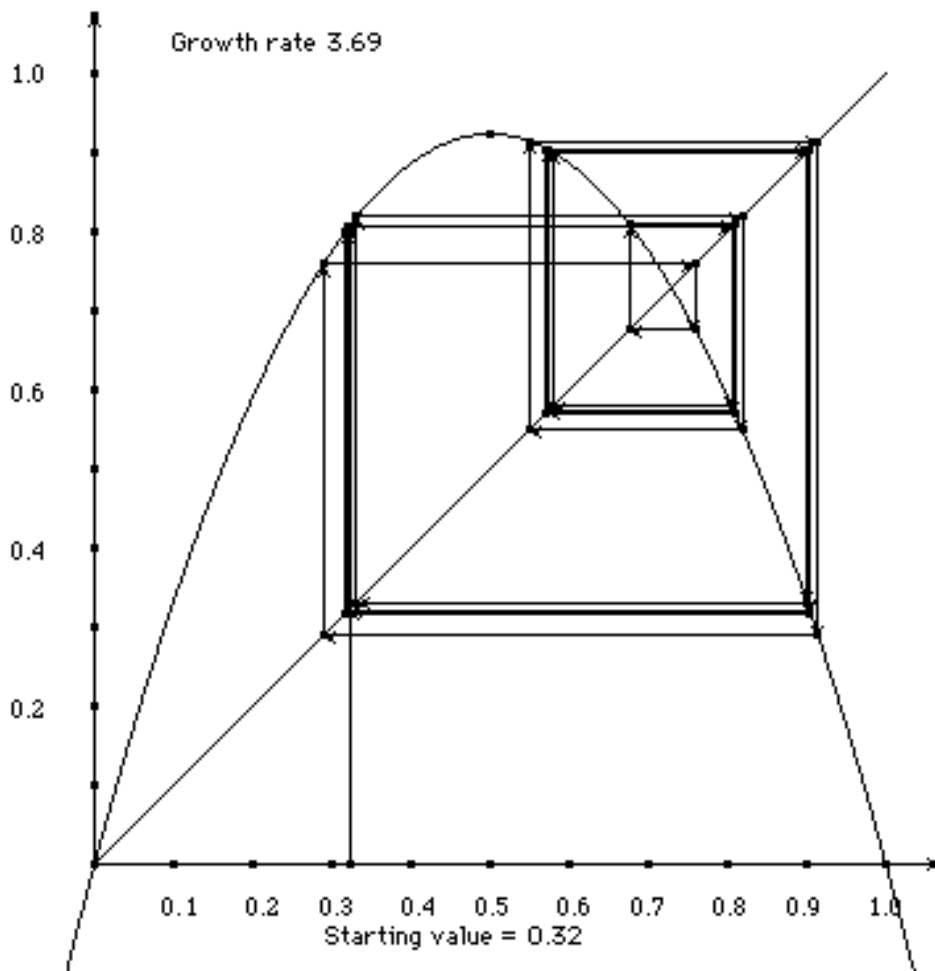


Figure 5. Convergence to a cycle of period four.

The iterates of $x_{n+1} = 3.85x_n(1 - x_n)$ with $x_0 = 0.31$

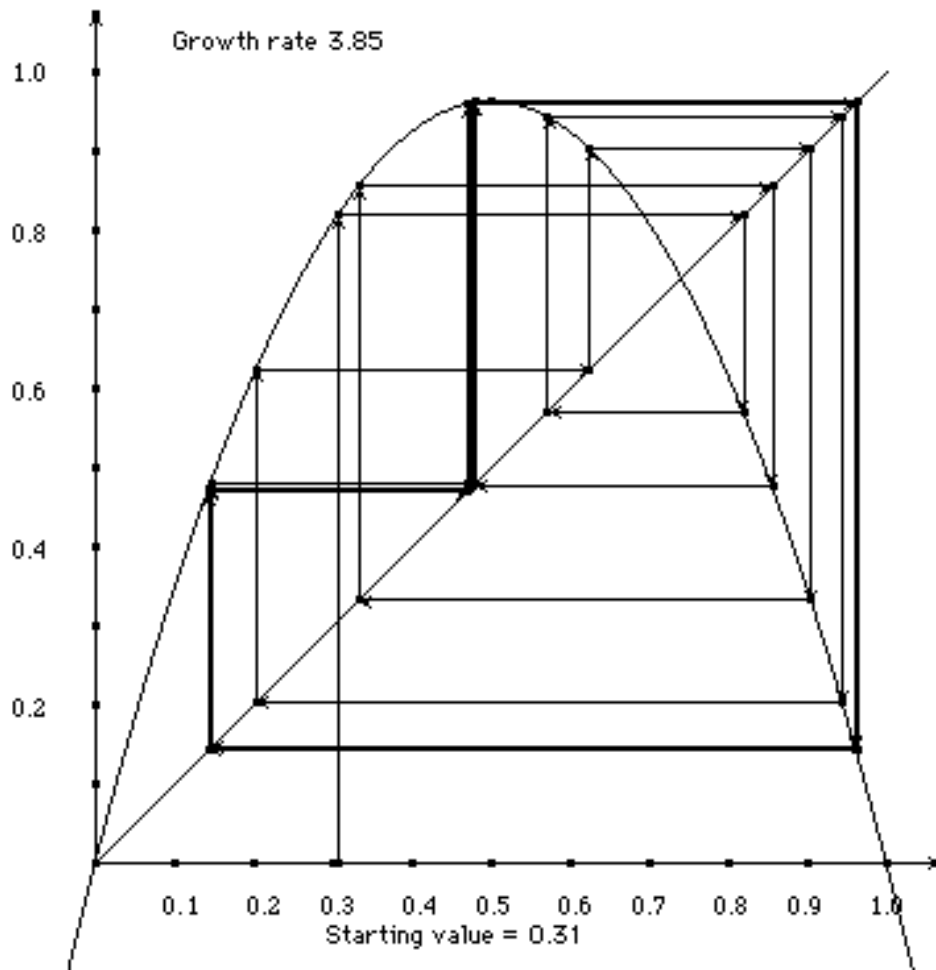


Figure 6. Convergence to a cycle of period three

Convergence to cycles

The iterates can be calculated and graphed using the function applet.

