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Title of chapter/article	A Classroom Investigation into the Catenary
Author(s)	Ed Staples
Copyright owner	The Australian Association of Mathematics Teachers (AAMT) Inc.
Published in	Australian Senior Mathematics Journal vol. 25 no. 2
Year of publication	2011
Page range	43–54
ISBN/ISSN	0819-4564

A classroom investigation into the catenary

Ed Staples
Erindale College, ACT
<ed.staples@ed.act.edu.au>

> The Catenary is the curve that an idealised hanging chain or cable assumes when supported at its ends and acted on only by its own weight… The word *catenary* is derived from the Latin word *catena*, which means "chain". Huygens first used the term *catenaria* in a letter to Leibniz in 1690… Hooke discovered that the catenary is the ideal curve for an arch of uniform density and thickness which supports only its own weight. (Wikipedia, catenary)

The quest to find the equation of a catenary makes an ideal investigation for upper secondary students. In the modelling exercise that follows, no knowledge of calculus is required to gain a fairly good understanding of the nature of the curve. This investigation[1] is best described as a scientific investigation—a 'hands on' experience that examines some of the techniques used by science to find models of natural phenomena. It was Joachim Jungius (1587–1657) who proved that the curve followed by a chain was not in fact a parabola, (published posthumously in 1669, Wikipedia, catenary). Wolfram MathWorld also provides relevant information and interesting interactive demonstrations (see http://mathworld.wolfram.com/Catenary.html).

The investigation, to be described here, is underpinned by the fact that the catenary's equation can be thought of as a real valued polynomial consisting of an infinite number of even-powered terms described as:

$$y = a_0 + a_2x^2 + a_4x^4 + \ldots + a_{2n}x^{2n} + \ldots \qquad (1)$$

where the coefficients can be estimated through experimental modelling. The catenary can also be described (see Maor, 1994) by the more familiar form:

$$y = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad (2)$$

Unaware of the existence of equation (2), students are led through a modelling exercise in an attempt to discover a polynomial function that describes the chain's shape. They first carefully measure a series of 31 ordinates from an actual hanging chain on a wall space of dimensions 1.2 m by 2 m.

1. I saw a similar investigation conducted with students by John Short at the time working at Rosny College in 2004. The chain hung along the length of the wall some four or five metres, with no more than a metre vertically from the chain's vertex to the horizontal line between the two anchor points. Students were asked to take 17 coordinate measurements along the length of the chain. They were then directed to model the curve using a simple quadratic equation. Examination of a residuals plot lead students to the conclusion that this equation was a poor approximation to reality and that a more sophisticated model would be necessary.

2. I have no affiliation with the software package "Autograph" but I do believe it is a wonderful resource for upper secondary mathematics students. More information can be found at www.autograph-maths.com

Students can plot these as coordinate points using a suitable graphing package such as Microsoft Excel or Autograph[2]. They then are guided through a series of tasks that ultimately leads them to deduce a degree six polynomial function as a suitable model.

The connection between equations (1) and (2) is then explained to them, using an argument similar to that in the next section (see below). The coefficients of their own degree six polynomial can be compared with the coefficients of a polynomial equation derived from these (discussed later).

On the face of it, the investigation is about the catenary. However, the investigation also points in the direction of deeper considerations about the difference between mathematical and physical research and the apparent convergence of the two, each discipline motivating the other in the search for an ultimate reality. One approach—the scientific one—concerns careful measurement and modelling and the other approach is about well-constructed mathematical arguments from reasonable premises.

From the point of view of mathematical physics, students may see that the mathematical model describing the shape of the hanging chain can be progressively improved from Galileo's quadratic, through even order polynomials of ever higher degree, eventually arriving at the hyperbolic cosine function—an essentially mathematical invention.

In the process of this investigation, students should also see relevance from their normal lesson work through the use of a number of ideas such as oddness and evenness of functions, simultaneous equations, function modelling, lines of best fit, and many other concepts.