Remember that $x_2 = f(x_1)$, $x_3 = f(x_2) = f(f(x_1))$, $x_4 = f(x_3) = f(f(x_2)) = f(f(f(x_1)))$ and so on.

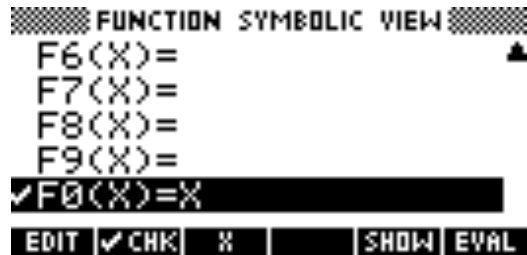
Whenever $x_k = x_j$ then a cycle has been encountered. The period of the cycle will be some divisor of $|j - k|$

Graphically, the intersections of $y = x$ and $y = f(f(f(...f(x)...)))$ correspond to the population values in the cycle.

Choose the Function applet from the LIBrary



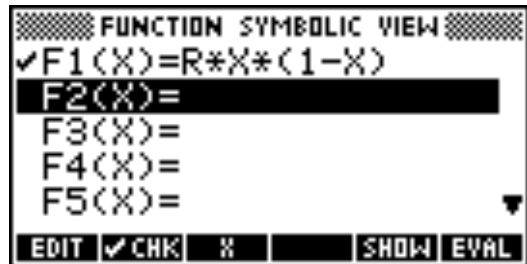
Set F0(X) to be X



Set R to be 3.3 in the HOME screen



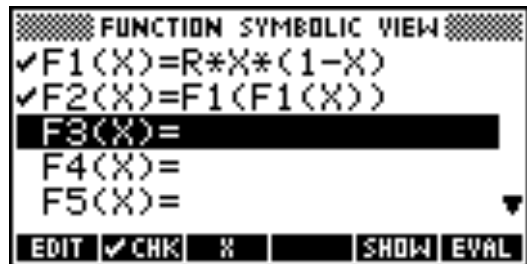
Set F1(X) to be R*X.*(1-X)



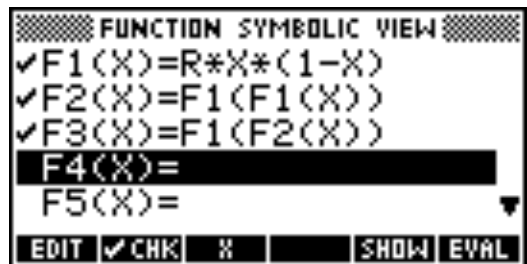
Set F2(X) to be F1(F1(X))

This is much easier than letting it be:

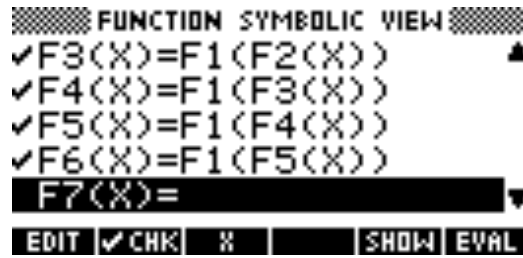
$$R*(R*X*(1-X))*(1-R*X*(1-X))$$



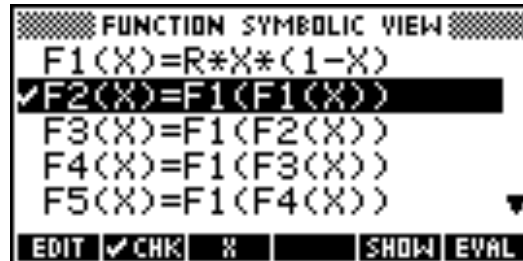
Set F3(X) to be F1(F2(X))



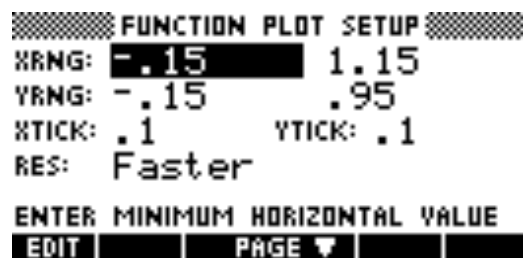
Continue and set F4(X), F5(X) and F6(X) as shown.



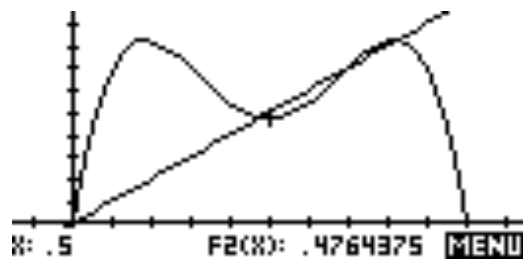
Make sure that only F0(X) and F2(X) are selected.



Press blue(shift) and PLOT and enter the parameters shown.



Press PLOT to produce the graphs of $y = x$ and $y = F2(X)$



The intersection points can be found using the TRACE and ZOOM features.

The graphs intersect near $X = 0.48, 0.70$ and 0.82

Another way is to use the Solve applet from the LIBrary.



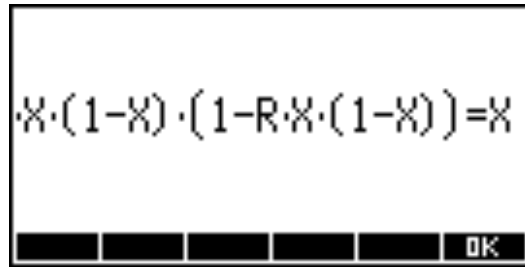
We need to solve the equation:

$$R * R * X * (1 - X) * (1 - R * X * (1 - X)) = X$$

Enter it and then press SK6 (OK)



To check that your equation is entered correctly highlight it and then press SK5 (SHOW) and scroll to the right.



Press SK6 (OK) and make any corrections.

Press NUM

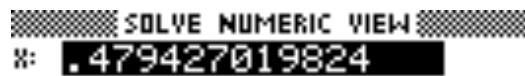
Input an approximate value for a point of intersection: the trace showed that they were near 0.5, 0.7 and 0.8



Press SK6 (SOLVE)



The value shown is the solution to the limit of the graphics calculator's discrimination.



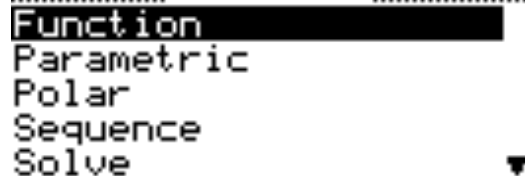
Inputting 0.7 and solving gives this value.



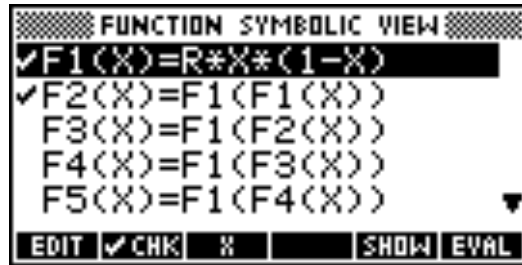
Inputting 0.8 and solving gives this value.



To see the effect of these numbers return to the LIBRARY and select the Function aplet

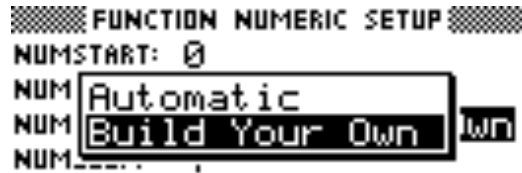


Press SK2 ($\sqrt{\text{CHK}}$) and select both F1(X) and F2(X).



Press blue (shift) and NUM.

Scroll down to NUM TYPE and press SK2 (CHOOSE).

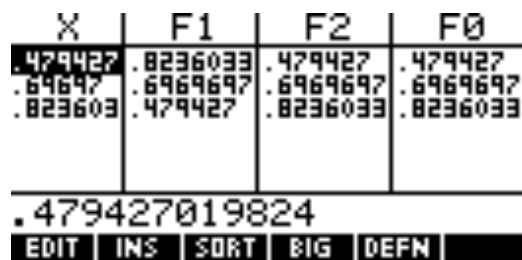


Highlight Build Your Own and press SK6 (OK) to select it.



Press NUM.

Input the solutions to $X = F2(X)$.



Activities

- Remembering that the solutions to $x = 3.3 * x * (1 - x)$ are also solutions to $x = 3.3 * 3.3 * x * (1 - x)(1 - 3.3 * x * (1 - x))$, find the exact values of the cycle of period two to which the population can converge.
- Change the value of r in $F1(x) = rx(1 - x)$ successively to 2.9, 3.1, 3.3, 3.5, 3.7 and 3.9 then find numerically and describe the solutions to $x = F1(F1(x))$.
- Solve $x = r * r * x * (1 - x)(1 - r * x * (1 - x))$ by hand for $r = 2.8, 2.9, 3.1, 3.3, 3.5, 3.7,$ and 3.9 and find the exact values of the cycles of period two to which the population can converge.
- Explain whether the value(s) to which the population converges depends on the initial population and/or the growth factor.
- At which value of the growth factor do cycles of period two seem to appear?

Seeking cycles with longer periods

The steps you have used in the previous part equip you to look for cycles with periods three, four, five and so on. The key to finding these cycles is the value of R , the growth factor.

Activities

While the following activities are numbered they should be seen as indicating parts of tasks rather than as tasks to be completed in order.

10. By trial and error try to find solutions to $x = F1(FN(x))$, checking that they are not members of shorter cycles. Finding solutions algebraically is difficult, even impossible.
11. Determine as well as you can an estimate for the members of the cycles.
12. Determine as well as you can an estimate for the value of the growth factor at which cycles of a particular period first occur.
13. Describe the behaviour of the graphs of $y = F1(F1(F1(...(x)...))$ and $y = x$ as the value of the growth factor increases.
14. Find some values for x_0 and r which appear to lead to very long, even endless cycles.
15. This investigation has been concerned with order, the opposite of which is chaos. What properties would you expect a chaotic sequence to have?