The equivalence of equations 1 and 2

Equation (2) can be shown to equal Equation (1) by considering the series expansions for e^x and e^{-x} given by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad (3a)$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad (3b)$$

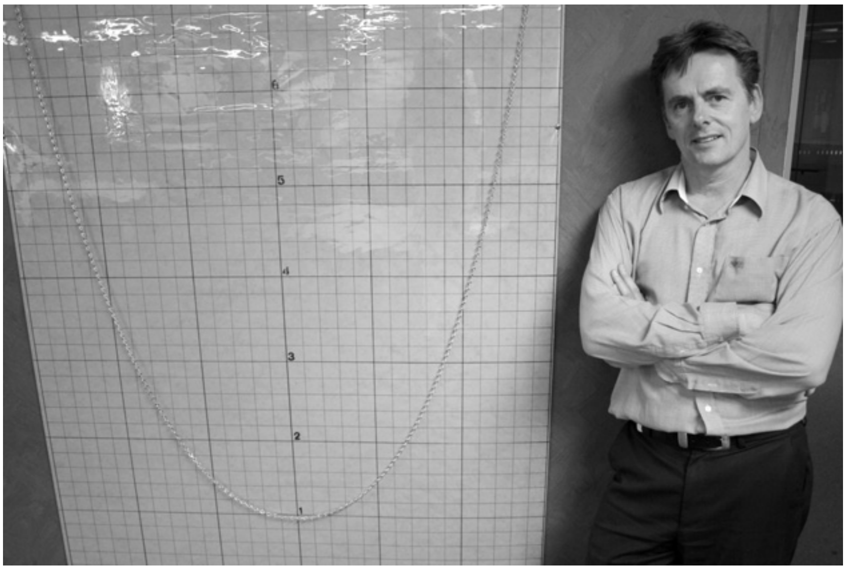
Substituting the equations (3a) and (3b) into (2) quickly establishes the result:

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}\dots \qquad (4)$$

The coefficients are simply the inverses of the factorials associated with each term's degree. For example the coefficient of the x^4 term is $\frac{1}{24}$, the inverse of 4!

Setting up the chain model of the catenary

With a little effort, teachers can set up a large coordinate system for the chain to hang against using a suitably-sized board that can be attached to a wall. (Note, you might consider constructing a board, perhaps made from MDF material of dimension 1.2 m by 2.2 metres high. The extra 0.2 m in height would leave 10 cm at the bottom to put *x* values on, and 10 cm at the top for a title.) The grid needs to be drawn up accurately within an area of dimensions 1.2 metres wide and 2 metres high as shown in Figure 1.



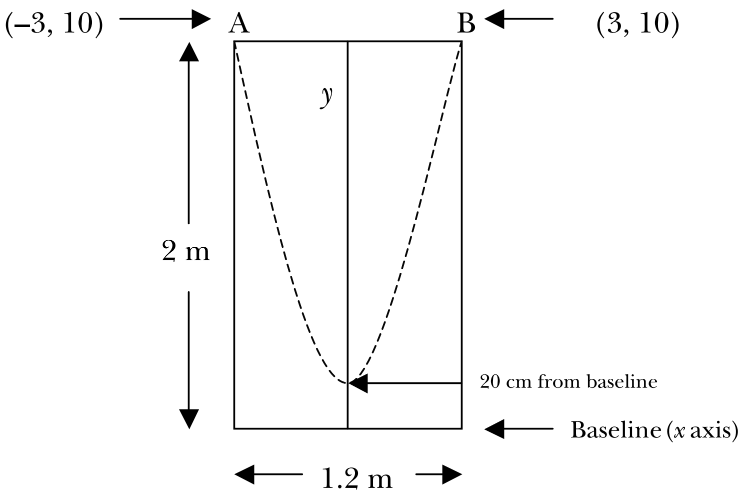


Figure 1

The coordinate system assumes an *x*-axis on the base line, and a central *y*-axis. The grid is then constructed using sharp vertical and horizontal lines 4 cm apart. The base line needs to represent the interval [–3,3] and the *y*-axis needs to represent the interval [0,10] so that means each coordinate unit is of length 20 cm in both the *x* and *y* directions. The grid lines are accordingly 0.2 coordinate units apart with five grid lines per unit. Points A and B are thus the coordinates (–3, 10) and (3, 10) respectively as shown in Figure 1.

Students can measure in centimetres (to 1 decimal place) using a standard metre rule, but they must convert all of their lengths to coordinates by the formula:

$$\text{coordinate values} = \frac{\text{measurement in cm}}{20}$$

The chain should hang from points A and B with its vertex 1 unit (20 cm) above the base line x-axis corresponding to the coordinate point (0,1).

After carefully measuring and calculating each of the 16 ordinates (in units to two decimal places) corresponding to the x values (–3,–2.8,–2.7…0), the students can fill out their y values in a table similar to Table 1.

A student sheet (without y values) is given in Appendix 1. Note that the 15 ordinates corresponding to the positive x values are exactly the same as the 15 ordinates corresponding to the negative x values. Table 2 shows an 'ideal set' of y values obtained from using:

$$y = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

The quest for the Holy Grail

Students will determine estimates of the coefficients in (5), the first 4 terms of equation (4), by progressively establishing polynomial models of ever increasing complexity. First they will attempt fitting a quadratic function, then a degree 4 polynomial, and finally a degree 6 polynomial, arriving at an expression that as near as possible resembles the true partial sum given by:

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} \tag{5}$$

Table 1. Student coordinate measurement table.

x	y	x	y
–3.0		0.2	
–2.8		0.4	
–2.6		0.6	
–2.4		0.8	
–2.2		1.0	
–2.0		1.2	
–1.8		1.4	
–1.6		1.6	
–1.4		1.8	
–1.2		2.0	
–1.0		2.2	
–0.8		2.4	
–0.6		2.6	
–0.4		2.8	
–0.2		3.0	
0			

Table 2. Ideal set of data.

x	y	x	y
–3.0	10.07	0.2	1.02
–2.8	8.25	0.4	1.08
–2.6	6.77	0.6	1.19
–2.4	5.56	0.8	1.34
–2.2	4.57	1.0	1.54
–2.0	3.76	1.2	1.81
–1.8	3.11	1.4	2.15
–1.6	2.58	1.6	2.58
–1.4	2.15	1.8	3.11
–1.2	1.81	2.0	3.76
–1.0	1.54	2.2	4.57
–0.8	1.34	2.4	5.56
–0.6	1.19	2.6	6.77
–0.4	1.08	2.8	8.25
–0.2	1.02	3.0	10.07
0	1.00		

Attempt 1: A parabola

In so far as the students are concerned, a first thought might be that the chain hangs as a parabola. Since the vertex is on the y-axis, this immediately implies that this simple model takes the form:

$$y = ax^2 + b \qquad (6)$$

As the curve passes through (0, 1), $b = 1$. If we assume that the two anchor points[3] are given by (–3, 10) and (3, 10), then we have that $9a + 1 = 10$ and this means that the parabola has the equation:

$$y = x^2 + 1 \qquad (7)$$

This is the first simple model that needs to be tested, and the best way to do this is with a simple overlaid graph as shown in Figure 2. I have sketched the parabola against an idealised 31 plotted points. Clearly the parabola is only a modest fit to the shape of the hanging chain, so a better model is sought.

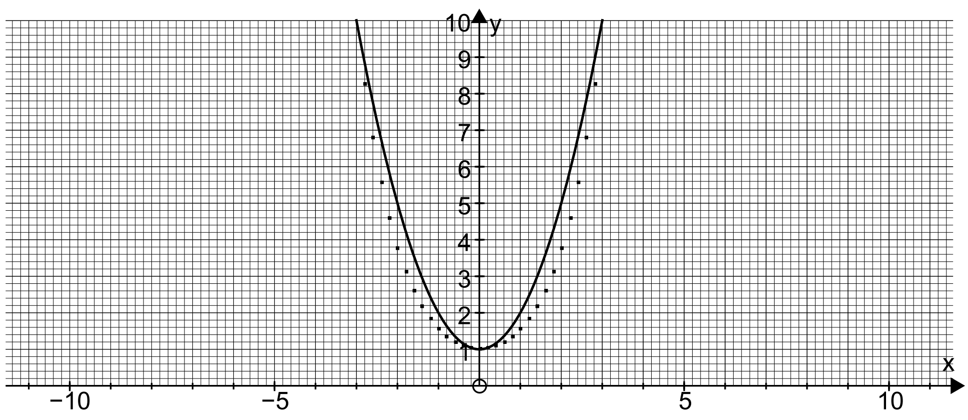


Figure 2. $y = x^2 + 1$, a modest fit for the catenary.