The iterates of $x_{n+1} = 3.84x_n(1 - x_n)$ with $x_0 = 0.33$

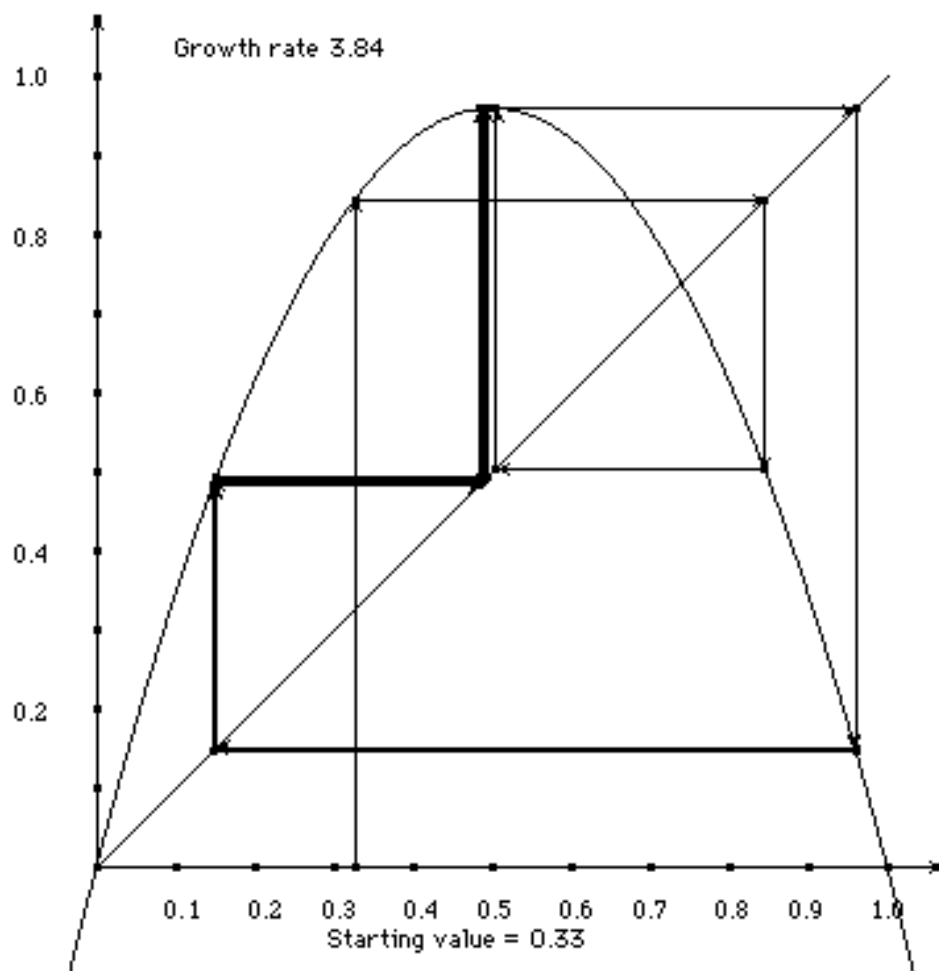


Figure 7. Convergence to a cycle of period three.