3. At $x = 3$, $y = \frac{1}{2}(e^3 + e^{-3})$ so the chain should be anchored slightly above 10 (slightly above the 2 m mark). In the 'first attempt' I have deliberately assumed this anchor point is (3,10) so that the coefficients of the fitting parabola are simple to calculate. Students may determine that the anchor points are somewhere around the coordinate (3,10.07) and (–3,10.07). Although I have used 10.07 as a measured data point in Table 2, these y values represent the best estimates based on the equation $y = \frac{1}{2}(e^3 + e^{-3})$. This column would be left blank for students to fill out through the investigation.

Attempt 2: A quartic polynomial

The next logical choice is to assume a quartic (degree 4) model that does not contain any cubic or linear term. The odd powers of the quartic are omitted from the model simply because of the symmetry of the catenary across the y-axis. Even powered polynomials are even functions. This means that we can say that for all $x \in \Re$, $f(-x) = f(x)$. Even polynomial functions are always symmetric about the y-axis.

This means finding coefficients a, b and c for the function $y = ax^4 + bx^2 + c$ and this can be done using three well spaced data points such as (3, 10), (2, 3.762) and (0, 1).

Again $c = 1$, and the other coefficients can be obtained by simultaneously solving:

$$81a + 9b = 9$$
$$16a + 4b = 2.76$$

A little bit of algebra yields $a = 0.07133$ and $b = 0.358$ and so the quartic becomes:

$$y = 0.07133x^4 + 0.358x^2 + 1 \qquad (8)$$

Figure 3 shows, with a scaled up *x*-axis, how good a fit this quartic is:

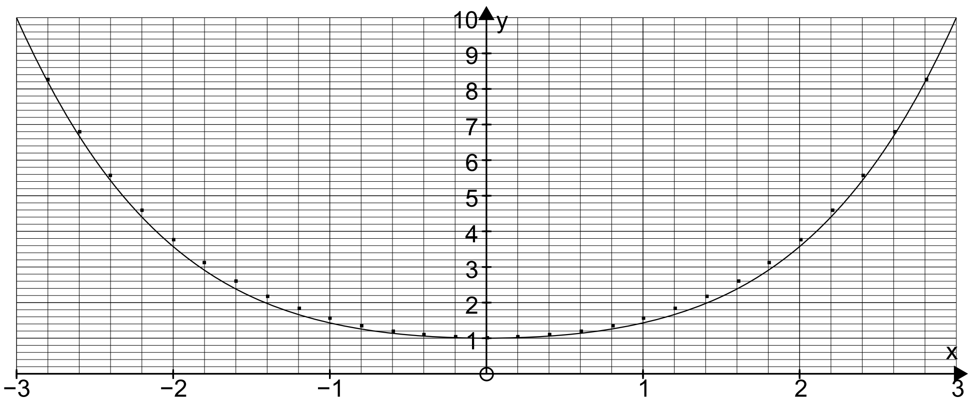


Figure 3. A quartic model of the catenary.

Attempt 3: Looking at the residuals to develop a degree 6 model

To proceed further, students are asked to look at the ordinate differences between their measured values collected in Table 2 and the quartic model estimates established in Attempt 2. These differences are generally known as *residuals*. These residuals are then graphed and inspected to identify any pattern. If there is a pattern, they then attempt to identify the function associated with that pattern. In a sense this curve 'explains' the variance between the measured estimates and the quartic model, and so adding this residual function to the quartic model must vastly improve the modelling.

This is a technique often used by any number of scientific investigations, where models are continually refined by the consideration of residuals between the actual data and some mathematical model trying to fit the data. In general terms, if these residuals show a well defined and perhaps recognisable pattern, they may indicate some other underlying mathematical curve that has not been included in previous models.