The iterates of $x_{n+1} = 3.91x_n(1 - x_n)$ with $x_0 = 0.63$

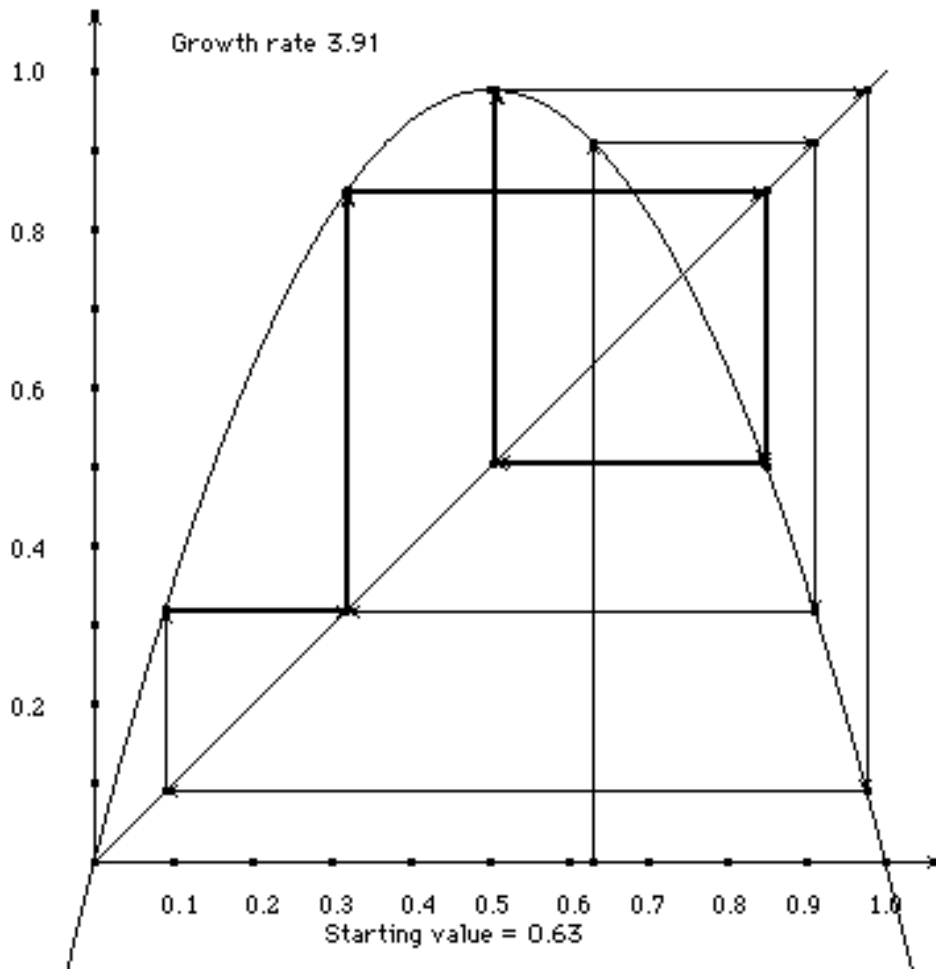


Figure 8. Convergence to a cycle of period five.

Some properties of the logistic difference equation

The logistic difference equation is the quadratic $y = rx(1 - x)$ where r is a parameter taking values $0 \leq r \leq 4$ which ensures that if $0 \leq x \leq 1$ then $0 \leq y \leq 1$.

For all values of r there are fixed points with values $x = 0$ and $x = \frac{(r-1)}{r}$ which can be shown by solving $rx(1 - x) = x$

Cycles of period two occur when $r > 3$ which can be shown by solving $f(f(x)) = x$

$f(f(x)) = rf(x)(1 - f(x)) = r^2x(1 - x)(1 - rx(1 - x))$ which leads to solving:

$$x(r^3x^3 - 2r^3x^2 + r^2(r+1)x - (r^2 - 1)) = 0$$

Note that $x = 0$ is one solution and $x = (r-1)/r$ will be another and hence $rx - (r-1)$ will be a divisor of the cubic.

Hence $(rx - (r - 1))(r^2x^2 - r(r + 1)x + (r + 1)) = 0$

Solving the quadratic gives:

$$\begin{aligned} x &= \frac{r(r + 1) \pm \sqrt{r^2(r + 1)^2 - 4r^2(r + 1)}}{2r^2} \\ &= \frac{(r + 1) \pm \sqrt{(r + 1)(r - 3)}}{2r} \end{aligned}$$

Since $0 \leq r \leq 4$ there will be real values of x only when $r \geq 3$

When $r = 3$, $x = (r + 1)/2r$ and it has the same value as $(r - 1)/r$

Graphically, as r increases towards 3, the dip in the graph of $y = f(f(x))$ becomes deeper until it is tangential to the line $y = x$: there is in fact an inflection point from a triple intersection. As r increases beyond 3, two of the intersection points diverge from $x = (r + 1)/2r$: these form the cycle of period two.

As r increases further, the dips in the graph of $y = f(f(f(x)))$ become deeper until again there is first tangential contact with the line $y = x$ and then distinct intersection. This occurs at about $r = 3.45$.

Further period doubling or bifurcation occurs when r is approximately 3.54, 3.564 and 3.569. When $r = 2\sqrt{2} + 1$, approximately 3.82843 a cycle of period three appears. This is particularly significant because it heralds the possibility of non-periodic sequences. Li and Yorke in their paper entitled *Period Three Means Chaos* published in 1975 were able to prove that if a mapping generated cycles of period three then not only were cycles of every other period generated but that uncountably many non-periodic cycles were also generated.

Shaping Mathematical Conflict in Australian Classrooms Using Peaceful and Humorous Means

Allan White

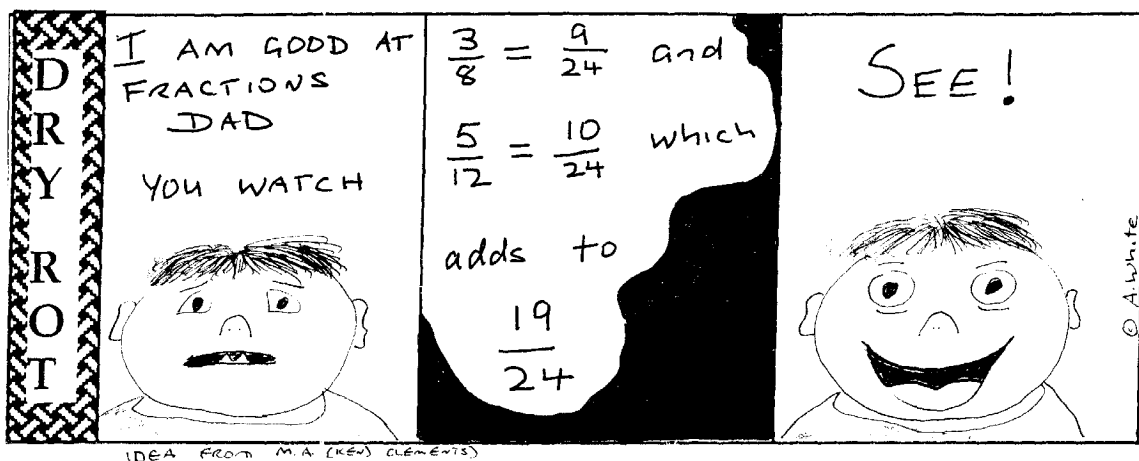
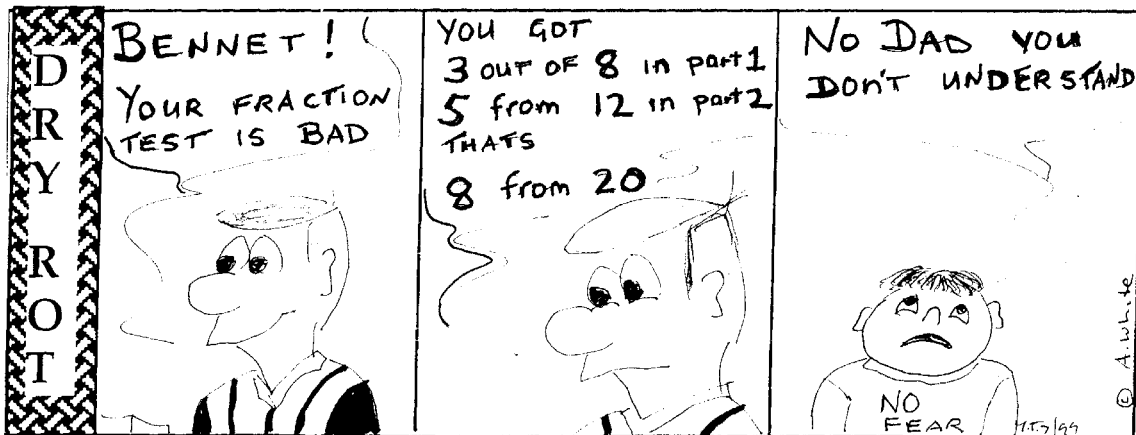
This is a hands-on workshop where participants can opt to work at their own level and pace, or remain as part of the main group. The workshop is suitable for teachers from primary to secondary. The presenter will encourage the investigation of common classroom misconceptions using humour via cartoons and their place in mathematics Grades 5–8. Issues covered in this session include cognitive conflict, research findings, humour, literacy, problem solving, working mathematically, and the Internet. Participants will have the opportunity to work with a set of worksheets suitable for use in the classroom.

There is a long tradition of using humour and cartoons in mathematics classrooms. A popular American journal titled *Mathematics Teaching in the Middle School* has a regular feature titled Cartoon Corner. The uses of cartoons in the teaching of mathematics are many. Cartoons have been used in the development of mathematical learning by challenging students' mathematical misconceptions (White, 2000 in press). Cartoons have been used: as a means of uncovering what students think (Fleener, Dupree, & Craven, 1997); as a stimulus for a discussion; as a springboard for posing mathematical investigations; and for fun, assisting in the creation of a friendly classroom learning environment (White, 2000). This paper addresses how cartoons can be used to provide a context for cognitive conflict and its peaceful resolution within a classroom that has a constructivist perspective.

It has been stated that constructivism 'is a popular position today not only in mathematics education but in developmental psychology, theories of the family, human sexuality, psychology of gender and even computer technology' (Noddings, 1990, p. 7). Thus it is not surprising to find that the Discipline Review of Teacher Education in Mathematics and Science (DEET, 1989) recommended the preparation of pre-service teachers should include constructivist strategies. Nor that, the National Statement on Mathematics for Australian Schools (AEC, 1991, p. 16) also recommended the application of constructivist learning principles in teaching practice. However, in spite of this popularity, there is still disagreement over the definition of constructivism (Ellerton & Clements, 1992), and what constitutes constructivist teaching (Simon, 1995).

It has been claimed that constructivism has a basis in the learning theories of Jean Piaget and Lev Vygotsky (Eggen & Kauchak, 1997). Piaget emphasised an individual's construction of knowledge and focused upon disrupting the equilibrium in order to promote learning. He stressed that the learner also needed to actively manipulate objects and ideas in order to learn. Vygotsky emphasised the social

transmission of language and culture and focused upon the need to scaffold and support individual learning. He stressed that the learner needed to be active in social contexts and interactions. While social interaction for Piaget provided the means for validating and testing schemas, for Vygotsky, it was a means for acquiring language and culture (Eggen & Kauchak, 1997). The use of cartoons in the classroom fits quite nicely with both researchers. The humour of the cartoon is the means by which disequilibrium or cognitive conflict is created (see Cartoon 1) and the group discussion of students' ideas provides the social context to encourage scaffolding.



Cartoon 1: How good are you at fractions?

Not surprisingly, other writers such as Helm and Clarke (1998) have suggested ways to optimise student cognitive engagement that resonate strongly with Piaget and Vygotsky. They listed four necessary conditions and the first referred to the context of the task being meaningful to the student. The Mathematics Destruction cartoon worksheets (See for example Cartoon 4) aimed to provide a context that encouraged student engagement. The second condition referred to the task recalling prior

personal experience. A cartoon could be a particularly successful means for evoking past connections and for provoking a fresh look at the 'taken for granted' aspects of mathematics. Often this shock value of the cartoon causes the disequilibrium and provokes a chuckle (see Cartoon 1).

The final two conditions were that the student should control the form of response and that the resolution of the task matters to the student. This is merely a matter of the teacher allowing the group to decide how they will present their reasoning, as the cartoon usually provides sufficient motivation for the task.

One issue that has attracted particular attention in the constructivist debate over teaching is the area of student errors. Student errors are a common classroom event and research suggests that something more is needed than just having the teacher carefully repeat the explanation of the correct method. The old formula of constant drill until the error resolves itself has been replaced by a range of strategies that concentrate upon the students' thinking and understanding of the underlying concept. These strategies usually involve the students discussing their thinking and various methods are used to encourage the discussion while managing the process. A common experience for the classroom teacher, however, is the difficulty that many students have in articulating their thought processes. The responses of 'I don't know' or 'I did it in my head' are commonly expressed by students. Cartoons can be used to assist students identify and explain strategies that they are using to solve problems.