The residuals determined by the students will vary according to the data point estimates collected and tabulated in Table 1. My residuals (using the *y* values of Table 2) are shown in Figure 4 and tabulated in Table 3. I have included as Appendix 2 a template that students can use for their own calculations.

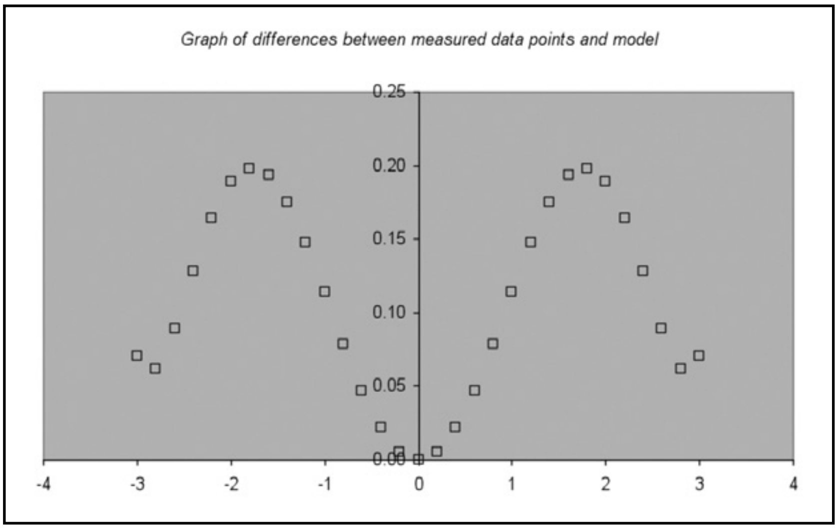


Figure 4. Plot of residuals of quartic model (differences show a symmetric pattern).

Table 3. Table of residuals between data points and the model $y = (0.07133)x^4 + (0.358)x^2 + 1$.

x	$f(x)$	model	difference
–3	10.07	10.00	0.07
–2.8	8.25	8.19	0.06
–2.6	6.77	6.68	0.09
–2.4	5.56	5.43	0.13
–2.2	4.57	4.40	0.16
–2	3.76	3.57	0.19
–1.8	3.11	2.91	0.20
–1.6	2.58	2.38	0.19
–1.4	2.15	1.98	0.18
–1.2	1.81	1.66	0.15
–1	1.54	1.43	0.11
–0.8	1.34	1.26	0.08
–0.6	1.19	1.14	0.05
–0.4	1.08	1.06	0.02
–0.2	1.02	1.01	0.01
0	1.00	1.00	0.00
0.2	1.02	1.01	0.01
0.4	1.08	1.06	0.02
0.6	1.19	1.14	0.05
0.8	1.34	1.26	0.08
1	1.54	1.43	0.11
1.2	1.81	1.66	0.15
1.4	2.15	1.98	0.18
1.6	2.58	2.38	0.19
1.8	3.11	2.91	0.20
2	3.76	3.57	0.19
2.2	4.57	4.40	0.16
2.4	5.56	5.43	0.13
2.6	6.77	6.68	0.09
2.8	8.25	8.19	0.06
3	10.07	10.00	0.07

Figure 4 clearly identifies a symmetric well defined pattern with the residuals. This was to be expected given that I used the idealised values. Students hopefully will derive a vaguely similar pattern depending of course on how accurately they made their first measurements. This is the main reason why the actual grid construction had to be sufficiently large. Figure 4 clearly shows the existence of five turning points shown in Table 4.

Table 4. Details of the five turning points.

x	y
–2.8	0.06
–1.8	0.20
0	0
1.8	0.20
2.8	0.06

From this evidence alone, we conclude that a residual function may have the form $y = ax^6 + bx^4 + cx^2$ and this can be expressed as $y = x^2(ax^4 + bx^2 + c)$ revealing the double root at $x = 0$. (Note: there are other methods to determine the values of a, b, and c. For example, by using three points we could set up a set of three simultaneous equations and solve them. You might note that $y = ax^6 + bx^4 + cx^2$ has a first derivative given by

$$\frac{dy}{dx} = 2x\left(3ax^4 + 2bx^2 + c\right)$$

providing us with a way to find the exact positions of the function's coordinates.