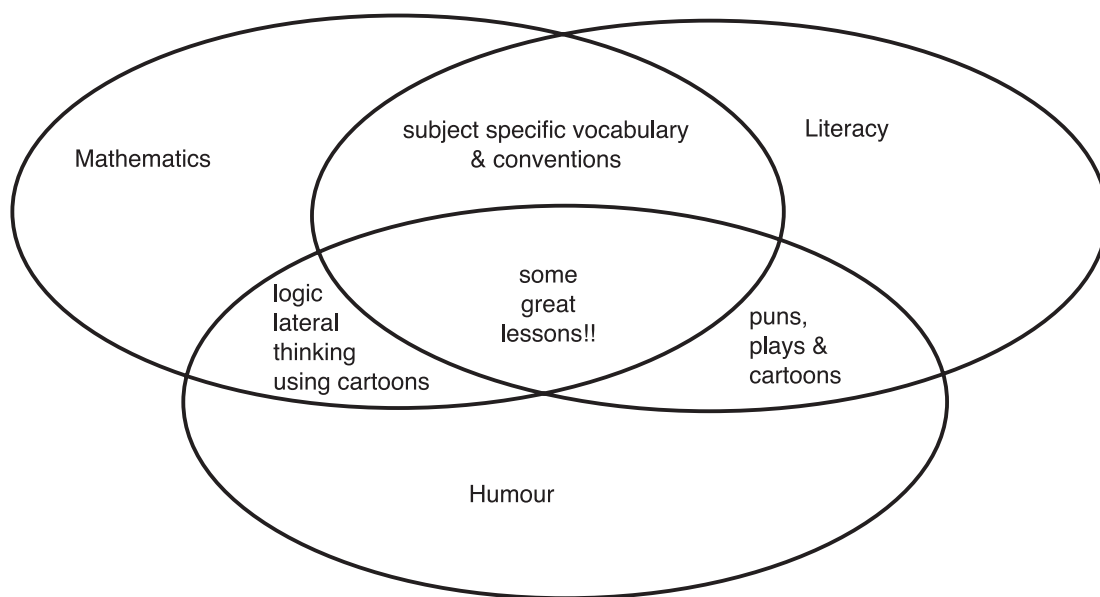


Figure 1. Mathematics, literacy and humour (adapted from Lannen, 1998)

There is another difficulty that arises from encouraging discussion of errors. It results from students investing some of their ego when solving problems. If the teacher explicitly exposes a student's error, particularly in a classroom setting, there is the opportunity for a negative effect upon the student's confidence and attitude towards

mathematics. A technique that has reported great success in overcoming this problem, concentrates upon getting the students to analyse other students' work, discovering errors and then giving the correct solution. It appears that students felt less threatened by their peers than by their teacher. However, there is still some 'loss of face' in front of a student's peers. It would seem logical that cartoons are less threatening as they replace the 'victim' or focus of attention. A student is able to change opinion or correct an error without the possible negative consequences

However, all is not rosy as there is a problem inherent in the use of cartoons, which classroom teachers would quickly identify. The English language proficiency of the students is a difficulty and I have included some examples in the following sections. Apart from the difficulties in learning general English usage, Mathematical English has its own structures where the word order is vitally important. As well, the humour in the cartoon is often a play on the meaning of a word. The resonance that exists between literacy, numeracy and humour is best expressed by means of a diagram (see Fig. 1).

In order to analyse students thinking it is interesting to consider work samples. Three cartoons were used on worksheets titled Mathematics Destruction, and spaces were left for the students to write their comments. While these sheets would normally provide the basis for a group discussion, in this case they are informative and interesting in their own right. The work samples that follow are those from a Grade 5/6 class at a large western metropolitan primary school and a small all girl Grade 7 class at a large inner city secondary school. The primary school children are predominantly from English speaking backgrounds whereas the secondary girls are exclusively from non-English speaking backgrounds (NESB). Not surprisingly, the NESB students struggled with the language and thus had difficulty in understanding the cartoons. This frustrated them and it showed in the responses, as one girl stated: 'Well it's not funny you should find new tactics or make them funnier. I don't get it that much so I can't explain it.' What was surprising was the quality of the answers from the primary students. A small selection of responses are included below.



Cartoon 2. Sharing drinks

Grade 7 NESB female secondary students

It is funny because the fat guy doesn't even know fraction.

This cartoon is meant to be funny because the boy on the right must of poured the water for them and doesn't know fractions. He had poured the water unequally into the cups, so one half is bigger than the other.

It's not funny. The guy who don't know how to do fraction told the guy who know how to do faction that he doesn't know how to do fraction.

Grade 5/6 primary students

If cartoon B knew his fractions he would realise that he doesn't have a smaller half but not a half at all. He has something that is more like a quarter. So this is why it is meant to be funny (drawing supplied).

Because there is no such thing as a smaller or bigger half. 1 half is always the same as the other half (drawing supplied).

Surprisingly there was no primary student who couldn't answer this question and there were many answers that relied upon correctly drawing the fractions. The students often labelled the characters in the cartoons. Asking why something was funny was a distracter and was later omitted. An interesting answer that arose a few times due to my poor cartooning drawing skills was the following answer.

A's glass is skinnier than B's because B's glass is wider his half seems like its less but it's not really. They both have exactly the same amount in their glasses.