Microsoft Excel has a best fit feature which when activated as a degree 6 polynomial provides an exceptionally good fit to this residual function. Ignoring the miniscule coefficients on the odd powered terms, including the constant term, Excel suggests:

$$y = 0.0019x^6 - 0.0326x^4 + 0.1473x^2 \qquad (9)$$

As a check on this residual function I substituted $x = 1.8$ into equation (9) and found that $y = 0.19965$, which is a very close estimate of the location of the turning point.

The new model arrived at by the addition of the residual function to the quartic model is given by:

$$y = [0.07133x^4 + 0.358x^2 + 1] + [0.0019x^6 - 0.0326x^4 + 0.1473x^2]$$

or

$$y = 0.0019x^6 + 0.033873x^4 + 0.5053x^2 + 1 \qquad (10)$$

The graphs of the model and the true chain curve are now virtually indistinguishable, but using Autograph, we can zoom in on a section of the curve (Figure 5) to realise that there is still a very small difference which is so close that it is difficult to tell which is which.

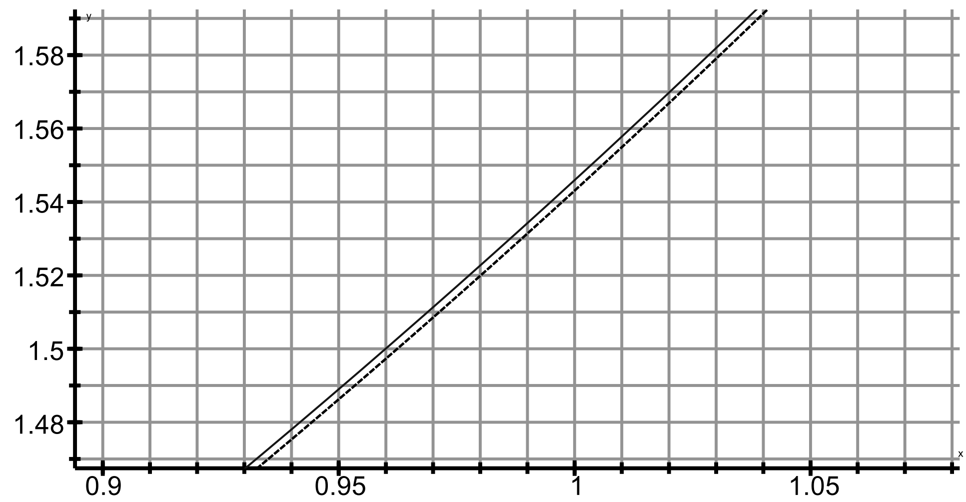


Figure 5. The degree 6 model and the catenary

We could repeat the differences approach to this new model, and find higher and higher degree polynomials that would keep minimising the gaps. Each of these polynomials would not only add new terms to the model, but would supplement existing terms in the way that we just saw.

Considering the catenary's equation (equation 4) if we decimalise the coefficients of the first four terms we see that:

$$y \approx 1 + 0.5x^2 + 0.04167x^4 + 0.0014x^6 \qquad (11)$$

Compare this to equation (9) shows that we are fairly close to the true representation of the catenary's curve.

Insight into student thinking

An extended assignment was conducted at Erindale College around this investigation. A large board was purchased to construct the grid and a chain was duly hung in what is known as the learning common. Students engaged well with the task set producing some outstanding work. It is thrilling as a teacher to see their efforts pay off particularly the way they attempted to solve issues around measurement error. I have included an image that was submitted by one student. Figure 6 shows his residuals—it was as if the truth around the degree six polynomial was just visible through the veil of measurement error.

This student decided to eliminate the odd-powered terms (that showed fairly small coefficient values anyway) on the basis that the asymmetry must be caused by measurement error because in his words "there was no reason to assume chain hung differently across its axis of symmetry". The insight is a profound one.