This answer would provoke a wonderful discussion over the issue of conservation of volume. I can feel Piaget smiling.



Cartoon 3. Decimal blindness.

Grade 7 NESB female secondary students

Again, the language was a problem. Many students just correctly set out a vertical algorithm to indicate the correct way of getting the answer.

The little boy haven't learn how to do decimals and just don't get the question.

It is funny because the girl thought he was blind instead of thinking that he have decimal blindness which means that most of the time he doesn't recognise the decimal point.

Grade 5/6 primary students

Once again most of the primary students were able to explain the mathematics contained within the cartoon.

When working with numbers and decimals you cannot add them like the decimal is not there. The numbers after the decimal are like a fraction of the number. So you always put the 3 under the 3 not the .6

The fat girl said that Craig doesn't see the point. The point of this question? Or the decimal point? That's what makes it funny. The last picture has two meanings.

18/10/00

12yrs

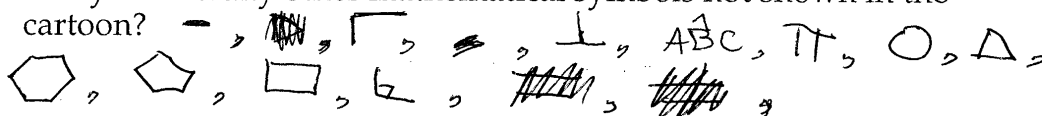
Mathematics Destruction Worksheet



Why is this cartoon supposed to be funny? How would you explain it to a friend or a younger brother or sister?

It's ~~not~~ supposed to be funny because he says symbol-minded and not simple minded. I would explain it by telling whoever it is by referring to the subject of mathematics and ~~its~~ its many symbols.

Can you find any other mathematical symbols not shown in the cartoon?



Cartoon 4. Symbol minded.

Year 7 NESB female secondary students

It's not really funny because its being mean and I don't get it that much so I can't explain it.

It is funny because you can't work out any mathematics problems with just mathematic symbols.

Grade 5/6 primary students

The primary responses were interesting due to the number of symbols that they invented mainly arising from their work in the Space strand (see cartoon 4).

Conclusion

In order to initiate change peacefully by considering and challenging students' mathematical thinking I feel I should prepare you for what you might uncover. As a work sample, I want you to consider the thinking expressed by the main female character in a novel by the famous Danish writer, Peter Høeg; additional comments of my own are shown in italics. To set the scene for this brief story, a man lives in a ground floor apartment and has been watching an attractive girl from Greenland pass his door each day. Finally, he has invited her in for dinner, and while he cooks she tells him of her love of ice, snow and mathematics. (The fact that the man is doing the cooking does not make this a fairy tale). It begins with the woman, who's name is Smilla, saying:

Do you know what the foundation of mathematics is?... The foundation of mathematics is numbers. If anyone asked me what makes me truly happy, I would say: numbers. Snow and ice and numbers. And do you know why?

Now how would you respond? Well, the man, he splits the claws with a nutcracker and pulls out the meat with curved tweezers.

Undaunted she continues,

Because the number system is like human life. First you have the natural numbers. The ones that are whole and positive. The numbers of a small child. But human consciousness expands. The child discovers a sense of longing, and do you know what the mathematical expression is for longing?

Obviously the strong silent type, he adds sour cream and several drops of orange juice to the soup. She continues,

The negative numbers. The formalisation of the feeling that you are missing something. And human consciousness expands and grows even more, and the child discovers the in between spaces. Between stones, between pieces of moss on the stones, between people. And between numbers. And do you know what that leads to? It leads to fractions. Whole numbers plus fractions produce rational numbers. And human consciousness doesn't stop there. It wants to go beyond reason. It adds an operation as absurd as the extraction of roots. And produces irrational numbers.

Overwhelmed by this insight he warms French bread in the oven and fills the pepper mill. She continues

It's a form of madness. Because the irrational numbers are infinite. They can't be written down. They force human consciousness out beyond the limits. And by adding irrational numbers to rational numbers, you get real numbers.

She notices that he is listening (although saying nothing), so she continues,

It doesn't stop. It never stops. Because now, on the spot, we expand the real numbers with imaginary square roots of negative numbers. These are the numbers we can't picture, numbers that normal human consciousness cannot comprehend. And when we add the imaginary numbers to the real numbers, we have the complex number system. The first number system in which it's possible to explain satisfactorily the crystal formation of ice. It's like a vast, open landscape. The horizons. That is Greenland, and that's what I can't be without!

Now what is a man supposed to do in such a situation, when he is confronted with someone bearing their soul and exposing their innermost feelings? In the novel he says, 'Smilla, can I kiss you?' (Høeg, 1992, pp. 121–122). However, an alternative, suggested by a friend, was to say, 'OK, Smilla, your number is up!'.

The use of humour and the mathematics classroom are not mutually exclusive. Cartoons are but one of the many means of adding humour to the classroom. What makes them particularly valuable is their ability to provide a non-threatening context where students can explore their errors in a peaceful and fun way. If you need more cartoons then *Square One* the primary journal of the Mathematics Association of NSW has a regular Dry Rot Cartoon Corner, as does the Cambridge University NRICH site in the United Kingdom. Go now and create conflict peacefully within your classroom.

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About the presenter

Allan taught in primary and secondary schools in three Australian states before obtaining his PhD in mathematics education. He has lectured at a number of local and overseas universities and is currently at the University of Western Sydney.

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