His final function was $y = 0.0026x^6 + 0.02963583x^4 + 0.51711641x^2 + 0.9903$ and when graphed proved remarkably close to the actual curve. He was also

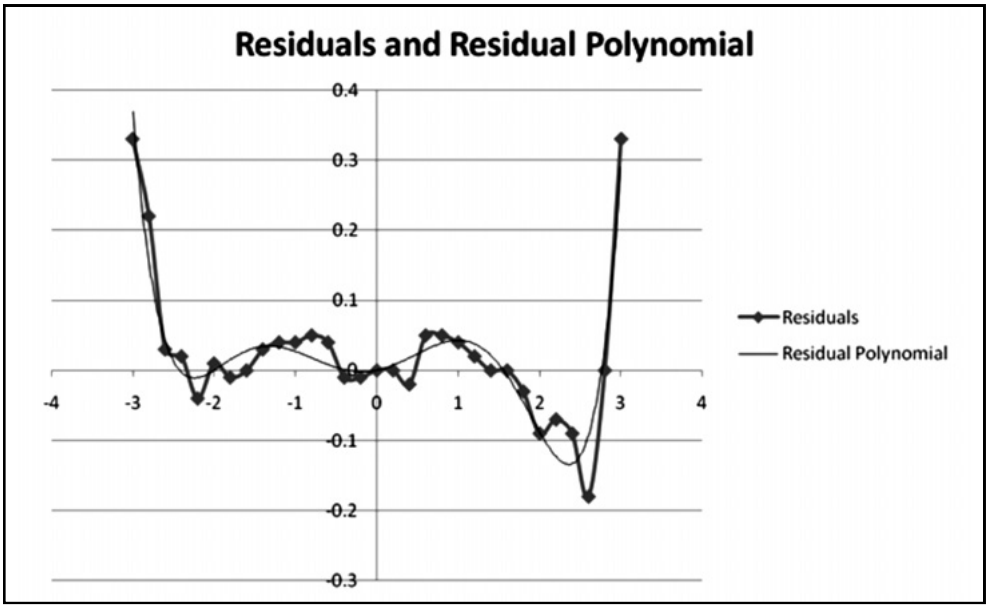


Figure 6. One student's response to the residuals.

able to explain that while the constant term and the coefficient of the squared term were close to 1 and $\frac{1}{2}$, adjustments to all of the coefficients could be expected if it was possible to derive a degree 8 residual polynomial. Again this indicated to me that this student had learnt how the residual polynomial supplemented simpler models.

As a teacher, given that you have got a little patience and time in constructing the chain, I can thoroughly recommend this as a worthwhile learning experience.

Conclusion

Using a real chain to make the measurements from is far more preferable than simply reading data points from a photocopied sheet. Students not only have to deal with the physical limitations of measuring the ordinates accurately, but it is in the doing that they gain a sense of real world connectedness. They see for themselves the physical reality of an *actual chain* bending in this beautiful way.

Depending on the care taken to get the measurements, and on how well the catenary is set up in the first place, the results may vary somewhat from the ideal measurements shown. There is always pedagogical value in result variability—things are never perfect in a less than ideal world. In addition to this, students have the opportunity to play with some of the features of Microsoft Excel and Autograph.

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Appendix 1: Student sheet

Table of values (y to 2 decimal places)

Table 1 (y values indicate ideal measurements).

x	y	x	y
–3		0.2	
–2.8		0.4	
–2.6		0.6	
–2.4		0.8	
–2.2		1.0	
–2.0		1.2	
–1.8		1.4	
–1.6		1.6	
–1.4		1.8	
–1.2		2.0	
–1.0		2.2	
–0.8		2.4	
–0.6		2.6	
–0.4		2.8	
–0.2		3.0	
0			

Appendix 2:
Student template for determining residuals

From table 1	Chain Curve	Model	Residuals
x values	Table 1 y values	Quartic y values	Table y – Quartic y
–3.0			
–2.8			
–2.6			
–2.4			
–2.2			
–2.0			
–1.8			
–1.6			
–1.4			
–1.2			
–1.0			
–0.8			
–0.6			
–0.4			
–0.2			
0			
0.2			
0.4			
0.6			
0.8			
1.0			
1.2			
1.4			
1.6			
1.8			
2.0			
2.2			
2.4			
2.6			
2.8			
3.0			